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A minimax theorem not involving convexities of the function

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Dedicated to Professor Ákos Császár for his 70th birthday

In this paper we prove the following

Theorem. *Let X be a compact topological space, Y be an arbitrary topological space and $f : X \times Y \to \mathbb{R}$ a function continuous in x for every fixed $y \in Y$ and lower semicontinuous in y for every fixed $x \in X$. This means that the level sets $H_c^y = \{x : f(x,y) \geq c\}$ and $H_x^c = \{y : f(x,y) \leq c\}$ are closed in X resp in Y and $\hat{H}_c^y = \{x : f(x,y) \succ c\}$ are open in X. Suppose further that*

a) *The intersection of any (not necessarily finite numbers of H_x^c sets), $x \in X$, $c \in \mathbb{R}$ is connected (may be empty).*

b) *The intersection of any finitely many $\hat{H}_c^y$ is connected (may be empty).*

Then

$$\sup_x \inf_y f(x,y) = \inf_y \sup_x f(x,y).$$

Proof of the Theorem. Define an interval structure on Y as follows: let $[y_1, y_2] = \bigcap_{x,c} \{H_x^c : y_1, y_2 \in H_x^c\}$.

If we choose the constant c sufficiently large, we obtain $y_1, y_2 \in H_x^c$. Hence $[y_1, y_2]$ is well-defined, closed connected and $y_1, y_2 \in [y_1, y_2]$. The sets H_x^c become convex in the sense that $y_1, y_2 \in H_x^c$ implies $[y_1, y_2] \in H_x^c$. Consequently we can apply Theorem B of Joó [1] stating that if

(1) $H_x^c \bigcap [y_1, y_2]$ is closed in $[y_1, y_2]$

(2) H_x^c is convex

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(3) $\bigcap_{i=1}^{n} \hat{H}_c^{y_i}$ is connected for every finite intersection

then $\bigcap_{i=1}^{n} \hat{H}_c^{y_i} \neq 0$ for every $c < c^y = \inf_y \sup_x f$ hence the sets H_c^y have also the finite intersection property. In our case (1) holds since H_x^c and $[y_1, y_2]$ are closed in Y, (2) was remarked above and (3) is identical to b), hence Theorem B of [1] applies. Since the sets H_c^y are compact, this implies $\bigcap_{y \in Y} H_c^y \neq 0$ i.e. there exist $x_0 \in X$ with $f(x_0, y) \geq c$ for all $y \in Y$. Hence $c < c^*$ implies $c \leq \sup_x \inf_y f$ i.e. $c^* \leq \sup_x \inf_y f$ which proves the desired minimax equality. □

References

[1] I. Joó, On a fixed point theorem, *Publicationes Mathematicae, Debrecen* **36** (1989), 127–128.

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