

Topologies and orders on function spaces

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Introduction

By Y and Z we denote two fixed topological spaces and by $C(Y, Z)$ we denote the set of all continuous maps of Y into Z . If τ is a topology on the set $C(Y, Z)$, then the corresponding topological space is denoted by $C_\tau(Y, Z)$.

Let X be a space and $F : X \times Y \rightarrow Z$ be a continuous map. By F_x , where $x \in X$, we denote the continuous map of Y into Z , for which $F_x(y) = F(x, y)$, for every $y \in Y$. By $\widehat{F}$ we denote the map of X into the set $C(Y, Z)$, for which $\widehat{F}(x) = F_x$, for every $x \in X$.

Let G be a map of the space X into the set $C(Y, Z)$. By $\widetilde{G}$ we denote the map of the space $X \times Y$ into the space Z , for which $\widetilde{G}(x, y) = G(x)(y)$, for every $(x, y) \in X \times Y$. It is easy to verify that $\widehat{\widetilde{G}} = G$ and $\widetilde{\widehat{F}} = F$.

A topology τ on $C(Y, Z)$ is called *splitting* (respectively, *jointly continuous*) (see [1]) if and only if for every space X , the continuity of a map $F : X \times Y \rightarrow Z$ (respectively, a map $G : X \rightarrow C_\tau(Y, Z)$) implies that of the map $\widehat{F} : X \rightarrow C_\tau(Y, Z)$ (respectively, of the map $\widetilde{G} : X \times Y \rightarrow Z$). If this condition is satisfied for the elements of the family $\mathcal{A}$ of spaces, then the topology is called $\mathcal{A}$ -*splitting* (respectively, $\mathcal{A}$ -*jointly continuous*). (See [2]). If $\mathcal{A} = \{X\}$, then instead of “ $\mathcal{A}$ -splitting” and “ $\mathcal{A}$ -jointly continuous” we write “ X -splitting” and “ X -jointly continuous”. The greatest $\mathcal{A}$ -splitting topology, which always exists, is denoted by $\tau(\mathcal{A})$.

We recall some notions. (See, for example, [3]). For every space X with a topology τ we define a preorder “ $\stackrel{\tau}{\leq}$ ” and an equivalence relation “ $\sim^\tau$ ” on X as follows: if $x, y \in X$, then we write $x \stackrel{\tau}{\leq} y$ (respectively, $x \sim^\tau y$) if and only if $x \in \text{Cl}_X(\{y\})$ (respectively, $x \in \text{Cl}_X(\{y\})$ and $y \in \text{Cl}_X(\{x\})$).

(By $\text{Cl}_X(Q)$ we denote the closure of the set Q in the space X). It is easy to see that the preorder “ $\leq^\tau$ ” on X is a partial order if and only if X is a T_0 -space. Also, the points x and y are $\sim^\tau$ -equivalent if and only if for every open subset U of X either $x, y \in U$ or $x, y \notin U$.

On the set $C(Y, Z)$ we define a preorder “ $\leq$ ” and an equivalence relation “ $\sim$ ” as follows: if $g, f \in C(Y, Z)$, then we write $g \leq f$ (respectively, $g \sim f$) if and only if $g(y) \leq^\tau f(y)$ (respectively, $g(y) \sim^\tau f(y)$), for every $y \in Y$, where τ is the topology of the space Z . Obviously, if Z is a T_0 -space, then the preorder “ $\leq$ ” on $C(Y, Z)$ is a partial order. Also, $g \sim f$ if and only if $g \leq f$ and $f \leq g$.

If X is a set equipped with a preorder “ $\leq$ ”, then we set $[y, \rightarrow)_\leq = \{x \in X : y \leq x\}$ and $(\leftarrow, y]_\leq = \{x \in X : x \leq y\}$.

Let $\mathcal{U}$ be a quasi-uniformity on the space Z . (See, for example, [8]). This quasi-uniformity defines on the set $C(Y, Z)$ a quasi-uniformity $\mathcal{Q} \equiv \mathcal{Q}(\mathcal{U})$ as follows (see [5]): the set of all subsets of $C(Y, Z)$ of the form $(Y, U) \equiv \{(f, g) \in C(Y, Z) \times C(Y, Z) : (f(y), g(y)) \in U, \text{ for every } y \in Y\}$,

where $U \in \mathcal{U}$, is a basis for the quasi-uniformity $\mathcal{Q}$. We denote by $\tau_{\mathcal{Q}}$ the topology on $C(Y, Z)$, which is defined by the quasi-uniformity $\mathcal{Q}$ and we say that $\tau_{\mathcal{Q}}$ is *generated* by the quasi-uniformity $\mathcal{U}$ on the space Z .

By $\mathbf{S}$ we denote the Sierpinski space, that is, the set $\{0, 1\}$ equipped with the topology $\tau(\mathbf{S}) \equiv \{\emptyset, \{0, 1\}, \{1\}\}$, and by $\mathbf{D}$ the set $\{0, 1\}$ with the trivial topology.

In the present paper we study the connections of the natural preorder “ $\leq$ ” and equivalence relation “ $\sim$ ” on the set $C(Y, Z)$ with the notions of X -splitting and X -jointly continuous topologies on this set, where X is either the space $\mathbf{S}$ or the space $\mathbf{D}$.

1. Theorem. *A topology τ on $C(Y, Z)$ is $\mathbf{S}$ -splitting if and only if from the condition $g \leq f$ it follows that $g \leq^\tau f$.*

PROOF. Let τ be an $\mathbf{S}$ -splitting topology on $C(Y, Z)$ and let $g \leq f$, where $g, f \in C(Y, Z)$. We prove that $g \leq^\tau f$.

Let $F : \mathbf{S} \times Y \rightarrow Z$ be a map for which $F(1, y) = f(y)$ and $F(0, y) = g(y)$, $y \in Y$. We prove that F is continuous. Let $F(1, y) = f(y)$ and let U be an open neighbourhood of $f(y)$ in Z . Since f is continuous, the set $f^{-1}(U)$ is open in Y . The set $V = \{1\} \times f^{-1}(U)$ is an open neighbourhood of $(1, y)$ in $\mathbf{S} \times Y$ and $F(V) \subseteq U$, which means that F is continuous at the point $(1, y) \in \mathbf{S} \times Y$.

Let $F(0, y) = g(y)$ and U be an open neighbourhood of $g(y)$ in Z . The set $V = \mathbf{S} \times g^{-1}(U)$ is an open neighbourhood of $(0, y)$ in $\mathbf{S} \times Y$. We prove that $F(V) \subseteq U$. Indeed, if $(0, y_1) \in V$, then $F(0, y_1) = g(y_1) \in U$.

If $(1, y) \in V$, then $F(1, y_1) = f(y_1)$. Since $g \leq f$ and $g(y_1) \in U$ we have that $f(y_1) \in U$. Hence, $F(V) \subseteq U$ and, therefore, F is continuous.

Since τ is $\mathbf{S}$ -splitting the map $\widehat{F} : \mathbf{S} \rightarrow C_\tau(Y, Z)$ is continuous. We have that $\widehat{F}(1) = f$ and $\widehat{F}(0) = g$. Let W be an open neighbourhood of g in $C_\tau(Y, Z)$. Then, $\widehat{F}^{-1}(W)$ is an open neighbourhood of $0 \in \mathbf{S}$. Since $0 \in \text{Cl}_{\mathbf{S}}(\{1\})$ we have that $1 \in \widehat{F}^{-1}(W)$, that is, $\widehat{F}(1) = f \in W$. This means that $g \in \text{Cl}_{C_\tau(Y, Z)}(\{f\})$. Hence $g \leq^\tau f$.

Conversely, let τ be a topology on $C(Y, Z)$ such that from the condition $g \leq f$ it follows that $g \leq^\tau f$. We prove that τ is $\mathbf{S}$ -splitting. Let $F : \mathbf{S} \times Y \rightarrow Z$ be a continuous map. Consider the map $\widehat{F} : \mathbf{S} \rightarrow C_\tau(Y, Z)$. Let $\widehat{F}(1) = f$ and $\widehat{F}(0) = g$. We prove that $g \leq f$. Indeed, let $y \in Y$ and let U be an open neighbourhood of $g(y)$ in Z . Since F is continuous, the set $F^{-1}(U)$ is an open neighbourhood of $(0, y)$ in $\mathbf{S} \times Y$. It is easy to see that every open neighborhood of $(0, y)$ contains the point $(1, y)$. Hence, $(1, y) \in F^{-1}(U)$, that is, $f(y) \in U$. Therefore, $g \leq f$. By the assumption, $g \leq^\tau f$.

Let U be an open subset of $C_\tau(Y, Z)$. If $f, g \in U$ or $f, g \notin U$, then $\widehat{F}^{-1}(U)$ is open in $\mathbf{S}$. If $f \in U$ and $g \notin U$, then $\widehat{F}^{-1}U = \{1\} \in \tau(\mathbf{S})$. Let $g \in U$. Since $g \leq^\tau f$ we have that $f \in U$ and, hence, the set $\widehat{F}^{-1}(U)$ is open in $\mathbf{S}$. Thus, $\widehat{F}$ is continuous and, therefore, the topology τ is $\mathbf{S}$ -splitting.

2. Corollary. *The discrete topology and, hence, every topology on $C(Y, Z)$ is $\mathbf{S}$ -splitting if and only if the space Z is a T_1 -space.*

PROOF. If Z is a T_1 -space, then by the condition $g \leq f$, where $g, f \in C(Y, Z)$, it follows that $g = f$. Hence, $g \leq^\tau f$, for every topology τ on $C(Y, Z)$. Therefore, by Theorem 1, every topology on $C(Y, Z)$ is $\mathbf{S}$ -splitting.

Conversely, suppose that every topology on $C(Y, Z)$ is $\mathbf{S}$ -splitting. If Z is not a T_1 -space, then there exist points $x, y \in Z$, $x \neq y$, such that $x \leq y$. Let $f, g \in C(Y, Z)$ such that $g(Y) = \{x\}$ and $f(Y) = \{y\}$. Then, $g \leq f$ and $g \neq f$. By Theorem 1, $g \leq^\tau f$, for every topology τ on $C(Y, Z)$. If τ is the discrete topology, then $g = f$, which is a contradiction. Hence, Z is a T_1 -space.

3. Theorem. *A topology τ on $C(Y, Z)$ is $\mathbf{S}$ -jointly continuous if and only if from the condition $g \leq^\tau f$ it follows that $g \leq f$.*

PROOF. Let τ be an $\mathbf{S}$ -jointly continuous topology on $C(Y, Z)$ and let $g \leq^\tau f$, where $g, f \in C(Y, Z)$. We prove that $g \leq f$.

Let $G : \mathbf{S} \rightarrow C(Y, Z)$ be a map for which $G(0) = f$ and $G(1) = g$. We prove that G is continuous. Let U be an open subset of $C(Y, Z)$. If $g \in U$, then since $g \leq^{\tau} f$ we have that $f \in U$ and, hence, the set $G^{-1}(U) = \mathbf{S}$ is open. Also, if $g \notin U$ and $f \notin U$, then the set $G^{-1}(U) = \emptyset$ is open. Hence, the map G is continuous. Since τ is $\mathbf{S}$ -jointly continuous, the map $\tilde{G} : \mathbf{S} \times Y \rightarrow Z$ is also continuous.

Let $y \in Y$ and let W be an open neighbourhood of $g(y)$ in Z . Then, $\tilde{G}^{-1}(W)$ is an open subset of $\mathbf{S} \times Y$ containing the point $(0, y)$. There exist an open neighbourhood V_1 of 0 in $\mathbf{S}$ and an open neighbourhood V_2 of y in Y such that $V_1 \times V_2 \subseteq \tilde{G}^{-1}(W)$. Since $1 \in V_1$ we have that $(1, y) \in \tilde{G}^{-1}(W)$, which means that $F(1, y) = f(y) \in W$. Thus, $g(y) \in \text{Cl}_Z(\{f(y)\})$. Hence, $g \leq f$.

Conversely, let τ be a topology on $C(Y, Z)$ such that from the condition $g \leq^{\tau} f$ it follows that $g \leq f$. We prove that τ is $\mathbf{S}$ -jointly continuous. Let $G : \mathbf{S} \rightarrow C(Y, Z)$ be a continuous map and let $G(1) = f$ and $G(0) = g$. We prove that $g \leq^{\tau} f$. Indeed, let U be an open neighbourhood of g in $C(Y, Z)$. Since G is continuous, the set $G^{-1}(U)$ is an open subset of $\mathbf{S}$ containing the point 0. Hence, $1 \in G^{-1}(U)$ and, therefore, $G(1) = f \in U$, which means that $g \leq^{\tau} f$.

Consider the map $\tilde{G} : \mathbf{S} \times Y \rightarrow Z$. Let W be an open subset of Z and let $(1, y) \in \tilde{G}^{-1}(W)$. Then, the set $\{1\} \times f^{-1}(W)$ is an open subset of $\mathbf{S} \times Y$ containing the point $(1, y)$ such that $\tilde{G}(\{1\} \times f^{-1}(W)) \subseteq W$.

Let $(0, y) \in \tilde{G}^{-1}(W)$. Consider the open set $\mathbf{S} \times g^{-1}(W)$ of $\mathbf{S} \times Y$. Obviously, $(0, y) \in \mathbf{S} \times g^{-1}(W)$. We prove that $\tilde{G}(\mathbf{S} \times g^{-1}(W)) \subseteq W$. Let $(0, y_1) \in \mathbf{S} \times g^{-1}(W)$. Then $\tilde{G}(0, y_1) = g(y_1) \in W$. Let $(1, y_2) \in \mathbf{S} \times g^{-1}(W)$. Then, $y_2 \in g^{-1}(W)$, that is, $g(y_2) \in W$. By the assumption, $f(y_2) \in W$ and, hence, $\tilde{G}(1, y_2) = f(y_2) \in W$. Thus, the map $\tilde{G}$ is continuous. Hence, τ is an $\mathbf{S}$ -jointly continuous topology.

4. Corollary. *The trivial topology and, hence, every topology on the set $C(Y, Z)$ is $\mathbf{S}$ -jointly continuous if and only if the topology of Z is trivial.*

PROOF. Suppose that the topology of Z is trivial. Then, $f \leq g$ for every $f, g \in C(Y, Z)$. By Theorem 3, it follows that every topology τ on $C(Y, Z)$ is $\mathbf{S}$ -jointly continuous.

Conversely, suppose that every topology on $C(Y, Z)$ is $\mathbf{S}$ -jointly continuous. Let τ be the trivial topology on $C(Y, Z)$. Then, for every $f, g \in C(Y, Z)$ we have $g \leq^{\tau} f$. If the topology of Z is not trivial, then there exist points $x, y \in Z$, $x \neq y$, such that $y \notin \text{Cl}_Z(\{x\})$. Let $f, g \in C(Y, Z)$

such that $f(Y) = \{x\}$ and $g(Y) = \{y\}$. Then $g \not\leq f$, which by Theorem 3 is a contradiction.

5. Corollary. *A topology τ on $C(Y, Z)$ is simultaneously $\mathbf{S}$ -splitting and $\mathbf{S}$ -jointly continuous if and only if the preorders “ $\leq^\tau$ ” and “ $\leq$ ” coincide.*

6. Theorem. *The subsets of $C(Y, Z)$ of the form $[g, \rightarrow)_\leq$, $g \in C(Y, Z)$, compose a basis for open sets in the space $C_{\tau(\{\mathbf{S}\})}(Y, Z)$.*

PROOF. Let τ be a topology on $C(Y, Z)$, for which the sets of the form $[g, \rightarrow)_\leq$, $g \in C(Y, Z)$, compose a subbasis for open sets. Obviously, if $f \in [g, \rightarrow)_\leq$, then $[f, \rightarrow)_\leq \subseteq [g, \rightarrow)_\leq$. Hence, if $g \in U \in \tau$, then $[g, \rightarrow)_\leq \subseteq U$. Thus, if $g \leq f$, then $g \leq^\tau f$. Hence, by Theorem 1, τ is an $\mathbf{S}$ -splitting topology. Therefore, $\tau \subseteq \tau(\{\mathbf{S}\})$.

Let $U \in (\{\mathbf{S}\})$ and $g \in U$. By Theorem 1, $[g, \rightarrow)_\leq \subseteq U$ and, hence, $U \in \tau$. This means that $\tau(\{\mathbf{S}\}) \subseteq \tau$ and the sets of the form $[g, \rightarrow)_\leq$, $g \in C(Y, Z)$, compose a basis for the topology $\tau(\{\mathbf{S}\})$.

7. Corollary. *The set of all subsets of $C(Y, Z)$ of the form $\{f \in C(Y, Z) : g^{-1}(U) \subseteq f^{-1}(U)\}$, for every $U \in \mathcal{O}(Z)\}$, $g \in C(Y, Z)$, is a basis for the greatest $\mathbf{S}$ -splitting topology on $C(Y, Z)$.*

The proof of this corollary follows by the relation $[g, \rightarrow)_\leq = \{f \in C(Y, Z) : g^{-1}(U) \subseteq f^{-1}(U)\}$, for every $U \in \mathcal{O}(Z)\}$.

8. Corollary. *The greatest $\mathbf{S}$ -splitting topology $\tau(\{\mathbf{S}\})$ on $C(Y, Z)$ has the following property: the intersection of any family of open sets is open, that is, every element f of $C(Y, Z)$ has a smallest open neighbourhood in the space $C_{\tau(\{\mathbf{S}\})}(Y, Z)$.*

9. Theorem. *The greatest $\mathbf{S}$ -splitting topology is $\mathbf{S}$ -jointly continuous.*

PROOF. Let $f, g \in C(Y, Z)$ and $g \leq^{\tau(\{\mathbf{S}\})} f$. Since $[g, \rightarrow)_\leq$ is an open neighbourhood of g in $C_{\tau(\{\mathbf{S}\})}(Y, Z)$ (see Theorem 6) we have that $f \in [g, \rightarrow)_\leq$, that is, $g \leq f$. By Theorem 3, $\tau(\{\mathbf{S}\})$ is $\mathbf{S}$ -jointly continuous.

10. Theorem. *In the set of all simultaneously $\mathbf{S}$ -splitting and $\mathbf{S}$ -jointly continuous topologies on $C(Y, Z)$ there exists a smallest topology denoted by $\tau_{\min}(\{\mathbf{S}\})$. Moreover, the set $C(Y, Z)$ and the subsets of $C(Y, Z)$ of the form $C(Y, Z) \setminus (\leftarrow, g]_\leq$, $g \in C(Y, Z)$, compose a subbasis for this topology.*

PROOF. Let τ be a topology on the set $C(Y, Z)$, for which the set $C(Y, Z)$ and the subsets of $C(Y, Z)$ of the form $C(Y, Z) \setminus (\leftarrow, g]_\leq$, $g \in C(Y, Z)$, compose a subbasis. We prove that τ is $\mathbf{S}$ -splitting. Indeed, let

$g \leq f$, that is, $f \in [g, \rightarrow)_{\leq}$ and let $g \in U \in \tau$. If $U = C(Y, Z)$, then $f \in U$. If $U \neq C(Y, Z)$, then there exist elements $g_1, \dots, g_k \in C(Y, Z)$ such that

$$g \in \bigcap \{C(Y, Z) \setminus (\leftarrow, g_i]_{\leq} : i \in \{1, \dots, k\}\} \subseteq U.$$

Hence,

$$g \notin \bigcup \{(\leftarrow, g_i]_{\leq} : i \in \{1, \dots, k\}\}.$$

Therefore,

$$\begin{aligned} [g, \rightarrow)_{\leq} &\subseteq C(Y, Z) \setminus (\bigcup \{(\leftarrow, g_i]_{\leq} : i \in \{1, \dots, k\}\}) \\ &= \bigcap \{C(Y, Z) \setminus (\leftarrow, g_i]_{\leq} : i \in \{1, \dots, k\}\} \subseteq U. \end{aligned}$$

Therefore, $f \in U$, that is $g \stackrel{\tau}{\leq} f$. By Theorem 1, τ is $\mathbf{S}$ -splitting.

We prove that τ is $\mathbf{S}$ -jointly continuous. Let $g \stackrel{\tau}{\leq} f$. By Theorem 3 it is sufficient to prove that $g \leq f$, that is, $f \in [g, \rightarrow)_{\leq}$. If $f \notin [g, \rightarrow)_{\leq}$, then the set $C(Y, Z) \setminus (\leftarrow, f]_{\leq}$ is an open neighbourhood of g , which does not contain the element f . Since $g \stackrel{\tau}{\leq} f$, this is a contradiction. Hence, $g \leq f$.

Now we prove that τ is the smallest $\mathbf{S}$ -splitting and $\mathbf{S}$ -jointly continuous topology on $C(Y, Z)$, that is $\tau = \tau_{\min}(\{\mathbf{S}\})$. Let τ' be an $\mathbf{S}$ -splitting and $\mathbf{S}$ -jointly continuous topology on $C(Y, Z)$. We prove that $\tau \subseteq \tau'$. Let $g, f \in C(Y, Z)$ and $f \in C(Y, Z) \setminus (\leftarrow, g]_{\leq}$. It is sufficient to prove that there exists an element $V \in \tau'$ such that

$$f \in V \subseteq C(Y, Z) \setminus (\leftarrow, g]_{\leq}.$$

We have that $f \notin (\leftarrow, g]_{\leq}$, that is, $f \not\leq g$. Since τ' is $\mathbf{S}$ -jointly continuous, by Theorem 3 it follows that $f \stackrel{\tau'}{\not\leq} g$. Therefore, there exists an element $V \in \tau'$ such that $f \in V$ and $g \notin V$. Since τ' is $\mathbf{S}$ -splitting, by Theorem 1 we have that $(\leftarrow, g]_{\leq} \cap V = \emptyset$. Hence,

$$f \in V \subseteq C(Y, Z) \setminus (\leftarrow, g]_{\leq}.$$

11. Theorem. *The pointwise topology on $C(Y, Z)$ is $\mathbf{S}$ -jointly continuous.*

PROOF. Let τ_p be the pointwise topology on $C(Y, Z)$. Then, the sets of the form

$$(y, U) = \{f \in C(Y, Z) : f(y) \in U\},$$

where $y \in Y$ and U is an open subset of Z , compose a subbasis for τ_p .

Let $f \stackrel{\tau_p}{\leq} g$, $y \in Y$ and U be an open neighbourhood of $f(y)$ in Z . Then, $f \in (y, U)$. Since $f \stackrel{\tau_p}{\leq} g$, we have that $g \in (y, U)$, that is, $g(y) \in U$.

This means that $f(y) \in \text{Cl}_Z(\{g(y)\})$ for every $y \in Y$, that is, $f \leq g$. By Theorem 3, τ_p is $\mathbf{S}$ -jointly continuous.

12. Corollary. *The compact-open topology and the Isbell topology (see, for example, [4]) on $C(Y, Z)$ $\mathbf{S}$ -jointly continuous.*

13. Corollary. *For the compact-open topology and Isbell topology the preorders " $\overset{\tau}{\leq}$ " and " $\leq$ " in $C(Y, Z)$ coincide.*

14. Remark. Propositions 3.6 of [6] and 3.2 of [7] follow immediately by Corollary 13.

15. Theorem. *For every $f \in C(Y, Z)$, the intersection of all neighbourhoods of f in the space $C_{\tau_{\min}(\{\mathbf{S}\})}(Y, Z)$ is the smallest open neighbourhood of f in the space $C_{\tau(\{\mathbf{S}\})}(Y, Z)$.*

PROOF. Let $f \in U \in \tau_{\min}(\{\mathbf{S}\})$. We prove that $[f, \rightarrow]_{\leq} \subseteq U$. It is sufficient to suppose that $U = C(Y, Z) \setminus (\leftarrow, g]_{\leq}$, for some $g \in C(Y, Z)$. Let $h \in [f, \rightarrow]_{\leq}$. Since $f \in U$, we have that $f \notin (\leftarrow, g]_{\leq}$. Hence, $h \notin (\leftarrow, g]_{\leq}$ and, therefore $h \in U$.

For the proof of the theorem, it is sufficient to prove that if $h \notin [f, \rightarrow]_{\leq}$, then there exists an element V of $\tau_{\min}(\{\mathbf{S}\})$ such that $f \in V$ and $h \notin V$. Obviously, the set $V = C(Y, Z) \setminus (\leftarrow, h]_{\leq}$ is the required open set.

16. Theorem. *Every topology on the set $C(Y, Z)$, which is generated by a quasi-uniformity on the space Z is $\mathbf{S}$ -splitting and $\mathbf{S}$ -jointly continuous.*

PROOF. Let $\mathcal{U}$ be a quasi-uniformity on the space Z and let $\tau_{\mathcal{Q}}$ be the corresponding topology on the set $C(Y, Z)$. Since $\tau_{\mathcal{Q}}$ is jointly continuous (see [5]), this topology is also $\mathbf{S}$ -jointly continuous.

We prove that $\tau_{\mathcal{Q}}$ is $\mathbf{S}$ -splitting. Let $g, f \in C(Y, Z)$ and $g \leq f$. Let H be a neighbourhood of g in the space $C_{\tau_{\mathcal{Q}}}(Y, Z)$. We can suppose that

$$H \equiv (Y, U)_{(g)} \equiv \{h \in C(Y, Z) : (g, h) \in (Y, U)\},$$

where U is an element of $\mathcal{U}$. We prove that $f \in H$, that is, $(g, f) \in (Y, U)$ or $(g(y), f(y)) \in U$, for every $y \in Y$. Let $y \in Y$. Since $g \leq f$ we have that $g(y) \in \text{Cl}_Z(\{f(y)\})$, that is, the point $f(y)$ belongs to any neighbourhood of $g(y)$. Hence, $f(y)$ belongs to the set

$$U_{(g(y))} \equiv \{z \in Z : (g(y), z) \in U\}.$$

Thus, $(g(y), f(y)) \in U$. Therefore, $g \overset{\tau_{\mathcal{Q}}}{\leq} f$. By Theorem 1, the topology $\tau_{\mathcal{Q}}$ is $\mathbf{S}$ -splitting.

17. Example. Let Z be a set equipped with a preorder “ $\leq$ ”. By $\tau(\leq)$ we denote the topology on Z , for which the sets of the form $[z, \rightarrow)_\leq$, $z \in Z$, compose a subbasis.

Let Y be a space, for which every continuous map of Y into Z is constant and let a be a fixed point of Y . Obviously, $f \leq g$, where $f, g \in C(Y, Z)$, if and only if $f(a) \leq g(a)$. Identifying every element f of $C(Y, Z)$ with the element $f(a)$ of Z , every topology on the set Z can be considered as a topology on $C(Y, Z)$. In particular, on the set $C(Y, Z)$ we can consider the topology $\tau(\leq)$. By Theorem 6, the topology $\tau(\leq)$ on $C(Y, Z)$ is the greatest $\mathbf{S}$ -splitting topology.

Let $\tau_{\min}(\leq)$ be a topology on Z , for which the set Z and the subsets of Z of the form $Z \setminus (\leftarrow, z]_\leq$, $z \in Z$, compose a subbasis. By Theorem 10, it follows that the topology $\tau_{\min}(\leq)$ on $C(Y, Z)$ is the smallest $\mathbf{S}$ -splitting and $\mathbf{S}$ -jointly continuous topology on $C(Y, Z)$.

18. Example. Let Z be the set of real numbers with the usual order “ $\leq$ ”. Then the sets of the form $[a, \infty)$, $a \in Z$ compose a basis of the topology $\tau(\leq)$ and the sets of the form (a, ∞) , $a \in Z$, compose a basis of the topology $\tau_{\min}(\leq)$. It is easy to see that $\tau_{\min}(\leq) \subseteq \tau(\leq)$ and $\tau_{\min}(\leq) \neq \tau(\leq)$.

Let τ be a topology on the set Z for which the sets of the form $Z \setminus [a, b]$, where $a, b \in Z$, $a \leq b$, compose a subbasis. It is easy to see that $\tau \not\subseteq \tau(\leq)$, that is, the topology τ on $C(Y, Z)$ is not $\mathbf{S}$ -splitting.

On the other hand, the topology τ on $C(Y, Z)$ satisfies the following condition: if $f \stackrel{\tau}{\leq} g$, then $f = g$. By Theorem 3 it follows that τ is $\mathbf{S}$ -jointly continuous. Obviously, $\tau_{\min}(\leq) \not\subseteq \tau$, which means that on the set $C(Y, Z)$ there is no smallest $\mathbf{S}$ -jointly continuous topology.

19. Example. Let $Z = \mathbf{S}$. Consider the set $\mathcal{O}(Y)$ of all open subsets of Y , the set $\mathcal{K}(Y)$ of all closed subsets of Y and the set $C(Y, \mathbf{S})$. If we identify every element U of $\mathcal{O}(Y)$ with the element $Y \setminus U$ of $\mathcal{K}(Y)$ and with the element f of $C(Y, \mathbf{S})$, for which $f(U) \subseteq \{1\}$ and $f(Y \setminus U) \subseteq \{0\}$, then for every topology on one of the above sets we can consider the corresponding topology on the other sets. In particular, on the sets $\mathcal{O}(Y)$, $\mathcal{K}(Y)$ and $C(Y, \mathbf{S})$ we can consider the Scott topology (see [3]), the Vietoris topology (see, for example, [M]), an $\mathcal{A}$ -splitting topology, for some family $\mathcal{A}$ of spaces, etc. Also, the preorder “ $\leq$ ”, which is defined on the set $C(Y, \mathbf{S})$, can be considered on the sets $\mathcal{K}(Y)$ and $\mathcal{O}(Y)$. It is easy to prove that if $K, F \in \mathcal{K}(Y)$, then $K \leq F$ if and only if $F \subseteq K$. Thus, by Theorems 6 and 10 we have the following corollaries:

20. Corollary. *The sets of the form*

$$\mathcal{K}(F) \equiv \{K \in \mathcal{K}(Y) : K \subseteq F\},$$

$F \in \mathcal{K}(Y)$, *compose a basis for the greatest $\mathbf{S}$ -splitting topology on $\mathcal{K}(Y)$.*

21. Corollary. *The set $\mathcal{K}(Y)$ and the sets of the form*

$$\mathcal{K}_{\min}(F) \equiv \{K \in \mathcal{K}(Y) : F \not\subseteq K\},$$

$F \in \mathcal{K}(Y)$, *compose a subbasis of open sets for the smallest $\mathbf{S}$ -splitting and $\mathbf{S}$ -jointly continuous topology on $\mathcal{K}(Y)$.*

22. Remark. We observe that the sets of the form

$$\mathcal{K}(Y) \setminus \mathcal{K}(F), \quad F \in \mathcal{K}(Y),$$

and the sets of the form

$$\mathcal{K}_0(F) \equiv \{K \in \mathcal{K}(Y) : K \cap F = \emptyset\},$$

$F \in \mathcal{K}(Y)$, *compose a subbasis of the Vietoris topology on $\mathcal{K}(Y)$. Since the element $\emptyset \in \mathcal{K}(Y)$ belongs to any open set of the greatest $\mathbf{S}$ -splitting topology on $\mathcal{K}(Y)$ while this element is an isolated element of $\mathcal{K}(Y)$ with Vietoris topology, we have that if $Y \neq \emptyset$, then the Vietoris topology is not $\mathbf{S}$ -splitting.*

The proofs of the following results concerning the space $\mathbf{D}$ are similar to the corresponding results concerning the space $\mathbf{S}$ and we omit them.

23. Theorem. *The following are true:*

(1) *A topology τ on $C(Y, Z)$ is $\mathbf{D}$ -splitting if and only if by the relation $f \sim g$, where $f, g \in C(Y, Z)$, it follows that $f \overset{\tau}{\sim} g$.*

(2) *A topology τ on $C(Y, Z)$ is $\mathbf{D}$ -jointly continuous if and only if by the relation $f \overset{\tau}{\sim} g$, where $f, g \in C(Y, Z)$, it follows that $f \sim g$.*

(3) *The set whose elements are the equivalence classes of the relation “ $\sim$ ” on the set $C(Y, Z)$, composes a basis for the greatest $\mathbf{D}$ -splitting topology $\tau(\{\mathbf{D}\})$ on $C(Y, Z)$.*

(4) *The greatest $\mathbf{D}$ -splitting topology is $\mathbf{D}$ -splitting continuous.*

(5) *In the set of all simultaneously $\mathbf{D}$ -splitting and $\mathbf{D}$ -jointly continuous topologies on $C(Y, Z)$ there exists a smallest topology denoted by $\tau_{\min}(\{\mathbf{D}\})$. Moreover, the set $C(Y, Z)$ and the subsets of $C(Y, Z)$ of the form $C(Y, Z) \setminus E$, where E is an equivalence class of the relation “ $\sim$ ” on the set $C(Y, Z)$, compose a subbasis for this topology.*

(6) *Every $\mathbf{S}$ -splitting (respectively, $\mathbf{S}$ -jointly continuous) topology on $C(Y, Z)$ is $\mathbf{D}$ -splitting (respectively, $\mathbf{D}$ -jointly continuous).*

24. Corollary. (1) The discrete topology and, hence, every topology on the set $C(Y, Z)$ is $\mathbf{D}$ -splitting if and only if the space Z is a T_0 -space.

(2) The trivial topology and, hence, every topology on the set $C(Y, Z)$ is $\mathbf{D}$ -jointly continuous if and only if the topology of Z is trivial.

(3) A topology τ on $C(Y, Z)$ is simultaneously $\mathbf{D}$ -splitting and $\mathbf{D}$ -jointly continuous if and only if the relations " $\sim^\tau$ " and " $\sim$ " coincide.

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