

Generalized power means for matrix functions

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§1. Introduction

Generalized power means are defined by [1]

$$(1) \quad \begin{aligned} M_{n,a}(x; w)_p &= \left\{ \frac{\sum_{i=1}^n w_i x_i^{a+p}}{\sum_{i=1}^n w_i x_i^p} \right\}^{1/a}, \quad a \neq 0 \\ &= \exp \left\{ \frac{\sum_{i=1}^n w_i x_i^p \log x_i}{\sum_{i=1}^n w_i x_i^p} \right\}, \quad a = 0 \end{aligned}$$

where $a, p \in \mathbb{R}$, $x, w \in \mathbb{R}_+^n$, $n \in \mathbb{N}$. Concerning inequalities for these means see [2–5]. Further generalizations of these means are given in ([6, 7]).

Let $\phi : I \rightarrow \mathbb{R}_+$ be a strictly positive function, $F : I \rightarrow \mathbb{R}$ be a strictly monotone function, $x \in I^n$, $w > 0$ ($\in \mathbb{R}^n$). Then define

$$(2) \quad F_n(x, \phi) = F^{-1} \left(\frac{\sum_{i=1}^n w_i \phi(x_i) F(x_i)}{\sum_{i=1}^n w_i \phi(x_i)} \right).$$

Remark. Note that in [8, 9] means are defined with $\phi_i(x_i)$ instead of $w_i \phi(x_i)$.

Concerning inequalities for these means see, for example, [8], [9] or the monograph [10, pp. 261–269].

In this paper, we give analogous means for matrix functions.

2. Preliminaries and definitions

Let $A \in C^{n \times n}$ be a normal matrix, i.e., $A^*A = AA^*$. Here A^* means $\bar{A}^t$, the transpose conjugate of A . There exists [11] a unitary matrix U such that

$$(3) \quad A = U^*[\lambda_1, \lambda_2, \dots, \lambda_n]U$$

where $[\lambda_1, \lambda_2, \dots, \lambda_n]$ is the diagonal matrix $(\lambda_j \delta_{ij})$, and where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A , each appearing as often as its multiplicity. A is Hermitian if and only if $\lambda_i, i \in I_n = \{1, 2, \dots, n\}$ are real. If A is Hermitian and all λ_i are strictly positive, then A is said to be positive definite.

Assume now that $f(\lambda_i) \in C, i \in I_n$ is well-defined. Then $f(A)$ may be defined by (see e.g. [11, p.71] or [12, p.90])

$$(4) \quad f(A) = U^*[f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)]U.$$

As before, if $f(\lambda_i), i \in I_n$ are real, then $f(A)$ is Hermitian. If, also, $f(\lambda_i) > 0, i \in I_n$, then $f(A)$ is positive definite.

We note that for the inner product

$$(5) \quad (f(A)x, x) = \sum_{i=1}^n |y_i|^2 f(\lambda_i)$$

where $y \in C^n, y = Ux$, and so $\sum_{i=1}^n |y_i|^2 = \sum_{i=1}^n |x_i|^2$. Thus if x is a unit vector, then so is y .

If A is positive definite, so that $\lambda_i > 0, i \in I_n$, and $f(t) = t^r$, where $t > 0$ and $r \in \mathbb{R}$, we have $f(A) = A^r$. This representation is used in [13] and [14] to obtain matrix inequalities involving powers of A .

It is obvious that the above representations can also be used to obtain matrix versions of means (1) and (2).

Definition 1. Let A be an $n \times n$ positive definite Hermitian matrix, $0 \neq x \in C^n; a, p \in \mathbb{R}$. Then the generalized power means of A is given by

$$(6) \quad \begin{aligned} M_a^p(A; x) &= \left[\frac{(A^{a+p}x, x)}{(A^p x, x)} \right]^{1/a}, \quad a \neq 0 \\ &= \exp \left\{ \frac{((A^p \log A)x, x)}{(A^p x, x)} \right\}, \quad a = 0. \end{aligned}$$

Definition 2. Let $\phi : I \rightarrow \mathbb{R}_+(I \subset \mathbb{R})$ be a positive function, $F : I \rightarrow \mathbb{R}$ a strictly monotone function, and let A be a Hermitian matrix with eigenvalues in I . The generalized quasi-arithmetic mean $F(A; x)$ is defined by

$$(7) \quad \tilde{F}(A; x; \phi) = F^{-1} \left(\frac{((\phi \cdot F)(A)x, x)}{(\phi(A)x, x)} \right)$$

where $x \in C^n$, $x \neq 0$.

3. Inequalities for generalized power means

Theorem 1. Let $a, b, p, q \in \mathbb{R}$ satisfy

$$(8) \quad ||a| - |b|| + a + 2p \leq b + 2q.$$

Then for every positive definite Hermitian matrix A and every $x \in C^n$, $x \neq 0$,

$$(9) \quad M_a^p(A; x) \leq M_b^q(A; x).$$

PROOF. This is a simple consequence of (5) and of the following ([2]):

Lemma 1. Let $a, b, p, q \in \mathbb{R}$. Then

$$M_{n,a}(x; w)_p \leq M_{n,b}(x; w)_q$$

holds for every $x \in \mathbb{R}^n (x \neq 0)$, if and only if (8) holds.

Theorem 2. Let $a, b_1, \dots, b_k, p, q_1, \dots, q_k \in \mathbb{R}$, $k \geq 2$. Further, let

$$Q_0 = a^- - p, \quad Q_i = b_i^+ + q_i \quad i = 1, \dots, k$$

$$Q_0^* = a^+ + p, \quad Q_i^* = b_i^- - q_i, \quad i = 1, \dots, k,$$

where $a^+ = (|a| + a)/2$ and $a^- = (|a| - a)/2$ and for $i = 0, 1, \dots, k$, let

$$H_i = \begin{cases} \left(\sum_{\substack{j=0 \\ j \neq i}}^k Q_j^{-1} \right)^{-1}, & \text{when } \prod_{\substack{j=0 \\ j \neq i}}^k Q_j \neq 0 \\ 0, & \text{when } \prod_{\substack{j=0 \\ j \neq i}}^k Q_j = 0. \end{cases}$$

Let A be a normal matrix with eigenvalues in $I (I \subset \mathbb{C})$; $f_j : I \rightarrow \mathbb{R}_+$ for $j = 1, \dots, k$. If

$$(10) \quad Q_i \geq 0 \text{ and } H_i \geq Q_i^* \quad (i = 0, \dots, k),$$

then

$$(11) \quad M_a^p((f_1 \dots f_k)(A); x) \leq M_{b_1}^{q_1}(f_1(A); x) \dots M_{b_k}^{q_k}(f_k(A); x).$$

PROOF. This is again a simple consequence of the following ([3]):

Lemma 2. *Let $a, b_1, \dots, b_k, p, q_1, \dots, q_k \in \mathbb{R}$, $k \geq 2$ and let Q_i, Q_i^*, H_i be defined as in Theorem 2. Then the inequality*

$$M_{n,a}(x_1 \dots x_k; w)_p \leq M_{n,b_1}(x_1; w)_{q_1} \dots M_{n,b_k}(x_k; w)_{q_k}$$

holds for all $x_1, \dots, x_n \in \mathbb{R}_+^n$, $n \in \mathbb{N}$, iff (10) holds.

Remark. If $x, y \in \mathbb{R}^n$ we use notation $xy \equiv (x_1 y_1, \dots, x_n y_n)$.

Theorem 3. *Let $a, b_1, \dots, b_k, p, q_1, \dots, q_k \in \mathbb{R}$, $k \geq 2$. Let A be a normal matrix with eigenvalues in I ($I \subseteq C$), and let $f_i : I \rightarrow \mathbb{R}_+$ for $i = 1, \dots, k$, $x \in C^n$, $x \neq 0$. If*

$$(12) \quad \max\{p + a^+, 1\} \leq q_i + b_i^+$$

and

$$(13) \quad \max\{p - a^-, 0\} \leq \min\{q_i - b_i^-, 1\}$$

hold for every $i = 1, \dots, k$, then

$$(14) \quad M_a^p((f_1 + \dots + f_k)(A); x) \leq M_{b_1}^{q_1}(f_1(A); x) + \dots + M_{b_k}^{q_k}(f_k(A); x).$$

The reverse inequality holds in (14) if

$$(15) \quad \min\{p + a^+, 1\} \geq \max\{q_i + b_i^+, 0\}$$

and

$$(16) \quad \min\{p - a^-, 0\} \geq q_i - b_i^-$$

hold for $i = 1, \dots, k$.

PROOF. This is a consequence of the following ([4]):

Lemma 3. *Let $a, b_1, \dots, b_k, p, q_1, \dots, q_k \in \mathbb{R}$, $k \geq 2$. Then the inequality*

$$M_{n,a}(x_1 + \dots + x_k; w)_p \leq M_{n,b_1}(x_1; w)_{q_1} + \dots + M_{n,b_k}(x_k; w)_{q_k}$$

holds for every $n \in \mathbb{N}$, $x_1, \dots, x_k \in \mathbb{R}_+^n$, iff (12) and (13) hold for every $i = 1, \dots, k$. The reverse inequality holds if (15) and (16) hold for $i = 1, \dots, k$.

Theorem 4. Let A be a positive definite Hermitian matrix with eigenvalues λ_i such that

$$0 < m \leq \lambda_i \leq M \quad (i = 1, \dots, n).$$

Then

$$(16) \quad M_b^q(A; x) \leq C(m, M) M_a^p(A; x)$$

where a, b, p, q are fixed numbers such that (8) holds, where $C(m, M)$ is defined by

$$(18) \quad C(m, M) = \Gamma_{b,q}(t_0, \gamma) / \Gamma_{a,p}(t_0, \gamma)$$

$\gamma = M/m$ and t_0 is the unique positive root of the equation

$$\lambda_{a,p}(\gamma)(\gamma^q + t)(\gamma^{b+q} + t) = \lambda_{b,q}(\gamma)(\gamma^p + t)(\gamma^{a+p} + t)$$

where, for $t > 0$,

$$\lambda_{a,p}(t) = \begin{cases} t^p \frac{t^{a-1}}{a}, & a \neq 0 \\ t^p \log t, & a = 0, \end{cases}$$

and

$$\Gamma_{a,p}(t, \gamma) = \begin{cases} ((\gamma^{a+p} + t)/(\gamma^p + t))^{1/a}, & a \neq 0 \\ \exp((\gamma^p \log \gamma)/(\gamma^p + t)), & a = 0. \end{cases}$$

PROOF. This is a simple consequence of the following ([5]):

Lemma 4. Let $0 < m < M < \infty$, let

$$C(m, M) = \sup_{x \in [m, M]^n} M_{n,b}(x; w)_q / M_{n,a}(x; w)_p$$

where a, b, p, q are fixed numbers such that (8) holds. Further, let $\lambda_{a,p}$, $\Gamma_{a,p}$, t_0 , γ be defined as in Theorem 4. Then $C(m, M)$ is given by (18).

Remark. Note that results in [3–5] are obtained in the non-weighted case, i.e., for $w = (1, \dots, 1)$. However, the previous lemmas can easily be obtained from the non-weighted case.

4. Inequalities for quasi-arithmetic means

Theorem 5. Let K, L, M be three differentiable strictly monotone functions from the closed interval I to $\mathbb{R}$; and let ϕ, ψ, χ be three functions

from I to $\mathbb{R}_+$, $f : I^2 \rightarrow I$ such that for all $u, v, s, t \in I$, the following inequality holds.

$$(19) \quad \left(\frac{M \circ f(u, v) - M \circ f(t, s)}{M' \circ f(t, s)} \right) \frac{\chi \circ f(u, v)}{\chi \circ f(t, s)} \leq \\ \leq \left(\frac{K(u) - K(t)}{K'(t)} \right) \frac{\phi(u)}{\phi(t)} f'_1(t, s) + \left(\frac{L(v) - L(s)}{L'(s)} \right) \frac{\psi(v)}{\psi(s)} f'_2(t, s).$$

Let A be a normal matrix with eigenvalues in J and $g, h : J \rightarrow I$ are given functions. Then for $x \in C^n$, ($x \neq 0$)

$$(20) \quad f(\tilde{K}(g(A); x, \phi), \tilde{L}(h(A); x, \psi)) \geq \tilde{M}(f(g(A), h(A)); x; \chi).$$

(Note that $f(g(A), h(A))$ is the matrix $U^*[f(g(\lambda_1), h(\lambda_1)), f(g(\lambda_2), h(\lambda_2)), \dots, f(g(\lambda_n), h(\lambda_n))]U$).

PROOF. This is a simple consequence of the following lemma [10, p.262].

Lemma 5. Let functions $K, L, M, \phi, \psi, \chi$ be defined as in Theorem 5 and let $a, b \in I^n$ then

$$f(K_n(a; \phi), L_n(b; \psi)) \geq M_n(f(a, b); \chi)$$

holds if and only if, for all $u, v, s, t \in I$, (19) holds. Here $f(a, b)$ is the vector whose j^{th} component is $f(a_j, b_j)$.

Remark. Note that in [8–10] a more general result than our Lemma 5 was proved. Of course, Lemma 5 follows from this result in the cases when $\phi_i = w_i \phi$, $\psi_i = w_i \psi$, and $\chi_i = w_i \chi$ ($i = 1, \dots, n$).

Theorem 6. With the notations of Theorem 5 with $\lambda_i \in I$ ($i = 1, \dots, n$) (λ_i are the eigenvalues of A),

$$(21) \quad \tilde{M}(A; x; \chi) \leq \tilde{K}(A; x; \phi)$$

if, for all $u, t \in I$,

$$\left(\frac{M(u) - M(t)}{M'(t)} \right) \frac{\chi(u)}{\chi(t)} \leq \left(\frac{K(u) - K(t)}{K'(t)} \right) \frac{\phi(u)}{\phi(t)}.$$

PROOF. Immediate from Theorem 5 taking $f(x, y) = x$, $g(x) = x$.

A very important particular case of Theorem 5 is when $f(x, y) = x + y$. Then, we get

$$\tilde{K}(g(A); x; \phi) + \tilde{L}(h(A); x; \psi) \geq \tilde{M}(g(A) + h(A); x; \chi)$$

holds if, for all $u, v, s, t \in I$,

$$\frac{M(u+v) - M(t+s)}{M'(t+s)} \frac{\chi(u+v)}{\chi(t+s)} \leq \frac{K(u) - K(t)}{K'(t)} \frac{\phi(u)}{\phi(t)} + \frac{L(v) - L(s)}{L'(s)} \frac{\chi(v)}{\chi(s)}.$$

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