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Spectral geometry on certain almost Hermitian Einstein manifolds

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Abstract. On compact Riemannian and Kähler manifolds the spectra of the real and complex Laplacians determine the geometry of the manifolds to a considerable extent, though not completely, as isospectral manifolds need not be isometric. The literature on the geometric consequences of isospectrality is extensive (e.g., [1], [2], [4]–[8], [10], [11]). In this paper we consider the spectrum of the real Laplacian on three classes of almost Hermitian Einstein manifolds which include almost Kähler and nearly Kähler Einstein manifolds. We prove that a compact almost Hermitian Einstein manifold of positive scalar curvature ϱ with $\varrho \le \varrho^*$, or negative scalar curvature ϱ with $\varrho \ge \varrho^*$, or nonzero scalar curvature ϱ with $\varrho = \varrho^*$ (where ϱ^* is the $*$–scalar curvature) and isospectral to a Kähler manifold of constant holomorphic sectional curvature is Kähler and the manifolds are holomorphically isometric.

1. Preliminaries

Let (M,g) be a Riemannian manifold of real dimension $m = 2n \ge 2$ with metric $g = (g_{ij})$. If $R = (R_{hijk})$ is the Riemann curvature tensor, $Rc = (R_{ij}) = g^{hk}R_{hijk}$ the Ricci curvature tensor and $\varrho = g^{ij}R_{ij}$ the scalar curvature, then the Einstein tensor $E = (E_{ij})$ is given by

$$(1.1)\qquad E_{ij} \equiv R_{ij} - \frac{\varrho}{m} g_{ij}\,,$$

where M is Einstein if $E = 0$.

If (M,g) is a compact connected C^∞ manifold and $\Delta = -(d\delta + \delta d)$ is the Laplace operator on p–forms, $0 \le p \le 2n$, (0–forms corresponding to differentiable functions on M) with respect to the metric g, then the spectrum of the Laplacian are the eigenvalues of Δ,

$$(1.2)\qquad \mathrm{Spec}^p(M,g) = \{\lambda_{ip} \mid 0 \ge \lambda_{1,p} \ge \lambda_{2,p} \ge \cdots \ge \lambda_{k,p} \ge \ldots \downarrow -\infty\}\,,$$

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where each eigenvalue is repeated as often as its multiplicity. Further, $\mathrm{Spec}^{2n-p}(M,g) = \mathrm{Spec}^p(M,g)$ when M is orientable.

Relevant to the study of the spectrum is the Minakshisundaram-Pleijel-Gaffney asymptotic formula

$$\sum_{k=0}^{\infty} \exp(\lambda_{k,p} t) \underset{t\to 0}{\sim} \frac{1}{(4\pi t)^n} \sum_{i=0}^{\infty} a_{i,p} t^i \,, \tag{1.3}$$

where the first three coefficients are given by [8]

$$a_{0,p} = \binom{2n}{p} \int_M dM = \binom{2n}{p} \mathrm{vol}(M)\,, \tag{1.4}$$

$$a_{1,p} = \left[\frac{1}{6}\binom{2n}{p} - \binom{2n-2}{p-1}\right] \int_M \varrho dM\,, \tag{1.5}$$

$$a_{2,p} = \int_M \left[c_1(2n,p)\varrho^2 + c_2(2n,p)|Rc|^2 + c_3(2n,p)|R|^2\right] dM; \tag{1.6}$$

where

$$c_1(2n,p) = \frac{1}{72}\binom{2n}{p} - \frac{1}{6}\binom{2n-2}{p-1} + \frac{1}{2}\binom{2n-4}{p-2}\,, \tag{1.7}$$

$$c_2(2n,p) = -\frac{1}{180}\binom{2n}{p} + \frac{1}{2}\binom{2n-2}{p-1} - 2\binom{2n-4}{p-2}\,, \tag{1.8}$$

$$c_3(2n,p) = \frac{1}{180}\binom{2n}{p} - \frac{1}{12}\binom{2n-2}{p-1} + \frac{1}{2}\binom{2n-4}{p-2}\,. \tag{1.9}$$

2. Almost Hermitian manifolds and the Bochner curvature tensor

Let (M,g,J) be an almost Hermitian manifold of real dimension $2n \geq 2$ with almost complex structure $J = (F_i{}^j)$ and almost Hermitian metric $g = (g_{ij})$; that is, $g(JX,JY) = g(X,Y)$ for all X,Y in the tangent space of M at p, $T_p(M)$. In dimension $2n = 2$, (M,g) is Kähler Einstein and has holomorphic sectional curvature $\kappa = \frac{\varrho}{2}$.

Define the Bochner curvature tensor $B = (B_{hijk})$ by

$$\begin{aligned} B_{hijk} \equiv R_{hijk} - \frac{1}{2n+4}\big(&R_{ij}g_{hk} - R_{ik}g_{hj} + R_{hk}g_{ij} - R_{hj}g_{ik} \\ &+F_{ij}F_h{}^r R_{rk} - F_{ik}F_h{}^r R_{rj} + F_{hk}F_i{}^r R_{rj} - F_{hj}F_i{}^r R_{rk} \\ &-2F_{jk}F_h{}^r R_{ri} - 2F_{hi}F_j{}^r R_{rk}\big) \end{aligned} \tag{2.1}$$

$$+\frac{\varrho}{(2n+2)(2n+4)}\left(g_{ij}g_{hk}-g_{ik}g_{hj}+F_{ij}F_{hk}-F_{ik}F_{hj}-2F_{hi}F_{jk}\right).$$

Lemma 2.1. (see, e.g., [3]). *If (M,g) is a Kähler manifold of nonzero constant holomorphic sectional curvature κ, then the Riemann curvature tensor, Ricci curvature tensor and scalar curvature are given, respectively, by*

$$R_{hijk}=\frac{\kappa}{4}\left(g_{hk}g_{ij}-g_{hj}g_{ik}+F_{hk}F_{ij}-F_{hj}F_{ik}-2F_{hi}F_{jk}\right), \tag{2.2}$$

$$R_{ij}=\frac{n+1}{2}\kappa g_{ij}, \tag{2.3}$$

$$\varrho=n(n+1)\kappa. \tag{2.4}$$

Hence, (M,g) is Einstein and $B=0$.

By Schur's theorem, a Kähler manifold of dimension $2n\geq 4$ with constant holomorphic sectional curvature κ is of constant holomorphic curvature; that is, κ is a global constant on the manifold.

Lemma 2.2. [3]. *An almost Hermitian manifold (M,g) of dimension $2n\geq 4$ with curvature tensor given by (2.2) is Kähler and of constant holomorphic sectional curvature $\kappa=\frac{\varrho}{n(n+1)}$.*

Consequently, we have

Corollary 2.3. *If (M,g) is an almost Hermitian Einstein manifold of dimension $2n\geq 4$ with $B=0$ and $\varrho\neq 0$, then the conclusion of Lemma 2.2 follows.*

Lemma 2.4. *If (M,g) is an almost Hermitian Einstein manifold, then the square length of the Bochner tensor is given by*

$$|B|^2=|R|^2+\frac{\varrho^2-3\varrho\varrho^*}{n(n+1)}, \tag{2.5}$$

where $\varrho^*\equiv F^{kq}F^{jr}R_{krqj}$.

Proof. The equation follows from a calculation of $|B_{hijk}|^2$ using (1.1) and (2.1).

3. Spectral geometry on almost Hermitian Einstein manifolds

An almost Kähler manifold (M,g,J) is an almost Hermitian manifold such that the differential form $\omega\equiv F_{ij}dx^i\wedge dx^j$ is closed; that is, $d\omega=0$

and hence, $\delta\omega = 0$ and ω is harmonic. This is equivalent to $\nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0$, where ∇ is the covariant derivative with respect to the Riemannian connection on M. A nearly Kähler manifold is an almost Hermitian manifold satisfying $\nabla_i {F_j}^k + \nabla_j {F_i}^k = 0$.

Lemma 3.1. (see, e.g., [3]). *On an almost Kähler manifold*

$$\varrho \le \varrho^* \,; \tag{3.1}$$

while on a nearly Kähler manifold

$$\varrho \ge \varrho^* \,. \tag{3.2}$$

Further, equality holds in each case if and only if the manifold is Kähler.

Theorem 3.2. *Suppose that (M, g, J) and (M', g', J') are compact almost Hermitian manifolds with $\mathrm{Spec}^p(M, g) = \mathrm{Spec}^p(M', g')$ for $p = 0, 1$ or 2.*

a) *In dimension $2n = 2$, (M, g) is of constant holomorphic curvature κ if and only if (M', g') is.*

b) *In dimension $2n \ge 4$, if (M', g') is Einstein and of positive scalar curvature ϱ' with $\varrho' \le {\varrho'}^*$, or negative scalar curvature ϱ' with $\varrho' \ge {\varrho'}^*$, or nonzero scalar curvature ϱ' with $\varrho' = {\varrho'}^*$, and (M, g) is Kähler and of constant holomorphic sectional curvature κ, then (M', g') is a Kähler manifold of constant holomorphic sectional curvature $\kappa' = \kappa$ in the following cases: $p = 0$ and $2n \ge 4$; $p = 1$ and $2n \ge 16$; $p = 2$ and $2n = 4, 6, 8, 14$ or $2n \ge 18$.*

Proof. Letting $p = 0$ in (1.4)–(1.9) gives

$$a_{0,0} = \int_M dM = \mathrm{vol}(M)\,, \tag{3.3}$$

$$a_{1,0} = \frac{1}{6}\int_M \varrho dM\,, \tag{3.4}$$

$$a_{2,0} = \frac{1}{360}\int_M \left[5\varrho^2 - 2|Rc|^2 + 2|R|^2\right] dM\,. \tag{3.5}$$

a) In dimension $2n = 2$, $|R|^2 = \varrho^2$ and since g is an Einstein metric, then by (1.1), $|Rc|^2 = \frac{\varrho^2}{2}$ and $a_{2,0} = \frac{1}{60}\int_M \varrho^2 dM$. Since $a_{0,0} = a'_{0,0}$, it follows that $\mathrm{vol}(M) = \mathrm{vol}(M')$. If, say, (M, g) has constant holomorphic curvature κ, since $a_{1,0} = a'_{1,0}$ and $a_{2,0} = a'_{2,0}$, then $\int_{M'} \varrho' dM' = 2\kappa\,\mathrm{vol}(M')$ and $\int_{M'} {\varrho'}^2 dM' = 4\kappa^2\,\mathrm{vol}(M')$. We then have equality in the Schwarz inequality $\left(\int_{M'} \varrho' dM'\right)^2 \le \left(\int_{M'} {\varrho'}^2 dM'\right)\left(\int_{M'} dM'\right)$, so that ϱ' is constant and $\varrho' = \varrho$.

The proofs in the remaining cases are similar upon taking $p = 1$ and 2, respectively, in (1.4)–(1.9). For $p = 2$ the result follows, as well, as a consequence of the case $p = 0$ since $\mathrm{Spec}^0(M, g) = \mathrm{Spec}^0(M', g')$ by duality.

b) Suppose (M', g') is almost Hermitian Einstein of either positive scalar curvature ϱ' with $\varrho' \le {\varrho'}^*$ or negative scalar curvature ϱ' with $\varrho' \ge {\varrho'}^*$. In dimension $2n \ge 4$, substitute (2.5) in (3.5). Then, since $E = 0$ and $a_{2,0} = a'_{2,0}$, we have

$$\begin{aligned}
&\int_M \left[\frac{(5n^2+4n-3)\varrho^2 + 6\varrho\varrho^*}{n(n+1)} + 2|B|^2\right] dM \\
&= \int_M \left[\frac{(5n^2+4n+3)\varrho^2}{n(n+1)} + 2|B|^2\right] dM \\
&= \int_{M'} \left[\frac{(5n^2+4n-3){\varrho'}^2 + 6\varrho'{\varrho'}^*}{n(n+1)} + 2|B'|^2\right] dM' \\
&\ge \int_{M'} \left[\frac{(5n^2+4n+3){\varrho'}^2}{n(n+1)} + 2|B'|^2\right] dM' .
\end{aligned} \tag{3.6}$$

Since (M, g, J) is Kähler with constant holomorphic sectional curvature κ, then by Lemma 2.1, $B = 0$. Further, since ϱ is constant, then $a_{0,0} = a'_{0,0}$, $a_{1,0} = a'_{1,0}$ and the Schwarz inequality imply that $\left(\int_{M'} {\varrho'}^2 dM'\right)\left(\int_{M'} dM'\right) \ge \left(\int_{M'} \varrho' dM'\right)^2 = \left(\int_M \varrho dM\right)^2 = \varrho^2[\mathrm{vol}(M)]^2 = \varrho^2\,\mathrm{vol}(M')\,\mathrm{vol}(M) = \mathrm{vol}(M')\int_M \varrho^2 dM$. Then by (3.6), $|B'|^2 = 0$. Hence, by Corollary 2.3, (M', g', J') is Kähler and of constant holomorphic sectional curvature $\kappa' = \kappa$.

Letting $p = 1$ in (1.4)–(1.9) gives

$$a_{0,1} = 2n\int_M dM = 2n\,\mathrm{vol}(M)\,, \tag{3.7}$$

$$a_{1,1} = \frac{n-3}{3}\int_M \varrho dM\,, \tag{3.8}$$

$$\begin{aligned} a_{2,1} = \frac{1}{180}\int_M \Big[&(5n-30)\varrho^2 + (-2n+90)|Rc|^2 \\ &+ (2n-15)|R|^2\Big] dM\,. \end{aligned} \tag{3.9}$$

In dimension $2n \geq 16$, substitute (2.5) in (3.9). Then, since $E = 0$ and $a_{2,1} = a'_{2,1}$, we have

$$\begin{aligned}
&\int_M \left[\frac{(5n^3 - 26n^2 + 12n + 60)\varrho^2 + (6n-45)\varrho\varrho^*}{n(n+1)} + (2n-15)|B|^2\right] dM \\
&\qquad = \int_M \left[\frac{(5n^3 - 26n^2 + 18n + 15)\varrho^2}{n(n+1)} + (2n-15)|B|^2\right] dM \\
&\qquad = \int_{M'} \left[\frac{(5n^3 - 26n^2 + 12n + 60){\varrho'}^2 + (6n-45)\varrho'{\varrho'}^*}{n(n+1)} \right. \\
&\qquad\qquad \left. + (2n-15)|B'|^2\right] dM' \\
&\qquad \geq \int_{M'} \left[\frac{(5n^3 - 26n^2 + 18n + 15){\varrho'}^2}{n(n+1)} + (2n-15)|B'|^2\right] dM' \,.
\end{aligned} \tag{3.10}$$

The result then follows in a similar way as in the case $p = 0$.

Letting $p = 2$ in (1.4)–(1.9) gives

$$a_{0,2} = (2n^2 - n)\int_M dM = (2n^2 - n)\,\mathrm{vol}(M)\,, \tag{3.11}$$

$$a_{1,2} = \frac{2n^2 - 13n + 12}{6}\int_M \varrho dM\,, \tag{3.12}$$

$$\begin{aligned}
a_{2,2} = \frac{1}{360}\int_M &\Big[(10n^2 - 125n + 300)\varrho^2 \\
&+ (-4n^2 + 362n - 1080)|Rc|^2 \\
&+ (4n^2 - 62n + 240)|R|^2\Big] dM\,.
\end{aligned} \tag{3.13}$$

In dimension $2n = 4, 6, 8, 14$ or $2n \geq 18$, substitute (2.5) in (3.13). Then, since $E = 0$ and $a_{2,2} = a'_{2,2}$, we have

$$\begin{aligned}
&\int_M \left[\frac{(10n^4 - 117n^3 + 350n^2 + 3n - 780)\varrho^2 + (12n^2 - 186n + 720)\varrho\varrho^*}{n(n+1)}\right. \\
&\qquad\qquad \left. + (4n^2 - 62n + 240)|B|^2\right] dM
\end{aligned}$$

$$= \int_M \left[\frac{(10n^4 - 117n^3 + 362n^2 - 183n - 60)\varrho^2}{n(n+1)} \right.$$

$$\left. +(4n^2 - 62n + 240)|B|^2 \right] dM \tag{3.14}$$

$$= \int_{M'} \left[\frac{(10n^4 - 117n^3 + 350n^2 + 3n - 780){\varrho'}^2 + (12n^2 - 186n + 720)\varrho' {\varrho'}^*}{n(n+1)} \right.$$

$$\left. +(4n^2 - 62n + 240)|B'|^2 \right] dM'$$

$$\geq \int_{M'} \left[\frac{(10n^4 - 117n^3 + 362n^2 - 183n - 60){\varrho'}^2}{n(n+1)} \right.$$

$$\left. +(4n^2 - 62n + 240)|B'|^2 \right] dM' .$$

The result then follows in a similar way as in the case $p = 0$.

If (M', g') is almost Hermitian Einstein of nonzero scalar curvature ϱ' with $\varrho' = {\varrho'}^*$, then we have equality in (3.6), (3.10) and (3.14) and the results follow in the same way.

Corollary 3.3. *If $(\mathbb{C}P^n, g_0, J_0)$ is complex projective space with the Fubini–Study metric, (M', g', J') a compact almost Hermitian Einstein manifold of either positive scalar curvature ϱ' with $\varrho' \leq {\varrho'}^*$ or nonzero scalar curvature ϱ' with $\varrho' = {\varrho'}^*$, and* $\mathrm{Spec}^p(M', g', J') = \mathrm{Spec}^p(\mathbb{C}P^n, g_0, J_0)$ *for $p = 0, 1$ or 2, then (M', g', J') is Kähler and holomorphically isometric to $(\mathbb{C}P^n, g_0, J_0)$ in the following cases: $p = 0$ and $2n \geq 2$, hence $(\mathbb{C}P^n, g_0, J_0)$ is characterized by the spectrum in every dimension in these classes of manifolds; $p = 1$ and $2n = 2$ or $2n \geq 16$; $p = 2$ and $2n = 2, 4, 6, 8, 14$ or $2n \geq 18$.*

Proof. Since $(\mathbb{C}P^n, g_0, J_0)$ is the only Kähler manifold with a metric of positive constant holomorphic sectional curvature κ, then (M', g', J') is Kähler with constant holomorphic sectional curvature κ and (M', g', J') and $(\mathbb{C}P^n, g_0, J_0)$ are holomorphically isometric.

Corollary 3.4. *If (M, g, J) is a compact Kähler manifold of constant holomorphic sectional curvature κ and (M', g', J') is either compact almost Kähler Einstein with positive scalar curvature or compact nearly Kähler*

Einstein with negative scalar curvature and $\mathrm{Spec}^p(M,g) = \mathrm{Spec}^p(M',g')$ *for* $p = 0, 1$ *or 2, then the conclusion of Theorem 3.2b holds.*

Proof. The result follows immediately from Lemma 3.1 and Theorem 3.2.

We note here that a compact almost Kähler Einstein manifold of non-negative scalar curvature is Kähler [9].

References

[1] M. Berger, P. Gauduchon and E. Mazet, Le spectre d'une variété riemannienne, Lecture Notes in Math., vol. 194, *Springer Verlag, Berlin and New York*, 1971.

[2] B.–Y. Chen and L. Vanhecke, The spectrum of the Laplacian of Kähler manifolds, *Proc. Amer. Math. Soc.* **79** (1980), 82–86.

[3] L. Friedland and C. C. Hsiung, A certain class of almost Hermitian manifolds, *Tensor N. S.* **48** (1989), 252–263.

[4] P. Gilkey, Spectral geometry and the Kähler condition for complex manifolds, *Inventiones* **26** (1974), 231–258.

[5] P. Gilkey, The spectral geometry of real and complex manifolds, *Proc. Symposia Pure Math., Amer. Math. Soc.* **27** (1975), 265–280.

[6] P. Gilkey and J. Sacks, Spectral geometry and manifolds of constant holomorphic sectional curvature, *ibid.*, 281–285.

[7] S. I. Goldberg, A characterization of complex projective space, *C. R. Math. Rep. Acad. Sci. Canada* **6** (1984), 193–198.

[8] V. K. Patodi, Curvature and the fundamental solution of the heat operator, *J. Ind. Math. Soc.* **34** (1970), 269–285.

[9] K. Sekigawa, On some compact Einstein almost Kähler manifolds, *J. Math. Soc. Japan* **39** (1987), 677–684.

[10] S. Tanno, Eigenvalues of the Laplacian of Riemannian manifolds, *Tôhoku Math. J.* **25** (1973), 391–403.

[11] S. Tanno, The spectrum of the Laplacian for 1-forms, *Proc. Amer. Math. Soc.* **45** (1974), 125–129.

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