

Stability properties of functional equations in several variables

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1. Introduction

Let S be a semigroup and we consider the functional equation

$$(1) \quad F(x, y) + F(xy, z) = F(x, yz) + F(y, z)$$

where $F : S \times S \rightarrow \mathbb{C}$ is a function and (1) is supposed to hold for all x, y in S . It is easy to check that (1) holds for any F of the form

$$(2) \quad F(x, y) = f(xy) - f(x) - f(y)$$

where $f : S \rightarrow \mathbb{C}$ is any function. The converse of this statement for symmetric F on any Abelian group S has been proved in [7]. The proof depends heavily on the commutative structure of S . The general solution of (1) has also been found on several classes of commutative semigroups, see e.g. [2]. Now we prove that any bounded solution F of (1) has a representation of the form (2) with a bounded f , if S is an amenable semigroup. Concerning amenable groups and semigroups the reader should refer to [5], [6]. Further we study the stability of (1). Again, let S be a semigroup, $F : S \times S \rightarrow \mathbb{C}$ a function, and suppose, that the three-place function

$$(x, y, z) \rightarrow F(x, y) + F(xy, z) - F(x, yz) - F(y, z)$$

is bounded on $S \times S \times S$. In the classical cases of Hyers–Ulam stability this implies that $F - K$ satisfies (1) with some bounded K . Here we prove that this is the case.

In the second part we study the functional equation

$$(3) \quad F(xy, z) + F(xy^{-1}, z) - 2F(y, z) = F(x, yz) + F(x, yz^{-1}) - 2F(x, y)$$

with $F : S \times S \rightarrow \mathbb{C}$, where S is a group. Equation (3) has been arisen in [9], where the question concerning (3) was, whether any solution F of (3) can be represented in the form

$$(4) \quad F(x, y) = f(xy) + f(xy^{-1}) - 2f(x) - 2f(y).$$

(Obviously, any F of the form (4) satisfies (3) if S is commutative.) This question has been answered in [3] in the negative, by presenting a counterexample. Nevertheless, the problem of the general solution of (3) remains open. In [4] it has been proved that in the case $S = \mathbb{R}$ any twice differentiable solution of (3) has the form (4) (with twice differentiable f). Here we show, that any bounded solution of (3) has the form (4) with bounded f , if the group S is amenable. Further, we show that equation (3) has the similar remarkable stability property, like equation (1): if the function $F : S \times S \rightarrow \mathbb{C}$ has the property, that the function

$$(x, y, z) \rightarrow F(xy, z) + F(xy^{-1}, z) - 2F(y, z) - F(x, yz) - F(x, yz^{-1}) + 2F(x, y)$$

is bounded, then $F - K$ is a solution of (3) with a bounded function K , supposing S is Abelian.

2. The functional equation (1)

Concerning (1) we first prove the following theorem:

Theorem 2.1. *Let S be a right amenable semigroup and let $F : S \times S \rightarrow \mathbb{C}$ be a bounded function satisfying (1). Then there exists a unique bounded function $f : S \rightarrow \mathbb{C}$ with*

$$F(x, y) = f(xy) - f(x) - f(y)$$

for all x, y in S .

PROOF. Let M denote any right invariant mean defined on the set of all bounded complex valued functions on S and we write M_x to indicate that M is applied on the argument as a function of x . Now apply M on both sides of (1) as functions of x , for any fixed y, z in S . We obtain

$$M_x[F(x, y)] + M_x[F(xy, z)] = M_x[F(x, yz)] + M_x[F(y, z)],$$

that is, by the right invariance of M and by $M(1) = 1$,

$$M_x[F(x, y)] + M_x[F(x, z)] = M_x[F(x, yz)] + F(y, z),$$

hence we may choose

$$f(y) = M_x[F(x, y)]$$

for all y in S . As F is bounded, it follows that f is bounded. For the uniqueness of f it is easy to see that the difference of two f 's is a complex homomorphism of S , which cannot be bounded unless it is zero. Hence our theorem is proved.

Now our stability theorem follows for (1).

Theorem 2.2. *Let S be a right amenable semigroup and let $F : S \times S \rightarrow \mathbb{C}$ be a function, for which the function*

$$(x, y, z) \rightarrow F(x, y) + F(xy, z) - F(x, yz) - F(y, z)$$

is bounded. Then there exists a function $\Phi : S \times S \rightarrow \mathbb{C}$ satisfying (1), for which $F - \Phi$ is bounded.

PROOF. We define the function $\Phi : S \times S \rightarrow \mathbb{C}$ by the formula

$$\Phi(y, z) = M_x[F(x, y) + F(xy, z) - F(x, yz)]$$

for all y, z in S , where M denotes an arbitrary right invariant mean defined on all bounded complex valued functions on S . Then we have for all y, z

$$\begin{aligned} & \Phi(y, z) + \Phi(yz, u) - \Phi(y, zu) - \Phi(z, u) = \\ &= M_x[F(x, y) + F(xy, z) - F(x, yz) + F(x, yz) + \\ &+ F(xyz, u) - F(x, yzu) - F(x, y) - F(xy, zu) + F(x, yzu) - \\ &- F(x, z) - F(xz, u) + F(x, zu)] = \\ &= M_x[\{F(xy, z) + F(xyz, u) - F(xy, zu)\} - \\ &- \{F(x, z) + F(xz, u) - F(x, zu)\}] = 0, \end{aligned}$$

by the right invariance of M . On the other hand, it follows

$$F(y, z) - \Phi(y, z) = M_x[F(y, z) + F(x, yz) - F(x, y) - F(xy, z)],$$

which is bounded, by the properties of M and F . Hence our theorem is proved.

3. The functional equation (3)

Concerning (3) we first prove the following theorem:

Theorem 3.1. *Let S be an amenable group and let $F : S \times S \rightarrow \mathbb{C}$ be a bounded function satisfying (3). Then there exists a unique bounded function $f : S \rightarrow \mathbb{C}$ with*

$$F(x, y) = f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)$$

for all x, y in S .

PROOF. Let M denote any invariant mean defined on the set of all bounded complex valued functions on S . If we apply M on both sides of (3) as functions of x , for any fixed y, z in S , then we obtain

$$\begin{aligned} M_x[F(xy, z)] + M_x[F(xy^{-1}, z)] - 2M_x[F(y, z)] &= \\ = M_x[F(x, yz)] + M_x[F(x, yz^{-1})] - 2M_x[F(x, y)], \end{aligned}$$

that is, by the properties of M

$$\begin{aligned} M_x[F(x, z)] + M_x[F(x, z)] - 2F(y, z) &= \\ = M_x[F(x, yz)] + M_x[F(x, yz^{-1})] - 2M_x[F(x, y)], \end{aligned}$$

which shows that our statement holds if we choose

$$f(y) = -\frac{1}{2}M_x[F(x, y)]$$

for all y in S . The boundedness of f follows from the properties of M . The uniqueness of f follows similarly as in Theorem 2.1.

For the stability theorem for equation (3) we need also commutativity on S .

Theorem 3.2. *Let S be an Abelian group and let $F : S \times S \rightarrow \mathbb{C}$ be a function, for which the function*

$$\begin{aligned} (x, y, z) \rightarrow F(x + y, z) + F(x - y, z) - 2F(y, z) - \\ - F(x, y + z) - F(x, y - z) + 2F(x, y) \end{aligned}$$

is bounded. Then there exists a function $\Phi : S \times S \rightarrow \mathbb{C}$ satisfying (3), for which $F - \Phi$ is bounded.

PROOF. We define the function $\Phi : S \times S \rightarrow \mathbb{C}$ by the formula

$$\Phi(y, z) = -\frac{1}{2}M_x[-F(x+y, z) - F(x-y, z) + F(x, y+z) + F(x, y-z) - 2F(x, y)]$$

for any y, z in S , where M is an invariant mean on S . Now we can compute as follows

$$\begin{aligned}
& \Phi(y+z, u) + \Phi(y-z, u) - 2\Phi(z, u) - \Phi(y, z+u) - \Phi(y, z-u) + 2\Phi(y, z) = \\
& = -\frac{1}{2}M_x[-F(x+y+z, u) - F(x-y-z, u) + F(x, y+z+u) + \\
& + F(x, y+z-u) - 2F(x, y+z) - F(x+y-z, u) - F(x-y+z, u) + \\
& + F(x, y-z+u) + F(x, y-z-u) - 2F(x, y-z) + 2F(x+z, u) + \\
& + 2F(x-z, u) - 2F(x, z+u) - 2F(x, z-u) + 4F(x, z) + \\
& + F(x+y, z+u) + F(x-y, z+u) - F(x, y+z+u) - F(x, y-z-u) + \\
& + 2F(x, y) + F(x+y, z-u) + F(x-y, z-u) - F(x, y+z-u) - \\
& - F(x, y+u-z) + 2F(x, y) - 2F(x+y, z) - 2F(x-y, z) + \\
& + 2F(x, y+z) + 2F(x, y-z) - 4F(x, y)] = \\
& = -\frac{1}{2}M_x[\{-F(x+y+z, u) - F(x+y-z, u) + F(x+y, z+u) + \\
& + F(x+y, z-u) - 2F(x+y, z)\} + \{F(x+z, u) + F(x-z, u) - \\
& - F(x, z+u) - F(x, z-u) + 2F(x, z)\}] - \frac{1}{2}M_x[\{-F(x-y+z, u) - \\
& - F(x-y-z, u) + F(x-y, z+u) + F(x-y, z-u) - 2F(x-y, z)\} + \\
& + \{F(x+z, u) + F(x-z, u) - F(x, z+u) - F(x, z-u) + 2F(x, z)\}] = 0
\end{aligned}$$

by the invariance of M . On the other hand,

$$\begin{aligned}
F(y, z) - \Phi(y, z) &= \frac{1}{2}M_x[2F(y, z) - F(x+y, z) - F(x-y, z) + \\
& + F(x, y+z) + F(x, y-z) - 2F(x, y)]
\end{aligned}$$

which is bounded, by the properties of M and F . Hence our theorem is proved.

We note, that actually we haven't used the commutativity of S , only the following property of F :

$$F(xyz, uvw) = F(xzy, uvw)$$

and the fact, that there exists a right invariant mean on S . A similar condition to this one has been used for functions of one variable in [1], [8], [10]. Hence we can prove the following corollary.

Corollary 3.3. *Let S be an amenable group and let $F : S \times S \rightarrow \mathbb{C}$ be a function, which satisfies*

$$F(xyz, uvw) = F(xzy, uvw)$$

for all x, y, z, u, v, w in S , and for which the function

$$(x, y, z) \rightarrow F(xy, z) + F(xy^{-1}, z) - 2F(y, z) - F(x, yz) - F(x, yz^{-1}) + 2F(x, y)$$

is bounded. Then there exists a function $\Phi : S \times S \rightarrow \mathbb{C}$ satisfying (3), for which $F - \Phi$ is bounded.

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