

On a family of set-valued functions

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Abstract. Let G be a linear continuous set-valued function defined on a closed convex cone C in a Banach space X . The aim of this paper is to show that for every $x \in C$ and $t \geq 0$ a series $B^t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x)$ is convergent in the space of non-empty compact convex subsets of X with the Hausdorff metric. Moreover the inclusion $(B^t \circ B^s)(x) \subset B^{t+s}(x)$ for $x \in C$ and $t, s \geq 0$ holds true.

1. Preliminaries

Throughout the paper vector spaces are always real. The symbols $\mathbb{R}$ and $\mathbb{N}$ denote the set of all real numbers and the set of positive integers, respectively.

Let X be a vector space and let A, B be subsets of X . The algebraic sum of A and B is the set defined as follows:

$$A + B = \{a + b : a \in A, b \in B\}.$$

For any $t \in \mathbb{R}$ the set tA contains all vectors of the form ta , $a \in A$ and only those. It is easily seen that the algebraic sum of convex sets is convex and if A, B are compact sets in a topological vector space X , then $A + B$ is compact as well.

It is clear that we have the following.

Lemma 1. *If A, B are subsets of a vector space X and $s, t \in \mathbb{R}$, then*

- (i) $s(A + B) = sA + sB$
- (ii) $(s + t)A \subset sA + tA$.

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If A is a convex subset of X , then $(s + t)A = sA + tA$ for all $s, t \geq 0$ or $s, t \leq 0$.

The next lemma can be found in [4].

Lemma 2. *Let A, B and C be subsets of a normed space such that*

$$A + C \subset B + C.$$

If B is closed and convex and C is non-empty and bounded, then $A \subset B$.

The above lemma allows us to get the following.

Corollary 1. *If A and B are closed and convex subsets of a normed space X and C is non-empty and bounded, then the equality*

$$A + C = B + C$$

implies $A = B$.

Let X be a normed space, and let in the sequel $n(X)$ be the family of all non-empty subsets of X . The families $bd(X), c(X), cc(X)$ consist of the bounded, closed, compact, and convex compact members of $n(X)$, respectively. Define the norm of a set $A \in n(X)$ in the natural way as

$$\|A\| := \sup\{\|a\| : a \in A\}.$$

It is easy to check that

$$\|A + B\| \leq \|A\| + \|B\| \quad \text{for } A, B \in n(X)$$

and

$$\|tA\| = |t| \|A\| \quad \text{for } A \in n(X) \text{ and } t \in \mathbb{R}.$$

Let A and B be members of $bd(X)$. The excess of A over B is defined as

$$e(A, B) = \sup\{d(x, B) : x \in A\},$$

where $d(x, B) = \inf\{d(x, y) : y \in B\}$. The Hausdorff distance of A and B is

$$h(A, B) = \max\{e(A, B), e(B, A)\}.$$

This function is a metric in the space $bd(X)$. If the normed space X is complete, then the space of all closed and bounded non-empty subsets with the Hausdorff metric is complete as well (see [1] and [2]).

It is not difficult to verify that

$$(1) \quad h(A, B) = \inf\{\varepsilon > 0 : A \subset B + \varepsilon S, B \subset A + \varepsilon S\},$$

where S is the closed unit ball in X .

The following equalities will be useful:

$$(2) \quad h(A + C, B + C) = h(A, B) \quad \text{for all } A, B, C \in cc(X)$$

and

$$(3) \quad h(\lambda A, \lambda B) = |\lambda| h(A, B) \quad \text{for all } A, B \in cc(X), \lambda \in \mathbb{R},$$

(cf. for example [2], [5]).

The relation $A_n \longrightarrow A$ means that the sequence $\{A_n\}_{n \in \mathbb{N}}$ is convergent to A with respect to the Hausdorff metric in the space $cc(X)$.

Properties of the above convergence may be collected in the following

Lemma 3. *If $A_n \longrightarrow A$, $B_n \longrightarrow B$, then*

- (i) $A_n + B_n \longrightarrow A + B$
- (ii) $\lambda A_n \longrightarrow \lambda A, \quad \lambda \in \mathbb{R}$
- (iii) *The inclusions $A_n \subset B_n$ for $n \in \mathbb{N}$, imply $A \subset B$.*

This lemma is known, e.g. (i) can be found in [3] in a general setting, but we will give its

PROOF. (i) follows in virtue of the triangle inequality and by (2)

$$\begin{aligned} h(A_n + B_n, A + B) &\leq h(A_n + B_n, A_n + B) + h(A_n + B, A + B) = \\ &= h(B_n, B) + h(A_n, A). \end{aligned}$$

(ii) is an obvious consequence of (3). Now we shall prove (iii). Let us fix an $\varepsilon > 0$. With respect to the convergence of $\{A_n\}_{n \in \mathbb{N}}$, $\{B_n\}_{n \in \mathbb{N}}$ one has

$$h(A_n, A) < \varepsilon \quad \text{and} \quad h(B_n, B) < \varepsilon$$

for large enough $n \in \mathbb{N}$, say $n \geq n_0$. Thus, (1) yields

$$A \subset A_n + \varepsilon S \quad \text{and} \quad B_n \subset B + \varepsilon S \quad \text{for } n \geq n_0$$

hence the inclusions $A_n \subset B_n$, $n \in \mathbb{N}$ imply

$$A \subset A_n + \varepsilon S \subset B_n + \varepsilon S \subset B + \varepsilon S + \varepsilon S = B + 2\varepsilon S,$$

because S is a convex set. The obtained inclusion $A \subset B + 2\varepsilon S$ gives the inequality

$$e(A, B) \leq 2\varepsilon.$$

Take an a belonging to A . Then

$$d(a, B) \leq \sup\{d(a, B) : a \in A\} = e(A, B) \leq 2\varepsilon.$$

Now $d(a, B) = 0$ in view of the unrestricted choice of $\varepsilon > 0$. Consequently $a \in B$ by the closedness of B . $\square$

Finally recall some definitions connected with set-valued functions (abbreviated to “s.v. functions” in the sequel).

Let X, Y, Z be vector spaces and let C be a convex cone in X . An s.v. function $A : C \rightarrow n(Y)$ is said to be additive (superadditive) iff it satisfies the condition

$$(4) \quad A(x + y) = A(x) + A(y) \quad (A(x + y) \supset A(x) + A(y)),$$

respectively, for all $x, y \in C$.

An s.v. function A is said to be linear iff it is additive and

$$(5) \quad A(tx) = tA(x) \quad \text{for all } x \in C \text{ and } t \in (0, +\infty).$$

An s.v. function is called $\mathbb{Q}_+$ -homogeneous iff (5) holds true for all $t \in \mathbb{Q} \cap (0, +\infty)$. For a given s.v. function $F : X \rightarrow n(Y)$ and sets $A \subset X, B \subset Y$ we define the sets

$$F(A) = \bigcup \{F(x) : x \in A\}$$

$$F^-(B) = \{x \in X; F(x) \cap B \neq \emptyset\}$$

$$F^+(B) = \{x \in X; F(x) \subset B\}.$$

They are called, respectively, the image of A , the lower inverse image of B and the upper inverse image of B under the s.v. function F .

The superposition $G \circ F$ of s.v. functions $F : X \rightarrow n(Y)$ and $G : Y \rightarrow n(Z)$ is the s.v. function defined as follows

$$(G \circ F)(x) := G(F(x)) \quad \text{for } x \in X.$$

Assume that X and Y are two topological vector spaces. We say that an s.v. function $F : X \rightarrow n(Y)$ is lower-semicontinuous (l.s.c.) iff the set $F^-(U)$ is open in X for every open set U in Y . We say that an s.v. function F is upper-semicontinuous (u.s.c.) iff the set $F^+(U)$ is open in X for every open set U in Y . F is said to be continuous iff it is both l.s.c. and u.s.c.

In what follows we shall apply the following lemma (cf. [5], Theorem 4).

Lemma 4. *Let X be a Banach space and let Y be a normed space. If C is a convex cone in X and $(F_i, i \in I)$ is a family of superadditive, l.s.c. in C and $\mathbb{Q}_+$ -homogeneous, s.v. functions $F_i : C \rightarrow n(Y)$, such that $\bigcup \{F_i(x) : i \in I\}$ is bounded for every $x \in C$, then there exists a constant $0 < M < +\infty$ such that*

$$\sup \{\|F_i(x)\| : i \in I\} \leq M\|x\|, \quad x \in C.$$

Assume that the family $(F_i, i \in I)$ contains exactly one element F . The least element of the set

$$\{M > 0 : \|F(x)\| \leq M\|x\|, x \in C\}$$

will be denoted by $\|F\|$.

2. Main result

The main objective of the paper is to prove the following

Theorem. *Let X be a Banach space and let C be a closed and convex cone in X . Assume that $G : C \rightarrow cc(C)$ is a linear and continuous s.v. function. Then for every $x \in C$ and $t \geq 0$ the series*

$$(6) \quad B^t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x)$$

is convergent in the metric space $(cc(C), h)$. Moreover, the s.v. functions $B^t, t \geq 0$ are linear and

$$(7) \quad (B^t \circ B^s)(x) \subset B^{t+s}(x) \quad \text{for } x \in C, s, t, \geq 0.$$

PROOF. To prove the convergence of the series (6) define the functions B_n^t on $C, n \in \mathbb{N}, t \geq 0$ by the formula

$$B_n^t(x) = \sum_{i=0}^n \frac{t^i}{i!} G^i(x) \quad , x \in C.$$

It is clear that the sets $B_n^t(x)$ are convex and compact for each $t \geq 0, n \in \mathbb{N}, x \in C$. Fix an $\varepsilon > 0$. We can find $n_0 \in \mathbb{N}$ such that

$$\sum_{i=m}^n \frac{t^i}{i!} \|G\|^i < \varepsilon$$

for all $n \geq m \geq n_0, n, m \in \mathbb{N}$. For such n and m we have

$$\begin{aligned} h(B_n^t(x), B_m^t(x)) &= h\left(B_m^t(x) + \sum_{i=m+1}^n \frac{t^i}{i!} G^i(x), B_m^t(x)\right) = \\ &= h\left(\sum_{i=m+1}^n \frac{t^i}{i!} G^i(x), \{0\}\right) = \left\| \sum_{i=m+1}^n \frac{t^i}{i!} G^i(x) \right\| \leq \\ &\leq \sum_{i=m+1}^n \frac{t^i}{i!} \|G\|^i \|x\| < \varepsilon \|x\|, x \in C, t \geq 0. \end{aligned}$$

The above inequality shows that the sequence $\{B_n^t(x)\}_{n \in \mathbb{N}}, t \geq 0, x \in C$ of partial sums of series (6) satisfies the Cauchy condition. Since a closed subspace of a complete space X is complete, the sequence $\{B_n^t(x)\}_{n \in \mathbb{N}}, x \in C, t \geq 0$ is convergent in $(cc(C), h)$. We write

$$B^t(x) := \lim_{n \rightarrow \infty} B_n^t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x).$$

Consequently the values of the function B^t belong to $cc(C)$ for $t \geq 0$. Furthermore, the inequality

$$(8) \quad h(B_n^t(x), B_m^t(x)) < \varepsilon \|x\|,$$

$t \geq 0, x \in C$ says that the series (6) is almost uniformly convergent to B^t for $t \geq 0$; this means that for each compact set K contained in C the series (6) is uniformly convergent on K .

Next we shall prove that the functions B^t are linear. Take any $x, y \in C$ and $t \geq 0$. By Lemma 3 we have

$$B_n^t(x) + B_n^t(y) \longrightarrow B^t(x) + B^t(y), \quad \text{as } n \rightarrow \infty.$$

The additivity of G yields

$$\begin{aligned} B_n^t(x) + B_n^t(y) &= \sum_{i=0}^n \frac{t^i}{i!} G^i(x) + \sum_{i=0}^n \frac{t^i}{i!} G^i(y) = \sum_{i=0}^n \frac{t^i}{i!} (G^i(x) + G^i(y)) = \\ &= \sum_{i=0}^n \frac{t^i}{i!} G^i(x+y) = B_n^t(x+y) \longrightarrow B^t(x+y), \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus

$$B^t(x+y) = B^t(x) + B^t(y).$$

To prove the homogeneity of $B^t, t \geq 0$ fix $x \in C$ and $\alpha \in (0, \infty)$. Again by Lemma 3 we have

$$(9) \quad \alpha B_n^t(x) \longrightarrow \alpha B^t(x), \quad \text{as } n \rightarrow \infty.$$

The linearity of G gives

$$B_n^t(\alpha x) = \alpha B_n^t(x),$$

hence we conclude from (9) that

$$(10) \quad \alpha B^t(x) = B^t(\alpha x).$$

Now we proceed to the proof of (7). Let us fix $n \in \mathbb{N}$, $x \in C$ and $s, t \geq 0$. We have

$$\begin{aligned} (B_n^t \circ B_n^s)(x) &= \left(\sum_{j=0}^n \frac{t^j}{j!} G^j \right) (B_n^s(x)) \subset \sum_{j=0}^n \frac{t^j}{j!} G^j \left(\sum_{i=0}^n \frac{s^i}{i!} G^i(x) \right) = \\ &= \sum_{j=0}^n \sum_{i=0}^n \frac{t^j s^i}{j! i!} G^{i+j}(x) = \sum_{\ell=0}^n \sum_{k=0}^{\ell} \frac{t^{\ell-k} s^k}{(\ell-k)! k!} G^{\ell}(x) + R_n(t, s, x), \end{aligned}$$

where

$$R_n(t, s, x) := \sum_{\ell=n+1}^{2n} \sum_{k=\ell-n}^n \frac{t^{\ell-k} s^k}{(\ell-k)! k!} G^{\ell}(x).$$

Observe that

$$\begin{aligned} \sum_{\ell=0}^n \sum_{k=0}^{\ell} \frac{t^{\ell-k} s^k}{(\ell-k)! k!} G^{\ell}(x) &= \sum_{\ell=0}^n \frac{1}{\ell!} \left(\sum_{k=0}^{\ell} \binom{\ell}{k} s^k t^{\ell-k} \right) G^{\ell}(x) = \\ &= \sum_{\ell=0}^n \frac{(s+t)^{\ell}}{\ell!} G^{\ell}(x) = B_n^{s+t}(x). \end{aligned}$$

Thus

$$(11) \quad (B_n^t \circ B_n^s)(x) \subset B_n^{s+t}(x) + R_n(t, s, x).$$

for every $x \in C$, $s, t \geq 0$ and $n \in \mathbb{N}$. With respect to Lemma 3 it suffices to prove that $R_n(t, s, x) \rightarrow \{0\}$ and $(B_n^t \circ B_n^s)(x) \rightarrow (B^t \circ B^s)(x)$ as $n \rightarrow \infty$. By the definition of R_n we have

$$\begin{aligned} \|R_n(t, s, x)\| &\leq \sum_{\ell=n+1}^{2n} \sum_{k=\ell-n}^{\ell} \frac{t^{\ell-k} s^k}{(\ell-k)! k!} \|G\|^{\ell} \|x\| \leq \\ &\leq \sum_{\ell=n+1}^{2n} \sum_{k=0}^{\ell} \frac{t^{\ell-k} s^k}{(\ell-k)! k!} \|G\|^{\ell} \|x\| \leq \sum_{\ell=n+1}^{2n} \frac{(t+s)^{\ell}}{\ell!} \|G\|^{\ell} \|x\| \end{aligned}$$

whence, on letting $n \rightarrow \infty$, we obtain

$$R_n(t, s, x) \rightarrow \{0\}.$$

Now, fix $\varepsilon > 0$, $x \in C$, $t, s \geq 0$. By the almost uniform convergence of the sequence $\{B_n^t\}_{n \in \mathbb{N}}$ there exists $n_1 \in \mathbb{N}$ such that

$$h(B_n^t(y), B^t(y)) < \frac{\varepsilon}{2} \quad \text{for } n > n_1, y \in B^s(x).$$

Thus

$$B_n^t(y) \subset B^t(y) + \frac{\varepsilon}{2}S \text{ and } B^t(y) \subset B_n^t(y) + \frac{\varepsilon}{2}S$$

for $y \in B^s(x)$, hence

$$B_n^t(y) \subset B^t(y) + \frac{\varepsilon}{2}S \subset B^t(B^s(x)) + \frac{\varepsilon}{2}S$$

for $y \in B^s(x)$ and

$$B_n^t(B^s(x)) \subset B^t(B^s(x)) + \frac{\varepsilon}{2}S \text{ for } n > n_1.$$

Similarly, one can show that

$$B^t(B^s(x)) \subset B_n^t(B^s(x)) + \frac{\varepsilon}{2}S \text{ for } n > n_1.$$

The two last inclusions yield

$$(12) \quad h(B_n^t(B^s(x)), B^t(B^s(x))) \leq \frac{\varepsilon}{2} \text{ for } n > n_1.$$

Note that

$$(13) \quad \|B_n^t(x)\| \leq \sum_{i=0}^n \frac{t^i}{i!} \|G\|^i \|x\| \leq M_t \|x\|,$$

where $M_t := e^{t\|G\|}$ for every $n \in \mathbb{N}$. Thus for every bounded set $B \subset C$ and $n \in \mathbb{N}$ we have the following relations:

$$\|B_n^t(B)\| = \sup\{\|B_n^t(y)\| : y \in B\} \leq \sup\{M_t \|y\| : y \in B\} \leq M_t \|B\|.$$

Since for every $n \in \mathbb{N}$ the set

$$\sum_{i=n+1}^{\infty} \frac{s^i}{i!} G^i(x)$$

is bounded, we get

$$\left\| B_n^t \left(\sum_{i=n+1}^{\infty} \frac{s^i}{i!} G^i(x) \right) \right\| \leq M_t \left\| \sum_{i=n+1}^{\infty} \frac{s^i}{i!} G^i(x) \right\|.$$

Thus

$$\begin{aligned} & h(B_n^t(B^s(x)), B_n^t(B_n^s(x))) = \\ & = h(B_n^t(B_n^s(x)) + B_n^t\left(\sum_{i=n+1}^{\infty} \frac{s^i}{i!} G^i(x)\right), B_n^t(B_n^s(x))) = \\ & = \left\| B_n^t\left(\sum_{i=n+1}^{\infty} \frac{s^i}{i!} G^i(x)\right) \right\| \leq M_t \left\| \sum_{i=n+1}^{\infty} \frac{s^i}{i!} G^i(x) \right\|, \end{aligned}$$

whence it follows that there exists $n_2 \in \mathbb{N}$ such that

$$(14) \quad h(B_n^t(B^s(x)), B_n^t(B_n^s(x))) < \frac{\varepsilon}{2}, \quad n \geq n_2.$$

Combining (12) with (14) we obtain

$$\begin{aligned} & h(B_n^t(B_n^s(x)), B^t(B^s(x))) \leq \\ & \leq h(B_n^t(B_n^s(x)), B_n^t(B^s(x))) + h(B_n^t(B^s(x)), B^t(B^s(x))) < \varepsilon \end{aligned}$$

for $n \geq \max\{n_1, n_2\}$. Consequently

$$B_n^t(B_n^s(x)) \rightarrow B^t(B^s(x)) \quad \text{as } n \rightarrow \infty$$

and the proof is complete. $\square$

Remark 1. If C is a closed cone in $\mathbb{R}$, then the family of functions satisfying all assumptions of the Theorem is a semigroup of linear functions.

PROOF. Let $C = [0, \infty)$. If $G : C \rightarrow cc(C)$ is a linear map, then there exist $0 \leq a \leq b$ such that

$$G(x) = [ax, bx] \quad \text{for } x \in [0, \infty).$$

In this case

$$\begin{aligned} B^t(x) &:= \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} [a^i x, b^i x] = \\ &= \left[\sum_{i=0}^{\infty} \frac{t^i a^i}{i!} x, \sum_{i=0}^{\infty} \frac{t^i b^i}{i!} x \right] = [e^{at} x, e^{bt} x] \end{aligned}$$

and

$$(B^t \circ B^s)(x) = B^t([e^{as} x, e^{bs} x]) = [e^{a(s+t)} x, e^{b(s+t)} x] = B^{s+t}(x)$$

for all $x \in C$, $t, s \geq 0$. $\square$

Remark 2. If the function $G : C \rightarrow cc(C)$ fulfils the conditions of the above Theorem and simultaneously the condition

$$(15) \quad G^2(x) = G(x) \quad \text{for } x \in C,$$

then the family of s.v. functions appearing in the assertion of Theorem is a semigroup of linear s.v. functions.

PROOF. Since G satisfies (15), the s.v. function

$$B^t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G(x) = e^t G(x)$$

for $t \geq 0$ defines a semigroup.

Example 1. The s.v. function $G : [0, \infty)^2 \rightarrow [0, \infty)^2$ of the form

$$G(x, y) = [0, x] \times [0, y]$$

is linear and fulfils equality (15).

Corollary 2. Under the hypotheses of Theorem the functions $B^t : C \rightarrow cc(C), t \geq 0$ are continuous on the set $\text{int } C$.

PROOF. The set $\text{int } C$ is an open convex cone. On account of (8), for any $t \geq 0, x \in C$ and $n \in \mathbb{N}$ large enough one has

$$h(B_n^t(x), B^t(x)) \leq \varepsilon \|x\|.$$

This, jointly with (13), shows that the inequalities

$$\|B^t(x)\| \leq h(B^t(x), B_n^t(x)) + \|B_n^t(x)\| \leq (\varepsilon + e^{\|G\|t})\|x\|$$

hold true. With respect to the unrestricted choice of $\varepsilon > 0$ the inequality

$$\|B^t(x)\| \leq e^{\|G\|t}\|x\|$$

is satisfied for all $x \in C, t \geq 0$. Consequently

$$(16) \quad B^t(x) \subset F^t(x) \quad \text{for } t \geq 0, x \in C,$$

where

$$(17) \quad F^t(x) = e^{\|G\|t}\|x\|S.$$

The s.v. functions $F^t : C \rightarrow cc(C)$ are continuous and the $B^t : C \rightarrow cc(C), t \geq 0$ are additive. The inclusion (16) and Theorem 5.2 from [3] together complete the proof. $\square$

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