

Some special subclasses of univalent starlike functions

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Abstract. Let $0 < \lambda < 1$, $\alpha > 1$, $\frac{2(\alpha-1)}{\lambda} < 1$ and $T(\lambda, \alpha)$ the class of homomorphic functions in the unit disc for which $f(0) = f'(0) - 1 = 0$ and $\operatorname{Re} \left[(1 - \lambda) z \frac{f'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] < \alpha$. We find sharp inequalities for the quantities $\operatorname{Re} \left\{ z \frac{f'(z)}{f(z)} \right\}$, $\operatorname{Re} \left\{ \frac{f(z)}{zf'(z)} \right\}$, $\left| \frac{zf'(z)}{f(z)} \right|$, $|f(z)|$ for $f \in T(\lambda, \alpha)$. A special result is that if $f \in T(\lambda, \alpha)$ then f is a univalent starlike function.

Introduction

If $\mathbf{H}(\mathbf{U})$ is the class of holomorphic functions defined in the unit disc $\mathbf{U} = \{z : |z| < 1\}$ then we denote by:

- (i) $\mathbf{A}$ the class of functions $f \in \mathbf{H}(\mathbf{U})$ for which $f(0) = f'(0) - 1 = 0$.
- (ii) $\mathbf{T}(\lambda, \alpha)$ the class of functions $f \in \mathbf{A}$ for which $\operatorname{Re} f_\lambda < \alpha$ where

$$f_\lambda(z) = (1 - \lambda) z \frac{f'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right), \quad 0 < \lambda < 1, \alpha > 1.$$

- (iii) $\mathbf{P}$ the class of functions $f \in \mathbf{H}(\mathbf{U})$ for which $f(0) = 1$ and $\operatorname{Re} f > 0$.
- NUNOKAWA [1] has proved the following Theorem.

Theorem 1. If $f \in \mathbf{T}(1, \frac{3}{2})$, then

$$0 < \operatorname{Re} \left[\frac{zf'(z)}{f(z)} \right] < \frac{4}{3} \quad \text{in } \mathbf{U}.$$

The inequalities in Nunokawa's Theorem are best possible. In the present paper we prove the following Theorem:

Theorem 2. If $f \in \mathbf{T}(\lambda, \alpha)$, $\frac{2(a-1)}{\lambda} < 1$ and

$$P(\lambda, \alpha, z) = \frac{1}{\lambda} \int_0^1 \left(\frac{1+tz}{1+z} \right)^{\frac{2(a-1)}{\lambda}} t^{\frac{1}{\lambda}-1} dt,$$

then

$$(i) \quad \frac{1}{P(\lambda, \alpha, -r)} < \left| \frac{zf'(z)}{f(z)} \right| < \frac{1}{P(\lambda, \alpha, r)} \text{ in } \mathbf{U}_r = \{z : |z| < r\}.$$

$$(ii) \quad 0 < \operatorname{Re} \left[\frac{zf'(z)}{f(z)} \right] < P(\lambda, \alpha, 1) \text{ in } \mathbf{U}.$$

$$(iii) \quad P(\lambda, \alpha, r) < \operatorname{Re} \left[\frac{f(z)}{zf'(z)} \right] < P(\lambda, \alpha, -r) \text{ in } \mathbf{U}_r.$$

$$(iv) \quad |f(z)| < u(\lambda, \alpha, r) \text{ in } U_r, \text{ where}$$

$$u(\lambda, \alpha, r) = r \cdot \exp \int_0^r [P^{-1}(\lambda, \alpha, \xi) - 1] \xi^{-1} d\xi.$$

All the above inequalities are best possible.

Remark. If $a = \frac{1}{2} + 1$ then

$$P(\lambda, \alpha, z) = \frac{1}{1+z} + \frac{z}{(\lambda+1)(z+1)}, u(\lambda, \alpha, r) = r \cdot \frac{(\lambda+1+r)^\lambda}{(\lambda+1)^\lambda}.$$

If $\lambda = 1$ then

$$\begin{aligned} P(\lambda, \alpha, z) &= \frac{1}{(2\alpha-1)z} \left[(1+z) - \frac{1}{(1+z)^{2\alpha-1}} \right], u(\lambda, \alpha, r) \\ &= \frac{1}{2\alpha-1} [(1+r)^{2\alpha-1} - 1]. \end{aligned}$$

In the special case $\lambda = 1$, $\alpha = \frac{3}{2}$ it is now obvious that (ii) coincides with Nunokawa's Theorem. In the same case we also have that

$$|f(z)| < r \left(1 + \frac{r}{2} \right) \text{ in } \mathbf{U}_r, \quad \text{and} \quad |f(z)| < \frac{3}{2} \text{ in } \mathbf{U}.$$

We now prove the following Lemma.

Lemma. If $f \in \mathbf{A}$ then $f_\lambda = q$, $q(0) = 1$ if and only if

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= \left[\frac{1}{\lambda} \int_0^1 \frac{F(tz)}{F(z)} t^{\frac{1}{\lambda}-1} dt \right]^{-1} \quad \text{where} \\ F(z) &= \exp \frac{1}{\lambda} \int_0^z [q(\omega) - 1] \omega^{-1} d\omega. \end{aligned}$$

PROOF. We consider a real interval $(0, \varepsilon)$ such that $\operatorname{Re} f(x) > 0$ in $(0, \varepsilon)$. We shall first prove the required relation in this interval. Then by the uniqueness Theorem for holomorphic functions we get the result in the general case.

From the relation

$$(1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left[1 + \frac{zf''(z)}{f'(z)} \right] = q(z)$$

we obtain

$$(1 - \lambda) \left[\frac{f'(z)}{f(z)} - \frac{1}{z} \right] + \lambda \frac{f''(z)}{f'(z)} = \frac{q(z) - 1}{z}$$

or

$$\left[(1 - \lambda) \operatorname{Log} \left(\frac{f(z)}{z} \right) + \lambda \operatorname{Log} f'(z) \right]' = \frac{q(z) - 1}{z}$$

or

$$(*) \quad \left(\frac{f(z)}{z} \right)^{1-\lambda} (f'(z))^\lambda = c \cdot \exp \int_0^z \frac{q(z_1) - 1}{z_1} dz_1.$$

Since $f'(0) = 1$ for $z \rightarrow 0$ we get $c = 1$. Therefore from the relation $(*)$ we have

$$(**) \quad \lambda (f^{\frac{1}{\lambda}})' = f^{\frac{1}{\lambda}-1} \cdot f' = z^{\frac{1}{\lambda}-1} F(z)$$

and

$$(***) \quad f^{\frac{1}{\lambda}} = \frac{1}{\lambda} \int_0^z u^{\frac{1}{\lambda}-1} F(u) du = \frac{1}{\lambda} z^{\frac{1}{\lambda}} \int_0^1 t^{1-\lambda} F(tz) dt.$$

Dividing $(*)^{\frac{1}{\lambda}}$ by $(***)$ we obtain the required result.

Conversely, if we set

$$Q(z) = \int_0^z u^{\frac{1}{\lambda}-1} F(u) du$$

we have

$$z \frac{f'(z)}{f(z)} = \lambda z \frac{Q'(z)}{Q(z)} \quad \forall z \in U.$$

Let $f_\lambda = q_1$. From q_1 we define, the functions F_1 and Q_1 , in the same manner as F and Q were defined from q . It is now obvious that:

$$z \frac{Q'(z)}{Q(z)} = z \frac{Q_1'(z)}{Q_1(z)} \quad \forall z \in \mathbf{U}.$$

From the above relation we obtain succesively:

$$\left(\frac{Q}{Q_1}\right)' = 0, Q' = cQ'_1, f = cF_1.$$

Since $F(0) = F_1(0) = 1$ then $c = 1$ and

$$\int_0^z (q(\omega) - 1)\omega^{-1}d\omega = \int_0^z (q_1(\omega) - 1)\omega^{-1}d\omega \quad \forall z \in \mathbf{U}.$$

Differentiating both sides of the above relation we have: $q = q_1$

PROOF of Theorem 2. If

$$p_n(z) = \sum_{k=1}^n \lambda_k \left(\frac{1 + \varepsilon_k z}{1 - \varepsilon_k z} \right), \quad |\varepsilon_k| = 1, \quad \lambda_k \geq 0 \quad \text{and} \quad \sum_{k=1}^n \lambda_k = 1$$

then we will prove the Theorem in case $f_\lambda = (1 - \alpha)p_n + \alpha$. In this case by simple calculations we obtain

$$(1) \quad F(z) = \prod_{k=1}^n (1 - z \cdot \varepsilon_k)^{\frac{2(a-1)}{\lambda} \cdot \lambda_k}.$$

Since the set

$$\left\{ \frac{1 + tz}{1 + z} : |z| < r \right\}$$

coincides with the open disc $S(R_0, R)$, where

$$R_0 = \frac{1}{2} \left(\frac{1 + tr}{1 + r} + \frac{1 - tr}{1 - r} \right), \quad R = \frac{1}{2} \left(\frac{1 - tr}{1 - r} - \frac{1 + tr}{1 + r} \right),$$

from the relation (1) follows

$$(2) \quad \left(\frac{1 + tr}{1 + r} \right)^{\frac{2(a-1)}{\lambda}} < \left| \frac{F(tz)}{F(z)} \right| < \left(\frac{1 - tr}{1 - r} \right)^{\frac{2(a-1)}{\lambda}}.$$

The conclusion (i) follows from Lemma and (2).

Since

$$|Arg \omega| < \frac{\pi}{2} \quad \forall \omega \in S(R_0, R), \quad \sum_{k=1}^n \lambda_k = 1 \quad \text{and} \quad \frac{2(a-1)}{\lambda} < 1$$

it follows

$$(3) \quad \operatorname{Re} \left[\frac{F(tz)}{F(z)} \right] > 0 \quad \text{in } \mathbf{U}.$$

The conclusion (ii) follows from (2) and (3).

The convexity of the function $\tau(\omega) = (R_0 + R\omega)^{\frac{2(a-1)}{\lambda}}$ in U implies

$$(4) \quad \operatorname{Re} \left(\frac{1+tz}{1+z} \right)^{\frac{2(a-1)}{\lambda}} > \left(\frac{1+tr}{1+r} \right)^{\frac{2(a-1)}{\lambda}}.$$

If $\tau_1(\omega) = \operatorname{Log} \tau(\omega)$ then

$$\operatorname{Re} \left[1 + \omega \frac{\tau_1''(\omega)}{\tau_1'(\omega)} \right] = \operatorname{Re} \left[\frac{R_0}{\tau(\omega)} \right] > 0 \quad \text{in } U.$$

The convexity of the function $\operatorname{Log} \tau_1(\omega)$ implies that for every $z \in \mathbf{U}_r$ there exists $\omega(z) \in \mathbf{U}_r$ such that

$$(5) \quad \frac{F(tz)}{F(z)} = \left(\frac{1+t\omega(z)}{1+\omega(z)} \right)^{\frac{2(a-1)}{\lambda}}.$$

The conclusion (iii) follows from (4), (5) and Lemma.

If $\frac{zf'(z)}{f(z)} = h(z)$ from the relation (*) of the Lemma for $\lambda = 1$ then we have

$$f(z) = z \exp \int_0^z (h(\omega) - 1) \omega^{-1} d\omega$$

or

$$|f(z)| = |z| \exp \int_0^1 \operatorname{Re}[h(tz) - 1] t^{-1} dt$$

Since $\operatorname{Re} h(tz) < \frac{1}{P(\lambda, \alpha, tr)}$ the relation (6) implies the conclusion (iv).

If $f_\lambda(z) = (1-\alpha)(1+z)(1-z)^{-1} + \alpha$ then $\frac{zf'(z)}{f(z)} = P(\lambda, \alpha, z)$.

It is now obvious that the inequalities of the theorem are best possible.

If $p \in \mathbf{P}$ then it is known that there exists a sequence of functions $p_n \in \mathbf{P}$ having the form we used such that $\lim_{n \rightarrow \infty} p_n(z) = p(z)$ in $\mathbf{U}$.

In the general case where $\operatorname{Re} f_\lambda > \alpha$ or $f_\lambda = (1-\alpha)p + \alpha$, $p \in \mathbf{P}$ we consider the sequence

$$(1-\lambda) \frac{zf_n'(z)}{f_n(z)} + \lambda \left[1 + z \frac{f_n''(z)}{f_n'(z)} \right] = (1-\alpha)p_n(z) + \alpha.$$

If $F(z)$ and $F_n(z)$ are the functions of Lemma corresponding to f and f_n , respectively, then from the relation $\left| \frac{(1-\alpha)p_n(z) + \alpha - 1}{z} \right| \leq \frac{2(a-1)}{r}$ in $\mathbf{U}_r$ it follows that $\lim_{n \rightarrow \infty} F_n(z) = F(z)$ in $\mathbf{U}_r$. Continuing in this manner we prove that

$$\lim_{n \rightarrow \infty} \left[\frac{zf_n'(z)}{f_n(z)} \right] = \frac{zf'(z)}{f(z)} \quad \text{and} \quad \lim_{n \rightarrow \infty} f_n'(z) = f'(z).$$

References

- [1] M. NUNOKAWA, A sufficient condition for Univalence and Starlikeness, *Proc. Japan Acad.* **65** Ser. A no. 6 (1989), 163–164.

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