

A note on the Diophantine equation $\binom{x}{4} = \binom{y}{2}$

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1. Introduction

Let n be a rational integer with $n \geq 3$. BRINDZA [B] proved that the equation

$$(1) \quad \binom{x}{n} = \binom{y}{2} \quad \text{in integers } x, y$$

has only finitely many solutions and all them can be, at least in principle, effectively determined. See also [Ki] and [P]. In 1967, AVANESOV [A] showed that for $n = 3$, the equation

$$\binom{x}{3} = \binom{y}{2} \quad \text{in integers } x \geq 3, y \geq 2$$

possesses only the solutions $(x, y) = (3, 2), (5, 5), (10, 16), (22, 56)$ and $(36, 120)$. The purpose of this note is to give a simple resolution of the equation

$$(2) \quad \binom{x}{4} = \binom{y}{2} \quad \text{in integers } x \geq 4, y \geq 2.$$

Theorem. *All the integer solutions (x, y) to the equation (2) are $(x, y) = (4, 2), (6, 6)$ and $(10, 21)$.*

This provides an answer to a question of GUY [G, Section D3].

For $n = 5$, it is easy to see that $\binom{15}{5} = \binom{78}{2}$ and $\binom{19}{5} = \binom{153}{2}$. However, it seems to be a harder problem to determine all solutions of the equation $\binom{x}{5} = \binom{y}{2}$ in positive integers x, y .

2. Proof of the Theorem

Equation (2) leads to

$$(3) \quad (x^2 - 3x + 1)^2 + 2 = 3(2y - 1)^2.$$

The left hand side of (3) can be factorized over $K = \mathbb{Q}(\sqrt{-2})$. Denote by O_K the ring of integers of K . As is known, $\{1, \sqrt{-2}\}$ is an integral basis for O_K and O_K is a unique factorization ring.

The greatest common divisor in O_K of the factors $x^2 - 3x + 1 + \sqrt{-2}$ and $x^2 - 3x + 1 - \sqrt{-2}$ divides $-2\sqrt{-2} = (\sqrt{-2})^3$. Hence we have

$$(4) \quad \begin{aligned} x^2 - 3x + 1 + \sqrt{-2} \\ = (\sqrt{-2})^\alpha \cdot (1 + \sqrt{-2})^\beta \cdot (1 - \sqrt{-2})^\gamma \cdot (-1)^\delta \cdot (a + b\sqrt{-2})^2, \end{aligned}$$

where $\alpha, \beta, \gamma, \delta \in \{0, 1\}$ and $a, b \in \mathbb{Z}$. On taking the norm with respect to $K/\mathbb{Q}$, in view of (3) we get $\alpha = \beta \cdot \gamma = 0$. Since $(a + b\sqrt{-2})^2 = a^2 - 2b^2 + 2ab\sqrt{-2}$, it is easy to exclude $(\alpha, \beta, \gamma, \delta) = (0, 0, 0, 0)$ and $(0, 0, 0, 1)$. Summarizing, we get four possibilities: $(\alpha, \beta, \gamma, \delta) = (0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 1, 0, 1)$ and $(0, 0, 1, 1)$. On equating the coefficients in the basis $\{1, \sqrt{-2}\}$ of the left and right hand sides of (4), after some straightforward calculations we get in the second and third cases the equations

$$(a - 13b)^2 + (2x - 3)^2 = 19(3b)^2,$$

and

$$(3a - b)^2 + (2x - 3)^2 = 19b^2,$$

respectively. Since $19 \equiv -1 \pmod{4}$, these equations are not solvable. If $(\alpha, \beta, \gamma, \delta) = (0, 1, 0, 0)$ then (4) yields

$$(5) \quad a^2 - 2b^2 - 4ab = x^2 - 3x + 1$$

and

$$(6) \quad a^2 - 2b^2 + 2ab = 1.$$

Thus

$$(7) \quad \begin{aligned} 4(a^2 - 2b^2 - 4ab) + 5(a^2 - 2b^2 + 2ab) \\ = (3a - b)^2 - 19b^2 = (2x - 3)^2. \end{aligned}$$

As is known (see [C]), the general solution of the equation $u^2 + 19v^2 = w^2$ in integers u, v, w can be written as

$$u = (m^2 - 19n^2)d, \quad v = 2mn \cdot d, \quad w = (m^2 + 19n^2)d,$$

where

$$(8) \quad m, n, d \in \mathbb{Z}, \quad m + n \equiv 1 \pmod{2} \text{ and } (m, n) = 1,$$

or

$$u = \frac{m^2 - 19n^2}{2}d, \quad v = mn \cdot d, \quad w = \frac{m^2 + 19n^2}{2}d$$

with

$$(9) \quad m, n, d \in \mathbb{Z}, \quad m \equiv n \equiv 1 \pmod{2} \text{ and } (m, n) = 1.$$

Using these formulas we have by (7)

$$2x - 3 = d(m^2 - 19n^2), \quad b = d \cdot 2mn, \quad 3a - b = d(m^2 + 19n^2)$$

with (8) or

$$2x - 3 = \frac{m^2 - 19n^2}{2}d, \quad b = d \cdot mn, \quad 3a - b = \frac{m^2 + 19n^2}{2}d$$

with (9). Substituting these values into the equation (6) we obtain

$$(10) \quad d^2(m^4 + 16nm^3 - 6n^2m^2 + 304n^3m + 361n^4) = 9$$

or

$$(11) \quad d^2(m^4 + 16nm^3 - 6n^2m^2 + 304n^3m + 361n^4) = 36.01$$

Using the program package KANT [Ka] and the BAKER–DAVENPORT reduction algorithm (see [BD]) we get the solutions $(d, m, n) = (\pm 3, \pm 1, 0)$ for (10) and $(d, m, n) = (\pm 1, 1, -1), (\pm 1, -1, 1)$ and $(\pm 6, \pm 1, 0)$ for (11), respectively. It is easy to see that they lead to $x = 3$ and $x = 6$.

In the remaining case $(\alpha, \beta, \gamma, \delta) = (0, 0, 1, 1)$, (4) yields

$$(12) \quad a^2 - 2b^2 - 2ab = 1$$

and

$$(13) \quad 2b^2 - a^2 - 4ab = x^2 - 3x + 1.$$

Following the argument above we have

$$(2x - 3)^2 + 19(3b)^2 = (a - 13b)^2,$$

whence

$$2x - 3 = (m^2 - 19n^2)d, \quad 3b = 2mn \cdot d, \quad a - 13b = (m^2 + 19n^2)d$$

with (8), or

$$2x - 3 = \frac{m^2 - 19n^2}{2}d, \quad 3b = mn \cdot d, \quad a - 13b = \frac{m^2 + 19n^2}{2}d$$

with (9). Substituting these values into the equation (12) we get

$$(14) \quad d^2(3m^4 + 48nm^3 + 302n^2m^2 + 912n^3m + 1083n^4) = 3$$

or

$$(15) \quad d^2(3m^4 + 48nm^3 + 302n^2m^2 + 912n^3m + 1083n^4) = 12.$$

Using KANT and the Baker–Davenport reduction algorithm again we have the solutions $(d, m, n) = (\pm 1, 6, 1)$, $(\pm 1, -6, 1)$, $(\pm 1, \pm 1, 0)$ for (14), and $(d, m, n) = (\pm 2, 6, -1)$, $(\pm 2, -6, 1)$, $(\pm 2, \pm 1, 0)$, $(\pm 1, 3, -1)$, $(\pm 1, -3, 1)$ for (15). They lead to $x = 2, 4$ and 10. By assumption $x \geq 4$, hence we have $x = 4, 6$ and 10. For $x = 4, 6$ and 10, we get from (2) that $y = 2, 6, 21$, respectively. This completes the proof of the theorem.

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Added in proof: The problem above has been solved independently by B. de Weger, A binomial diophantine equation, to appear in Quart. J. Math. Oxford.

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