

Oscillation criteria for second order half-linear differential equations with functional arguments

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Abstract. Oscillation criteria for the second order half-linear differential equations with functional arguments of the form

$$(*) \quad [r(t)|y'(t)|^{\alpha-1}y'(t)]' + p(t)f(y(t), y(g(t))) = 0,$$

are established, where $\alpha > 0$ is a constant and $g(t) \rightarrow \infty$ as $t \rightarrow \infty$. These results exhibit a surprising similarity in the oscillatory behavior existing between $(*)$ and the corresponding differential equation

$$y''(t) + p(t)f(y(t), y(g(t))) = 0.$$

1. Introduction

Consider the following three second order differential equations

$$(E) \quad [r(t)|y'(t)|^{\alpha-1}y'(t)]' + p(t)f(y(t), y(g(t))) = 0,$$

$$(E_1) \quad [r(t)|y'(t)|^{\alpha-1}y'(t)]' + p(t)|y(g(t))|^{\beta-1}y(g(t)) = 0,$$

$$(E_2) \quad [r(t)|y'(t)|^{\alpha-1}y'(t)]' + \lambda p(t)|y(t)|^{\alpha-1}y(t) = 0,$$

where

- (a) $p, g \in C([t_0, \infty); \mathbb{R})$ for some $t_0 \geq 0$ and $\lim_{t \rightarrow \infty} g(t) = \infty$;
- (b) $r \in C^1([t_0, \infty), (0, \infty))$;

- (c) $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $f(x, y)$ has the same sign of x and y when they have the same sign, that is,

$$f(x, y) \begin{cases} > 0 & \text{if } x > 0, y > 0, \\ < 0 & \text{if } x < 0, y < 0; \end{cases}$$

- (d) α and β are positive constants.

Throughout this paper, we define

$$\pi(t) := \int_t^\infty (r(s))^{-\frac{1}{\alpha}} ds, \quad t \geq t_0.$$

In [1], ELBERT established the existence and uniqueness of solutions to the initial value problem for equation (E_2) on $[t_0, \infty)$. Note that any constant multiple of a solution of (E_2) is also a solution of (E_2) . A nontrivial solution is called oscillatory if it has arbitrarily large zeros; otherwise it is said to be nonoscillatory. The equation (E_2) is nonoscillatory [resp. oscillatory] if all of its solutions are nonoscillatory [resp. oscillatory].

Surprisingly, some similar properties between equation (E_2) and the linear equation

$$(E_0) \quad (p(t)y')' + q(t)y = 0, \quad t \geq 0$$

have been observed by ELBERT [1, 2], MIRZOV [9, 10, 11], KUSANO and NAITO [4], KUSANO and YOSHIDA [5], KUSANO et al [6], LI and YEH [7, 8]. For example, Sturmian theory for (E_0) has been extended in a natural way to (E_2) by ELBERT [1], LI and YEH [7]. They showed that the zeros of two linearly independent solutions of (E_2) separate each other. If $\alpha = 1$ and $r(t) = 1$, then TRAVIS [12] and YEH [13] established some sufficient conditions on the oscillation of (E) . TRAVIS [12] also gave a sufficient condition on the oscillation of $y'(t)$ for any solution of (E_2) . For the other result, we refer to KUSANO and LALLI [3].

The purpose of this paper is to extend the results of TRAVIS [12] and YEH [13] to the equation (E) by using a result (lemma 3 below) of LI and YEH [7].

2. Oscillations of Equation (E)

Theorem 1. *Let $\pi(t_0) = \infty$; and*

(C₁) there exist a constant k and a function $h(t) \in C([t_0, \infty), \mathbb{R})$ such that $h(t) \leq g(t)$ and $0 < k \leq h'(t) \leq 1$

(C₂) there exist a constant $M > 0$ such that $|y| \geq M$ implies

$$\liminf_{|w| \rightarrow \infty} \left| \frac{f(y, w)}{|w|^{\alpha-1}w} \right| \geq \varepsilon > 0$$

for some $\varepsilon > 0$.

(C₃) $p(t) \geq 0$ and $\limsup_{t \rightarrow \infty} A(t, t_0)^{-\lambda} \int_{t_0}^t A(t, s)^\lambda p(s) ds = \infty$,

where $A(t, s) = \int_s^t (r(u))^\frac{-1}{\alpha} du$; $\lambda > 1$.

Then (E) is oscillatory.

PROOF. Assume the contrary. Then (E) has a nonoscillatory solution $y(t)$. Without loss of generality, we may assume that $y(t) > 0$ on $[T, \infty)$ for some $T \geq t_0$. It is easily to verify that $y'(t) > 0$ for large t . Let

$$w(t) = \frac{r(t)|y'(t)|^{\alpha-1}y'(t)}{|y(h(t))|^{\alpha-1}y(h(t))}.$$

Then $w(t)$ satisfies

$$(1) \quad w'(t) = -p(t) \frac{f(y(t), y(g(t)))}{|y(h(t))|^{\alpha-1}y(h(t))} - \alpha \frac{y'(h(t))}{y(h(t))} h'(t) w(t),$$

for $t \geq T$.

Since $y'(t) > 0$ for large t , $\lim_{t \rightarrow \infty} y(t)$ exists either as a finite or infinite limit. If $\lim_{t \rightarrow \infty} y(t) = b$ is finite, then

$$\lim_{t \rightarrow \infty} \frac{f(y(t), y(g(t)))}{|y(g(t))|^{\alpha-1}y(g(t))} = \frac{f(b, b)}{b^\alpha} > 0.$$

If $\lim_{t \rightarrow \infty} y(t) = \infty$, then, by (C₂), we have that

$$\frac{f(y(t), y(g(t)))}{|y(g(t))|^{\alpha-1}y(g(t))} \geq \varepsilon > 0$$

for all large t . Let $\varepsilon_1 = \min\{\varepsilon, \frac{f(b,b)}{2b^\alpha}\}$. Since $y(t)$ is increasing for large t , we have that

$$(2) \quad p(t) \frac{f(y(t), y(g(t)))}{|y(h(t))|^{\alpha-1} y(h(t))} \geq p(t) \frac{f(y(t), y(g(t)))}{|y(g(t))|^{\alpha-1} y(g(t))} \geq \varepsilon_1 p(t),$$

and

$$(3) \quad \alpha \frac{y'(h(t))}{y(h(t))} h'(t) w(t) \geq \alpha k r^{\frac{-1}{\alpha}}(t) w^{\frac{1+\alpha}{\alpha}}(t),$$

for all large t . It follows from (1), (2) and (3) that

$$(4) \quad \begin{aligned} w'(t) &\leq -\varepsilon_1 p(t) - \alpha k r^{\frac{-1}{\alpha}}(t) w^{\frac{1+\alpha}{\alpha}}(t) \\ &\leq -\varepsilon_1 p(t) \leq 0 \end{aligned}$$

for $t \geq T$, where T is large enough. This implies

$$(5) \quad \int_T^t A^\lambda(t, s) w'(s) ds \leq -\varepsilon_1 \int_T^t A^\lambda(t, s) p(s) ds.$$

Since

$$\int_T^t A^\lambda(t, s) w'(s) ds = \lambda \int_T^t A^{\lambda-1}(t, s) r^{\frac{-1}{\alpha}}(s) w(s) ds - w(T) A^\lambda(t, T),$$

we get, by (5)

$$\varepsilon_1 A^{-\lambda}(t, t_0) \int_T^t A^\lambda(t, s) p(s) ds \leq w(T) \left\{ \frac{A(t, T)}{A(t, t_0)} \right\}^\lambda \leq w(T).$$

Hence

$$\limsup_{t \rightarrow \infty} \varepsilon_1 A^{-\lambda}(t, t_0) \int_T^t A^\lambda(t, s) p(s) ds \leq w(T),$$

which contradicts condition (C₃). Thus, our proof is complete.

We say that (E₂) is strongly oscillatory if (E₂) is oscillatory for every $\lambda > 0$. In order to discuss the next two theorems, we need the following three lemmas:

Lemma 2 (KUSANO et al. [6]).

If $\int^\infty (r(s))^{\frac{-1}{\alpha}} ds = \infty$, then (E₂) is strongly oscillatory if and only if (C₄) $p \geq 0$ is integrable on $[t_0, \infty)$ and $\limsup_{t \rightarrow \infty} A^\alpha(t, t_0) \int_t^\infty p(s) ds = \infty$, where $A(t, t_0)$ is defined as in Theorem 1.

Lemma 3 (KUSANO and NATIO [4]). *If $\int^\infty (r(s))^{-\frac{1}{\alpha}} ds < \infty$, then (E_2) is strongly oscillatory if and only if (C_5) $p \geq 0$ is integrable on $[t_0, \infty)$, $\int_{t_0}^\infty (\pi(t))^{\alpha+1} p(t) dt < \infty$, and*

$$\limsup_{t \rightarrow \infty} \pi^{-1}(t) \int_t^\infty (\pi(s))^{\alpha+1} p(s) ds = \infty.$$

Lemma 4 (LI and YEH [7]). *Equation (E) is nonoscillatory if and only if there is a function $\omega \in C^1[T, \infty)$ for some $T \geq t_0$, satisfying*

$$\omega'(t) + p(t) + \alpha r^{-\frac{1}{\alpha}}(t) |\omega(t)|^{\frac{\alpha+1}{\alpha}} \leq 0.$$

Theorem 5. *Let (C_1) and (C_2) hold and $\pi(t_0) = \infty$. If condition (C_4) holds, then (E) is oscillatory.*

PROOF. Assume the contrary. Then (E) has a nonoscillatory solution $y(t)$. With loss of generality, we may assume that $y(t) > 0$ on $[T, \infty)$ for some $T \geq t_0$. Let

$$w(t) = \frac{r(t) |y'(t)|^{\alpha-1} y'(t)}{|y(h(t))|^{\alpha-1} y(h(t))}.$$

As in the proof of theorem 1, we have

$$w'(t) + \varepsilon_1 p(t) + \alpha k r^{-\frac{1}{\alpha}}(t) w^{\frac{1+\alpha}{\alpha}}(t) \leq 0.$$

If $u(t) = k^\alpha w(t)$, then

$$u'(t) + \varepsilon_1 k^\alpha p(t) + \alpha r^{-\frac{1}{\alpha}}(t) u^{\frac{1+\alpha}{\alpha}}(t) \leq 0.$$

It follows from lemma 4 that

$$[r(t) |u'(t)|^{\alpha-1} u'(t)]' + \varepsilon_1 k^\alpha p(t) |u(t)|^{\alpha-1} u(t) = 0$$

is nonoscillatory. However, this contradicts the fact that (E_2) is strongly oscillatory (by Lemma 2).

Similiarly, using Lemma 3 and Lemma 4, we can prove the following

Theorem 6. *Let (C_1) and (C_2) hold and $\pi(t_0) < \infty$. If condition (C_5) holds, then (E) is oscillatory.*

3. Oscillation of the derivative of a solution of (E₁)

Theorem 7. Assume $g(t)$ is differentiable, $g'(t) \geq 0$, and $\int^\infty p(t)dt = \infty$. If $\int^\infty r^{\frac{-1}{\alpha}}(t)dt = \infty$, then $y'(t)$ is oscillatory for any solution $y(t)$ of (E₁).

PROOF. If $y(t)$ oscillates, then there is nothing to prove. If $y(t)$ is ultimately positive, then so is $y(g(t))$. Suppose $y'(t) > 0$ for all large t . Then

$$w(t) = \frac{r(t)|y'(t)|^{\alpha-1}y'(t)}{|y(g(t))|^{\beta-1}y(g(t))}$$

satisfies the equation

$$w'(t) = -p(t) - w(t) \frac{\beta[y(g(t))]^{\beta-1}y'(g(t))g'(t)}{|y(g(t))|^{\beta-1}y(g(t))} \leq -p(t).$$

Integrating the above inequality, we obtain

$$w(x) \leq w(\alpha) - \int_\alpha^x p(t)dt.$$

It follows from $\int^\infty p(t)dt = \infty$ that $y'(t) < 0$ for all large t , which is a contradiction. Suppose now $y'(t) < 0$ for all large t . It is easy to see that $\int^\infty p(t)dt = \infty$ implies that there exists a positive constant T such that

$$\int_T^t p(t)dt \geq 0$$

for $t \geq T$. Hence, we have

$$\begin{aligned} & \int_T^t p(s)(|y(g(s))|^{\beta-1}y(g(s)))ds \\ (6) \quad & = |y(g(t))|^{\beta-1}y(g(t)) \int_T^t p(s)ds \\ & - \beta \int_T^t (y(g(s)))^{\beta-1}y'(g(s))g'(s) \int_T^s p(r)dr ds \geq 0, \quad t \geq T. \end{aligned}$$

Now integrating equation (E₁) and using (6), we have

$$r(t)|y'(t)|^{\alpha-1}y'(t) \leq r(T)|y'(T)|^{\alpha-1}y'(T) := |c|^{\alpha-1}c < 0,$$

i.e.,

$$|y'(t)|^{\alpha-1}y'(t) \leq \frac{r(T)|y'(T)|^{\alpha-1}y'(T)}{r(t)} < 0,$$

thus

$$y'(t) \leq cr^{\frac{-1}{\alpha}}(t) < 0$$

for some $c < 0$. Integrating it from T to t ($\geq T$), we obtain

$$y(t) - y(T) \leq c \int_T^t r^{\frac{-1}{\alpha}}(s) ds.$$

Thus $y(t) < 0$ for t large enough, which contradicts the fact that $y(t)$ is positive for large t . This completes the proof.

Example 8. Let $y(t)$ be a solution of

$$\frac{d}{dt}\phi(y') + \frac{\sin t}{2 - \sin t}\phi(y) = 0$$

for $t \geq 0$, where $\phi(u) = |u|^{\alpha-1}u$, then, by theorem 7, $y'(t)$ is oscillatory because

$$\int^{\infty} \frac{\sin t}{2 - \sin t} dt = \infty.$$

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(Received June 2, 1994)