

A sharp inequality for Bergman-Nevanlinna functions

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Abstract. In this note we prove inequalities, one for harmonic functions in the unit disc Δ which are representable as the Poisson integral of a finite measure on $\partial\Delta$, and another one for Bergman-Nevanlinna functions. These are used to characterize those entire functions f whose associated autonomous nonlinear superposition operator transforms Nevanlinna functions into Bergman-Nevanlinna functions.

1. Introduction

Let f be an entire function and $H(\Delta)$ the space of analytic functions in the unit disc Δ . By h^1 we mean the Banach space of harmonic functions in Δ which are equal to the difference of two positive harmonic functions. The nonlinear superposition operator F_f is defined by $F_f(u) = f \circ u$, whenever $u \in H(\Delta)$. General information about this operator may be found in [1]. About the action of F_f between Bergman spaces the reader may consult [5]. We shall denote by N the well known Nevanlinna space of functions u in $H(\Delta)$ such that $\log^+|u(z)|$ has a harmonic majorant. The symbol BN will denote the Bergman-Nevanlinna space of functions u in $H(\Delta)$ such that

$$\iint_{\Delta} \log^+|u(z)| dx dy < \infty.$$

It is natural to ask what are the entire functions f such that F_f transforms N into BN . In other words, characterize those entire functions f for which $f \circ u \in BN$ whenever $u \in N$. In this paper we solve this problem. To do this we need an inequality for functions in h^1 which may

be of independent interest. We shall also need an inequality for functions in BN which, in a certain way, sharpens the usual one.

We shall use the following notation: for $0 < p < \infty$

$$M_p(r, h) = \left(\frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}},$$

$$\|h\|_p = \sup_{r < 1} M_p(r, h).$$

When $p = \infty$ we shall denote by $M_\infty(r, h)$ the usual maximum modulus of h on the circle of radius r .

In [4] I proved the following inequality

Theorem A. *If $h \in h^1$ then*

$$\int_0^1 M_{2-\varepsilon}^{2-\varepsilon}(r, h) dr \leq C \|h\|_1^{2-\varepsilon}, \quad 0 < \varepsilon < 1.$$

For $\varepsilon = 0$ the inequality is no longer true as shown by the function $h(z) = \frac{1-|z|^2}{|1-z|^2}$.

We shall improve this inequality by changing the function $t^{2-\varepsilon}$ to a convex nondecreasing function $\phi(t)$ which grows more slowly than t^2 . More precisely we have

Theorem 1. *If $h = h_1 - h_2$, where h_1 and h_2 are two positive harmonic functions in Δ , and $\phi : [1, \infty) \rightarrow \mathbb{R}^+$ is a non-decreasing convex function such that*

$$\int_1^\infty \frac{\phi(t)}{t^3} dt < \infty,$$

then

$$\iint_\Delta \phi(|h(z)|) dx dy < \infty.$$

This theorem sharpens the result $h^1 \subset b_q$, $\forall q < 2$, where b_q is the Banach space of Bergman harmonic functions ([2], page 167).

The subharmonicity of $\log^+ |u(z)|$ for $u \in BN$ easily implies the well known inequality

$$\log^+ |u(z)| \leq \frac{C}{(1-|z|)^2}.$$

We shall improve this inequality in the following manner.

Theorem 2. *Let $u \in BN$. Then there exists a non-decreasing convex function $\phi : [1, \infty) \rightarrow \mathbb{R}^+$ with*

$$\int_1^\infty \frac{\phi(t)}{t^3} dt < \infty,$$

such that

$$\log^+ |u(z)| \leq \phi \left(\frac{1}{1 - |z|} \right).$$

As an application of Theorems 1 and 2 we characterize those entire function f for which the nonlinear operator F_f acts from N to BN .

Theorem 3. *Let f be an entire function. Then F_f acts from N to BN if and only if there exists a non-decreasing convex function $\phi : [0, +\infty) \rightarrow \mathbb{R}^+$ with*

$$\int_1^\infty \frac{\phi(t)}{t^3} dt < \infty$$

and $\log^+ M_\infty(r, f) \leq \phi(\log^+ r)$.

The way we have followed to prove Theorem 3 has forced us to state it in terms of a convex function ϕ . Actually we can get rid of ϕ and state Theorem 3 in the following equivalent form.

Theorem 3'. *Let f be an entire function. Then F_f acts from N to BN if and only if*

$$\int_1^\infty \frac{\log M_\infty(e^t, f)}{t^3} dt < \infty.$$

This result follows easily from Theorem 3 by the convexity (as a function of t) of the function $\log M_\infty(e^t, f)$ (Hadamard's Three Circles Theorem).

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1. Proof of Theorem 1

Set $A = \iint_{B(0, \frac{1}{2})} \phi(|h(z)|) dx dy$. By the Harnack inequality A is bounded above by $C_1 = \frac{\pi}{4} \phi(C(h_1(0) + h_2(0)))$, where C is an absolute constant. Therefore

$$\iint_{\Delta} \phi(|h(z)|) dx dy \leq \iint_{\Delta \setminus B(0, \frac{1}{2})} \phi(|h(z)|) dx dy + C_1.$$

By the Riesz-Herglotz Theorem ([6], Theorem 1.1) there is a function $\mu(t)$ of bounded total variation $\|\mu\| = \|h\|_1$ such that

$$h(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{|e^{it} - re^{i\theta}|^2} d\mu(t), \quad z = re^{i\theta} \in \Delta.$$

Using Jensen's inequality at the appropriate step we can write

$$\begin{aligned} & \iint_{\Delta \setminus B(0, \frac{1}{2})} \phi(|h(z)|) dx dy \\ & \leq \iint_{\Delta \setminus B(0, \frac{1}{2})} \phi \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\|\mu\|(1-r^2)}{|e^{it} - re^{i\theta}|^2} \frac{d|\mu|(t)}{\|\mu\|} \right) dx dy \\ & \leq \iint_{\Delta \setminus B(0, \frac{1}{2})} \left(\int_{-\pi}^{\pi} \phi \left(\frac{\|\mu\|(1-r^2)}{2\pi|e^{it} - re^{i\theta}|^2} \right) \frac{d|\mu|(t)}{\|\mu\|} \right) dx dy \\ & = \iint_{\Delta \setminus B(0, \frac{1}{2})} \phi \left(\frac{\|\mu\|(1-r^2)}{2\pi|1 - re^{i\theta}|^2} \right) dx dy. \end{aligned}$$

If we set

$$I := \iint_{\Delta} \phi \left(\frac{\|\mu\|(1-|z|^2)}{|1-z|^2} \right) dx dy$$

it suffices to prove that

$$I \leq C(\phi) \|\mu\|^2.$$

Let $z = \frac{w-1}{w+1}$ with $w = t + is$. Then

$$\begin{aligned} I &= 4 \int_{\operatorname{Re} w > 0} \phi(\|\mu\| \operatorname{Re} w) \frac{dt ds}{|1+w|^4} \\ &= 4 \int_0^\infty \phi(\|\mu\|t) dt \int_{-\infty}^\infty \frac{ds}{[(1+t)^2 + s^2]^2} = 2\pi \int_0^\infty \frac{\phi(\|\mu\|t)}{(1+t)^3} dt \\ &\leq 2\pi \|\mu\|^2 \int_0^\infty \frac{\phi(t)}{(\|\mu\| + t)^3} dt < \infty, \end{aligned}$$

as required.

Remark. This results is sharp in the following sense. If $\theta : [0, \infty) \rightarrow \mathbb{R}^+$ is a non-decreasing function such that

$$\int_1^\infty \frac{\phi(t)}{t^3} dt = +\infty$$

then

$$\iint_{\Delta} \phi \left(\frac{1 - |z|^2}{|1 - z|^2} \right) dx dy = \infty.$$

The exponent 2 on the right hand side of the inequality cannot be improved.

In [4] I proved the following lemma.

Lemma 2. *If $u \in N$ then*

$$\int_0^1 dr \int_0^{2\pi} (\log^+ |u(re^{i\theta})|)^{2-\varepsilon} d\theta < \infty, \quad 0 < \varepsilon < 1.$$

As an application of Theorem 1 we have the following corollary which sharpens this result.

Corollary. *If $u \in N$ and $\phi : [0, \infty) \rightarrow \mathbb{R}^+$ is a non-decreasing convex function which satisfies*

$$(1.1) \quad \int_1^\infty \frac{\phi(s)}{s^3} ds < \infty$$

then

$$(1.2) \quad \int_0^1 r dr \int_0^{2\pi} \phi(\log^+ |u(re^{i\theta})|) d\theta < \infty.$$

PROOF. We assume first that $u \neq 0$ in Δ . Then $\log |u| \in h^1$. Therefore

$$\int_0^1 r dr \int_0^{2\pi} \phi(\log^+ |u(re^{i\theta})|) d\theta \leq \int_0^1 r dr \int_0^{2\pi} \phi(|\log |u||) d\theta < \infty$$

in view of Theorem 1. For a general $u \in N$ we take $v = \frac{u}{B_u}$, where is the Blaschke product associated to the zeroes of u . Then $v \in N$, $v \neq 0$ in Δ and $|u| \leq |v|$. Thus

$$\int_0^1 r dr \int_0^{2\pi} \phi(\log^+ |u(re^{i\theta})|) d\theta \leq \int_0^1 r dr \int_0^{2\pi} \phi(\log^+ |v(re^{i\theta})|) d\theta < \infty$$

by means of what was proved first.

Remark. This corollary is sharp in the sense that if $\phi : [0, \infty) \rightarrow \mathbb{R}^+$ is a non-decreasing function which satisfies (1.1) then (1.2) diverges when $u(z) = e^{\frac{1+z}{1-z}} \in N$.

3. Proof of Theorem 2

We shall need the following fundamental lemma.

Lemma 3. *If $u(z)$ is analytic in $\{|z| \leq R\}$ then*

$$(3.1) \quad \log^+ M_\infty(s, u) \leq \frac{R+s}{R-s} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |u(Re^{i\theta})| d\theta, \quad 0 \leq s < R.$$

A proof of this can be found in [7, pag. 18].

PROOF of Theorem 2. Let $\phi_1(r) = \log^+ M_\infty(1 - \frac{1}{r}, u)$. Clearly

$$\log^+ |u(z)| \leq \phi_1 \left(\frac{1}{1 - |z|} \right).$$

Now we define

$$\phi(r) = 4 \int_1^{2r} T \left(1 - \frac{1}{2t}, u \right) dt,$$

where

$$T(s, u) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |u(se^{i\theta})| d\theta$$

is the Nevanlinna characteristic function of u . Since $T(s, u)$ is a positive non-decreasing function of s then $\phi(r)$ is an increasing convex function of r . We also have that $\phi_1(r) \leq \phi(r)$. In fact, if we set $R = 1 - \frac{1}{2r}$ and $s = 1 - \frac{1}{r}$ in (3.1) we obtain

$$\log^+ M_\infty \left(1 - \frac{1}{r}, u \right) \leq (4r - 3) T \left(1 - \frac{1}{2r}, u \right).$$

Hence

$$\begin{aligned} \phi(t) &\geq 4 \int_r^{2r} T \left(1 - \frac{1}{2t}, u \right) dt \geq 4r T \left(1 - \frac{1}{2r}, u \right) \\ &\geq \frac{4r}{4r - 3} \log^+ M_\infty \left(1 - \frac{1}{r}, u \right) \geq \phi_1(r). \end{aligned}$$

Finally,

$$\begin{aligned} \int_1^\infty \frac{\phi(r)}{r^3} dr &= 4 \int_1^\infty \frac{dr}{r^3} \int_1^{2r} T \left(1 - \frac{1}{2t}, u \right) dt \\ &= 2 \int_1^\infty T \left(1 - \frac{1}{2t}, u \right) dt \int_{\frac{t}{2}}^\infty \frac{dr}{r^3} = 8 \int_1^\infty \frac{T(1 - \frac{1}{2t}, u)}{t^2} dt \end{aligned}$$

$$= 16 \int_{\frac{1}{2}}^1 T(s, u) ds < \infty,$$

since $u \in BN$.

4. Proof of Theorem 3

First of all let us assume that $u \in N$ and

$$\log^+ M_\infty(r, f) \leq \phi(\log^+ r) + C, \quad r \geq 0,$$

for a function ϕ satisfying the conditions stated in the theorem. Then

$$\begin{aligned} \iint_{\Delta} \log^+ |F_f(u)(z)| dx dy &= \int_0^1 r dr \left(\int_0^{2\pi} |f(u(re^{i\theta}))| d\theta \right) \\ &\leq \int_0^1 r dr \left(\int_0^{2\pi} \log^+ M_\infty(|u(re^{i\theta})|, f) d\theta \right) \\ &\leq \int_0^1 r dr \left(\int_0^{2\pi} \phi(\log^+ |u(re^{i\theta})|) d\theta \right) + O(1) \\ &= \iint_{\Delta} \phi(\log^+ |u(z)|) dx dy + O(1) < \infty, \end{aligned}$$

in view of the corollary to Theorem 1. Therefore, F_f acts from N to BN . Next, let us suppose that F_f acts from N to BN . The function $w = u(z) = \exp \left\{ \frac{1+z}{1-z} \right\}$ belongs to N . Thus $f \circ u$ belongs to BN . As a consequence of Theorem 2 we have a function $\phi : [1, \infty) \rightarrow \mathbb{R}^+$, convex and non-decreasing which satisfies

$$\int_1^\infty \frac{\phi(r)}{r^3} dr < \infty$$

and

$$(4.1) \quad \log^+ |f(w)| \leq \phi \left(\frac{1}{1-|z|} \right).$$

We shall confine ourselves to those z in the Stolz angle $S = \{z : |1-z| \leq C_1(1-|z|), \operatorname{Re} z \geq c_0\} \cap \Delta$. If S is big enough then $u(S) \supseteq \{w : |w| > R\}$, for some $R > 1$. Since $|w| = \exp \left\{ \frac{1-|z|^2}{1-|z|} \right\}$ we can write

$$(4.2) \quad \frac{1}{1-|z|} = \frac{|1-z|^2}{(1-|z|)^2(1+|z|)} \log |w| \leq C_1^2 \log |w|, \quad z \in S.$$

Since ϕ_1 is non-decreasing we obtain from (4.1) and (4.2) that

$$\log^+ |f(w)| \leq \phi_1(C_1^2 \log |w|) \quad |w| > R.$$

The desired result is obtained by choosing $\phi(r) = \phi_1(C_1^2 r)$.

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