

On the number of solutions of the generalized Ramanujan-Nagell equation $x^2 - D = k^n$

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Abstract. Let $D, k \in \mathbb{N}$ be such that $D > 1$, $k > 1$ and $\gcd(2D, k) = 1$. In this paper we prove that the titled equation has at most $3 \cdot 2^{\omega(k)-1} + 1$ positive integer solutions (x, n) , where $\omega(k)$ is the number of distinct prime factors of k . Moreover, if $\max(D, k) > 10^{60}$, then the equation has at most $3 \cdot 2^{\omega(k)-1}$ solutions (x, n) .

1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ be the sets of integers, positive integers and rational numbers respectively. Let $D, k \in \mathbb{N}$ be such that $D > 1$, $k > 1$ and $\gcd(D, k) = 1$, and let $\omega(k)$ be the number of distinct prime factors of k . Further, let $N(D, k)$ be the number of solutions (x, n) of the generalized Ramanujan-Nagell equation

$$(1) \quad x^2 - D = k^n, \quad x, n \in \mathbb{N}.$$

There are many works concerned with the upper bounds for $N(D, k)$, including the following:

- 1 (BEUKERS [1]). $N(D, 2) \leq 4$.
- 2 (LE [5]). If $D = 2^{2m} - 3 \cdot 2^{m+1} + 1$ for some $m \in \mathbb{N}$ with $m \geq 3$, then $N(D, 2) = 4$. Otherwise, $N(D, 2) \leq 3$.
- 3 (BEUKERS [2]). If k is an odd prime, then $N(D, k) \leq 4$.
- 4 (LE [4]). If k is an odd prime and $\max(D, k) \geq 10^{240}$, then $N(D, k) \leq 3$.

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The last result basically confirmed a conjecture posed by BEUKERS [2]. In this paper, we extend the result as follows.

Theorem. *If $\gcd(2D, k) = 1$, then $N(D, k) \leq 3 \cdot 2^{\omega(k)-1} + 1$. If moreover $\max(D, k) > 10^{60}$, then $N(D, k) \leq 3 \cdot 2^{\omega(k)-1}$.*

This last upper bound is best possible while k is an odd prime.

2. Preliminaries

In this section, we assume that $2 \nmid k$, $k \geq 15$ and D is nonsquare.

Lemma 1 ([7, Theorems 1 and 2]). *If the equation*

$$(2) \quad X^2 - DY^2 = kZ, \quad X, Y, Z \in \mathbb{Z}, \quad \gcd(X, Y) = 1, \quad Z > 0,$$

is solvable in integers X, Y, Z , then we have:

(i) *For a fixed solution (X, Y, Z) , there exists a unique $\ell \in \mathbb{N}$ such that*

$$(3) \quad \begin{aligned} X &\equiv \pm \ell Y \pmod{k}, & \ell^2 &\equiv D \pmod{k}, \\ \ell &< \frac{k}{2}, & \gcd\left(k, 2\ell, \frac{\ell^2 - D}{k}\right) &= 1. \end{aligned}$$

(ii) *All solutions of (2) can be put into $2^{\omega(k)-1}$ classes in such a way that each solution (X, Y, Z) in a class has the same value of ℓ in (3).*

(iii) *For a fixed class, say S , there exists a unique solution (X_1, Y_1, Z_1) in S which satisfies $X_1 > 0$, $Y_1 > 0$, $Z_1 \leq Z$ and*

$$(4) \quad 1 < \frac{X_1 + Y_1\sqrt{D}}{X_1 - Y_1\sqrt{D}} < \left(u_1 + v_1\sqrt{D}\right)^2,$$

where Z runs over all solutions (X, Y, Z) in S , $u_1 + v_1\sqrt{D}$ is the fundamental solution of the equation

$$(5) \quad u^2 - Dv^2 = 1, \quad u, v \in \mathbb{Z}.$$

The solution (X_1, Y_1, Z_1) is called the least solution of S .

(iv) *If (X_1, Y_1, Z_1) is the least solution of S , then every solution (X, Y, Z) in S can be expressed as*

$$\begin{aligned} Z &= Z_1 t, & X + Y\sqrt{D} &= \left(X_1 + \lambda Y_1\sqrt{D}\right)^t \left(u + v\sqrt{D}\right), \\ & & t &\in \mathbb{N}, \quad \lambda \in \{-1, 1\}, \end{aligned}$$

where (u, v) is a solution of (5).

Clearly, if (x, n) is a solution of (1), then $(X, Y, Z) = (x, 1, n)$ is a solution of (2).

Lemma 2. *Let (x, n) be a solution of (1). If $(x, 1, n)$ belongs to the class S and (X_1, Y_1, Z_1) is the least solution of S , then we have:*

$$(6) \quad n = Z_1 t, \quad x + \delta \sqrt{D} = \varepsilon^t \bar{\varrho}^s, \quad \delta \in \{-1, 1\},$$

where

$$(7) \quad \begin{aligned} \varepsilon &= X_1 + Y_1 \sqrt{D}, & \bar{\varepsilon} &= X_1 - Y_1 \sqrt{D}, \\ \varrho &= u_1 + v_1 \sqrt{D}, & \bar{\varrho} &= u_1 - v_1 \sqrt{D}, \end{aligned}$$

$s, t \in \mathbb{Z}$ satisfy $t > 0$, $t \geq s \geq 0$ and $\gcd(s, t) = 1$.

PROOF. By (iv) of Lemma 1, (6) holds for some $s, t \in \mathbb{Z}$ with $t > 0$. If $s < 0$, then

$$\varepsilon^t = X + Y \sqrt{D}, \quad \bar{\varrho}^s = u + v \sqrt{D}, \quad X, Y, u, v \in \mathbb{N},$$

and $\delta = Xv + Yu \geq 2$, a contradiction. So we have $s \geq 0$. Moreover, by [2, Lemma 3] we have $\gcd(s, t) = 1$.

If $\delta = 1$, then we have

$$1 < \frac{x + \sqrt{D}}{x - \sqrt{D}} = \left(\frac{\varepsilon}{\bar{\varepsilon}} \right)^t \bar{\varrho}^{2s} < \varrho^{2t-2s},$$

by (4). Hence $t > s$. If $\delta = -1$, then in view of $\varrho > 2\sqrt{D}$ we have

$$\frac{x}{\sqrt{D}} = \left(1 + \frac{k^n}{D} \right)^{1/2} > \begin{cases} \sqrt{2}, & \text{if } k^n > D \\ 1 + k^n/2D > 1 + 1/D, & \text{if } k^n < D \end{cases} > \frac{\varrho^2 + 1}{\varrho^2 - 1}.$$

Hence

$$\frac{x + \sqrt{D}}{x - \sqrt{D}} = \left(\frac{\varepsilon}{\bar{\varepsilon}} \right)^{-t} \varrho^{2s} < \varrho^2.$$

Together with (4) this implies $t + 1 > s$. So in both cases we obtain $t \geq s$. This completes the proof of Lemma 2. $\square$

Let $(x, n), (x', n')$ be two solutions of (1). If $(x, 1, n)$ and $(x', 1, n')$ are solutions of (2) which belong to the same class, then this will be denoted by $(x, n) \sim (x', n')$. The pair (D, k) will be called exceptional if

$$(8) \quad k = 4a^2 + \lambda, \quad D = \left(\frac{k^m - \lambda}{4a} \right)^2 - k^m$$

for some $a, m \in \mathbb{N}$, $\lambda \in \{-1, 1\}$ with $m > 1$ and the additional condition $2 \nmid m$ if $\lambda = -1$. If (D, k) satisfies (8), then (1) has three solutions (x, n) , (x', n') , (x'', n'') given by

$$(9) \quad \begin{aligned} (x, n) &= \left(\frac{k^m - \lambda}{4a} - 2a, 1 \right), & (x', n') &= \left(\frac{k^m - \lambda}{4a}, m \right), \\ (x'', n'') &= \left(2ak^m + \lambda \frac{k^m - \lambda}{4a}, 2m + 1 \right). \end{aligned}$$

The solutions in (9) satisfy $(x, n) \sim (x', n') \sim (x'', n'')$.

Lemma 3. *Let (x, n) , (x', n') , (x'', n'') be three solutions of (1) such that $n < n' < n''$ and $(x, n) \sim (x', n') \sim (x'', n'')$. If (D, k) satisfies (8), then we have either (9) or $n'' \geq 2n' + \max(3, n, 2n'/3 - 2/3)$. If (D, k) does not satisfy (8), then we have $n'' \geq 2n' + \max(3, n, 2n'/3 - 2/3)$.*

PROOF. Under the assumptions, $(x, 1, n)$, $(x', 1, n')$ and $(x'', 1, n'')$ are solutions of (2) satisfying

$$x^2 \equiv D \pmod{k^n}, \quad x'^2 \equiv D \pmod{k^{n'}}, \quad x''^2 \equiv D \pmod{k^{n''}},$$

and

$$\begin{aligned} x &\equiv \delta \ell \pmod{k}, & x' &\equiv \delta' \ell \pmod{k}, & x'' &\equiv \delta'' \ell \pmod{k}, \\ \delta, \delta', \delta'' &\in \{-1, 1\}, \end{aligned}$$

for the same $\ell \in \mathbb{N}$. So we have $x \equiv \pm x' \pmod{k^n}$ and $x' \equiv \pm x'' \pmod{k^{n'}}$. Recalling that $2 \nmid k$ and $k \geq 15$. By much the same argument as in the proof of [2, Lemma 5], we can prove the lemma without any difficulty. $\square$

Let $\alpha = (\log(\varepsilon/\bar{\varepsilon}))/\log \varrho^2$, and let $\Lambda(x, n) = \log \left((x + \sqrt{D})/(x - \sqrt{D}) \right)$ for any solution (x, n) of (1). Lemmas 4 and 5 stated below can be proved similarly as Lemmas 9 and 10 of [6], respectively.

Lemma 4. *If (x, n) is a solution of (1) satisfying $k^n \geq 3D$ and (6), then s/t is a convergent of α .*

Lemma 5. *Let (x, n) , (x', n') be two solutions of (1) such that $k^{n'} > k^n \geq 3D$, $(x, n) \sim (x', n')$ and $(x, 1, n)$ belongs to the class S . Further let $(X_1, Y_1, Z_1,)$ be the least solution of S . Then we have $n + n' > Z_1 \log \varrho^2 / \Lambda(x, n)$.*

Lemma 6. *Equation (1) has at most one solution (x, n) with $k^n < \sqrt{D}$.*

PROOF. This follows immediately from [3, Theorem 10.8.2]. $\square$

Lemma 7. *If (x, n) is a solution of (1) such that k^n is a square, then $k^n < D^2/4$.*

PROOF. Under the assumption, we have $x + k^{n/2} = D_1$ and $x - k^{n/2} = D_2$, where $D_1, D_2 \in \mathbb{N}$ with $D_1 D_2 = D$. It implies that $k^{n/2} = (D_1 - D_2)/2 \leq (D - 1)/2 < D/2$. The lemma is proved. $\square$

Lemma 8. *If (1) has a solution (x, n) such that k^n is a non-square and $k^n \geq 4^{1+s/r} D^{2+s/r}$ for some $r, s \in \mathbb{N}$, then we have*

$$\left| \frac{y}{2k^{n'/2}} - 1 \right| > \frac{8}{2187} \left(\frac{81}{4} \right)^{1/s} k^{n/s - n(3+\nu/2) - n'(1+\nu)/2}$$

for any $y, n' \in \mathbb{N}$ with $2 \nmid n'$, where

$$\nu = \frac{r}{s} + \frac{1}{\log k^n} \left(\log 9 + \frac{r}{s} \log \frac{81}{4} \right).$$

PROOF. This follows immediately from [8, Theorem I.2]. $\square$

3. Proof of Theorem

By [2] and [6], it suffices to prove the theorem while k is not a prime power. We may assume that $k \geq 15$. If D is a square, then $D = D_1^2$ and

$$\begin{aligned} x + D_1 &= k_1^n, & x - D_1 &= k_2^n, & k &= k_1 k_2, \\ D_1, k_1, k_2 &\in \mathbb{N}, & \gcd(k_1, k_2) &= 1, \end{aligned}$$

by (1). Since the number of such pairs (k_1, k_2) does not exceed $2^{\omega(k)-1}$ we have $N(D, k) \leq 2^{\omega(k)-1}$. Hence we may assume also that D is not a square.

Let (x, n) be a solution of (1). Then $(X, Y, Z) = (x, 1, n)$ is a solution of (2). By (ii) of Lemma 1, we may assume that $(x, 1, n)$ belongs to a certain class S . Let (X_1, Y_1, Z_1) be the least solution of S , and let $N(S)$ be the number of solutions (x, n) of (1) such that $(x, 1, n)$ belongs to S . We now suppose that $N(S) > 4$. Then (1) has five solutions (x_i, n_i) ($i = 1, \dots, 5$) such that $n_1 < \dots < n_5$ and $(x_1, n_1) \sim \dots \sim (x_5, n_5)$. If

the pair (D, k) is exceptional and the solutions (x_j, n_j) ($j = 1, 2, 3$) satisfy (9), then we have

$$(10) \quad \begin{aligned} \Lambda(x_3, n_3) &= \frac{\sqrt{D}}{x_3} = \sum_{i=0}^{\infty} \frac{1}{2i+1} \left(\frac{D}{x_3^2} \right)^i < 1.01 \frac{\sqrt{D}}{x_3} < \frac{1.01}{8a^2 - 1} \\ &= \frac{1.01}{2k - 1} < \frac{1}{28}, \end{aligned}$$

by (8). Notice that $k^{n_3} > 3D$ by (9). On using Lemma 5 with (10), we get

$$(11) \quad n_3 + n_4 > 28 \log \varrho^2 > 28 \log 4D.$$

On the other hand, since $k^{n_3} < (4D)^{1.8}$ by (9), we obtain from (11) that

$$(12) \quad k^{n_4} > (4D)^{55}.$$

By Lemma 7, we see from (12) that k is not a square and $2 \nmid n_4 n_5$. Let $n = n_4$, $n' = n_5$, $y = 2x_5$, $r = 1$ and $s = 53$. On applying Lemma 8 with (12), we get

$$(13) \quad \left| \frac{x_5}{k^{n_5/2}} - 1 \right| > \frac{8}{2187} \left(\frac{81}{4} \right)^{1/53} k^{n_4/53 - (3+\nu/2)n_4 - (1+\nu)n_5/2},$$

where

$$(14) \quad \begin{aligned} \nu &= \frac{1}{53} + \frac{1}{\log k^{n_4}} \left(\log 9 + \frac{1}{53} \log \frac{81}{4} \right) \\ &< \frac{1}{53} + \frac{1}{55 \log 4D} \left(\log 9 + \frac{1}{53} \log \frac{81}{4} \right) < 0.0364. \end{aligned}$$

Notice that

$$(15) \quad \left| \frac{x_5}{k^{n_5/2}} - 1 \right| = \frac{D}{k^{n_5/2} (x_5 + k^{n_5/2})} < \frac{D}{2k^{n_5}}.$$

The combination of (13), (14) and (15) yields

$$(16) \quad 138Dk^{2.9994n_4} > k^{0.4818n_5}.$$

Since $4D < k^{n_4/55}$ by (12), we get from (16) that

$$(17) \quad 6.316n_4 > n_5.$$

On the other hand, by Lemma 5 we see from (10) that

$$(18) \quad n_4 + n_5 > \frac{k^{n_4/2} \log 4D}{1.01\sqrt{D}} > k^{0.496n_4}.$$

The combination of (17) and (18) yields

$$(19) \quad 7.316n_4 > k^{0.496n_4}.$$

But (18) is impossible, since $k \geq 15$ and $n_4 > 4$. Therefore, we obtain $N(S) \leq 4$ if the pair (D, k) is exceptional and the solutions (x_j, n_j) ($j = 1, 2, 3$) satisfy (9).

We now prove the inequality (12) for the other cases. By Lemma 3, if (D, k) satisfies (8) and solutions (x_j, n_j) ($j = 1, 2, 3$) do not satisfy (9) or (D, k) does not satisfy (8), then $n_3 \geq 2n_2 + 3$. Further, by Lemma 6, we have $k^{n_2} > \sqrt{D}$. Hence $k^{n_3} \geq Dk^3$ and

$$(20) \quad \begin{aligned} \Lambda(x_3, n_3) &= \log \frac{x_3 + \sqrt{D}}{x_3 - \sqrt{D}} = \log \frac{\sqrt{k^{n_3} + D} + \sqrt{D}}{\sqrt{k^{n_3} + D} - \sqrt{D}} \\ &< \log \frac{k^{3/2} + 1}{k^{3/2} - 1} < \frac{1}{28.9855}. \end{aligned}$$

By Lemma 5, we see from (20) that

$$(21) \quad n_3 + n_4 > 28.9855Z_1 \log \varrho^2 > 28.9855 \log 4D.$$

This follows that

$$(22) \quad k^{n_3+n_4} > (4D)^{28.9855 \log k}.$$

On the other hand, by Lemma 3, we have $n_4 \geq 2n_3 + 2n_3/3 - 2/3$. Hence $n_3 \leq 3n_4/8 + 1/4$ and

$$(23) \quad k^{n_4+2/11} > (4D)^{57.0877}.$$

By (23), if $k^{2/11} \leq (4D)^{2.0877}$, then (12) holds. If $k^{2/11} > (4D)^{2.0877}$, then $k > (4D)^{11.4823}$ and $k^{n_4} > (4D)^{57.4115}$. Therefore, (12) holds in any case. Thus, using the same method, we can prove that

$$(24) \quad N(S) \leq \begin{cases} 4, & \text{if (1) has a solution } (x, n) \text{ such that} \\ & k^n < \sqrt{D} \text{ and } (x, 1, n) \text{ belongs to } S, \\ 3, & \text{otherwise.} \end{cases}$$

Notice that (1) has at most one solution (x, n) with $k^n < \sqrt{D}$ by Lemma 6. Thus, by (ii) of Lemma 1, we get from (24) that $N(D, k) \leq 3 \cdot 2^{\omega(k)-1} + 1$.

On the other hand, by much the same argument as in the proof of [6, Theorem], we can prove that $N(S) \leq 3$ if $\max(D, k) > 10^{60}$. Then we have $N(D, k) \leq 3 \cdot 2^{\omega(k)-1}$. The proof is complete.

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