

Equivariant embeddings of smooth G -pseudomanifolds

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Abstract. Given a compact Lie group G acting smoothly (C^∞) on a compact manifold M , then by a well known result of Mostow, there is an equivariant smooth embedding of M into some Euclidean space with an orthogonal G -action.

In this work we extend Mostow's result to smooth G -pseudomanifolds (in the sense of GORESKEY and MAC PHERSON).

§0. Introduction

Given a compact Lie group G acting smoothly on a compact manifold M , then a well known result of MOSTOW [4] states the following.

Theorem. *There is a Euclidean space $\mathbb{R}^n$ with an orthogonal G -action, together with an equivariant smooth embedding $\theta : M \rightarrow \mathbb{R}^n$.*

The objective of this work is to extend Mostow's equivariant embedding theorem to smooth G -pseudomanifolds.

A smooth G -pseudomanifold is a certain G -space which is a topological pseudomanifold with the orbit type filtration [2,3], and whose strata are smooth manifolds.

In §1 we define smooth G -pseudomanifolds and give some examples.

In §2 we prove a theorem on equivariant embeddings into Euclidean space, for smooth G -pseudomanifolds.

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§1. Smooth G -pseudomanifolds

First we recall some definitions given in [5], which will be used in this paper.

Let G be a compact Lie group. By a G -space we mean a completely regular space X , together with a continuous action $G \times X \rightarrow X$.

Let $G_x = \{g \in G : g \cdot x = x\}$ be the isotropy subgroup of G at $x \in X$. Denote by X/G the corresponding orbit space with the quotient topology induced by the canonical projection $\pi : X \rightarrow X/G$.

Also let $cX = X \times [0, 1)/(x, 0) \sim (x', 0)$ be the open cone of X . Then there is a canonical action on cX , given as follows $g \cdot [x, r] = [g \cdot x, r]$ where $g \in G$, $x \in X$, $r \in [0, 1)$. Here $[x, r]$ denotes the equivalence class of (x, r) in cX . Notice that the distinguished point $* = [x, 0]$ of cX is G -fixed.

Now let X be a G -space.

Definition 1.1. Given an orbit P in X , then a slice S_x in X at $x \in P$ is called a *conical slice* of P , if it satisfies the following condition.

There is a compact H -space L (possibly empty), without fixed points, called a *link* of P , where $H = G_x$, together with an H -equivariant homeomorphism $\phi : S_x \rightarrow \mathbb{R}^{i_0} \times cL$, for an integer $i_0 \geq 0$, where H acts trivially on $\mathbb{R}^{i_0}$.

For the definition of slices, and their existence in any G -space, see [1] pp. 82–86.

We now define G -pseudomanifolds.

Definition 1.2. A (-1) -dimensional G -pseudomanifold is the empty set.

A n -dimensional G -pseudomanifold ($n \geq 0$) is a G -space X which satisfies the following conditions.

(C1) Each orbit P in X has a conical slice $S_x \simeq^\phi \mathbb{R}^{i_0} \times cL$, such that L is an $n - i - 1$ -dimensional H -pseudomanifold, where $i = i_0 + \dim G/H \neq n - 1$.

(C2) There is a family of orbits with an empty link in X , containing an orbit over each connected component of X/G , such that all the orbits in this family have the same type.

For the definition of orbit type, see [1] p. 42.

If X is a G -pseudomanifold, define on X the following equivalence relation $x \sim y \iff G_x \sim G_y$ i.e. the corresponding isotropy subgroups are

conjugate. The equivalence classes of this relation are denoted by $X_{(H)}$, which is the union of all orbits of type G/H in X .

Given an orbit P of type G/H in X and S_x a conical slice at $x \in P$, let $\Phi^{-1} : G \times_H S_x \rightarrow X$ be the G -equivariant embedding onto an open set in X , given by $[g, s] \mapsto g \cdot s$ for $g \in G, s \in S_x$. The image Γ of Φ^{-1} is called [1, p. 82] a *tubular neighborhood* of P in X . Clearly we have

$$\Gamma \cap X_{(H)} \simeq^{\Phi} \{G \times_H S_x\}_{(H)} \simeq^{[1, \phi]} \{G \times_H (\mathbb{R}^{i_0} \times cL)\}_{(H)} \simeq G/H \times \mathbb{R}^{i_0},$$

since H acts on L without fixed points. Therefore the connected components of the sets $X_{(H)}$, called *strata* of X , are topological manifolds embedded in X , with dimension $i = i_0 + \dim G/H \leq n$, ($i \neq n-1$).

Consider the following canonical filtration of the G -pseudomanifold X , called *the orbit type filtration*,

$$X = X_n \supset X_{n-1} = X_{n-2} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset,$$

where each X_i is the union of the strata of X with dimension less than, or equal to i . Note that $x \in X - X_{n-2}$ if and only if $G(x)$ has an empty link. It is shown in [5] that the connected components of each $X_i - X_{i-1}$ coincide with the i -dimensional strata of X , for $i = 0, \dots, n$.

Now let P be an orbit in a stratum $X_i - X_{i-1}$, with a given conical slice $S_x \simeq^{\phi} \mathbb{R}^{i_0} \times cL$, where $G_x = H$.

Let $\sigma : W \rightarrow G$ be a local section of $\pi_0 : G \rightarrow G/H$, with W a chart of G/H , $eH \in W$ and $\sigma(eH) = e$.

Then $N = \pi_0^{-1}(W) \cdot S_x$ is called a *distinguished neighborhood* of x .

It is shown in [5] that there is a stratum-preseving homeomorphism $\varphi : N \rightarrow \mathbb{R}^i \times cL$, where N has the relative filtration, and $\mathbb{R}^i \times cL$ the canonical filtration induced by the H -orbit type filtration of L , given explicitly as follows $\varphi(g \cdot z) = (gH, p_1\phi(z), \sigma(gH)^{-1}g \cdot p_2\phi(z))$, for $g \in \pi_0^{-1}(W)$ and $z \in S_x$. Clearly $\varphi|_{S_x} = \phi$.

Also $g \cdot N$ is a *distinguished neighborhood* of $g \cdot x$ for $g \in G$, with a similar homeomorphism.

It is proved in [5] that a G -pseudomanifold X , with the orbit type filtration, is a topological pseudomanifold as defined by Goresky and Mac Pherson [2,3].

Now let X, Y be two non-empty filtered spaces, whose strata are smooth (C^∞) manifolds.

Definition 1.3. Let $\text{len } X = \sup\{p - q : X_p - X_{p-1} \neq \emptyset \neq X_q - X_{q-1}\}$ be the *length* of X . Put $\text{len } \emptyset = -1$.

Definition 1.4. A stratum-preserving map $f : X \rightarrow Y$ is said to be *smooth*, or a *smooth embedding*, if f is smooth, or a smooth embedding, on the strata of X .

Definition 1.5. A stratum-preserving homeomorphism $h : X \rightarrow Y$ is said to be a *diffeomorphism*, if h and h^{-1} are smooth.

We now define the concept of a smooth G -pseudomanifold.

Definition 1.6. Let X be a G -pseudomanifold.

For $\text{len } X = -1$, $X = \emptyset$ is a *smooth G -pseudomanifold*.

For $\text{len } X \geq 0$, X is a *smooth G -pseudomanifold*, if it satisfies the following conditions.

(C1) Each $X_i - X_{i-1}$ is a smooth manifold, with a smooth restricted G -action.

(C2) Each orbit P in any $X_i - X_{i-1}$ has a conical slice $S_x \simeq^\phi \mathfrak{R}^{i_0} \times cL$, with L a smooth H -pseudomanifold, such that $\varphi : N \rightarrow \mathfrak{R}^i \times cL$ is a diffeomorphism for some distinguished neighborhood $N = \pi_0^{-1}(W) \cdot S_x$.

Since $\phi : S_x \rightarrow \mathfrak{R}^{i_0} \times cL$ is an H -equivariant homeomorphism, then S_x is a smooth H -pseudomanifold (see 2.7 in [5]), which is smoothly embedded into the tubular neighborhood $\Gamma \simeq G \times_H S_x$ of the orbit P .

We now give some examples.

(1) Let X be a torus in $\mathfrak{R}^3$ which is pinched at the points $q_{\pm 1} = (0, \pm 1, 0)$ and intersects the xz plane in $L = S_1^1 \cup S_2^1$, the union of two circles centered at the points $(0, \pm 2)$.

There is a canonical Z_2 -action on X given by reflection through the xy plane. Clearly T is a smooth Z_2 -pseudomanifold embedded in $\mathfrak{R}^3$.

Notice in particular the embedding in $\mathfrak{R}^2$ of the smooth Z_2 -pseudomanifold L , which is a link of the orbits $q_{\pm 1}$, and the corresponding embeddings of the smooth Z_2 -pseudomanifolds $c_{q_{\pm 1}} L$ in $\mathfrak{R}^3$.

(2) Let G be a compact Lie group acting smoothly on a paracompact manifold M . Assume that the orbit type filtration of M has no strata of codimension one, and M/G is connected. Claim that M is a smooth G -pseudomanifold.

The proof is by induction on $\text{len } M$. For $\text{len } M = 0$ it is trivial. Assume that the claim is true for smooth G -manifolds of length strictly smaller than $\text{len } M$.

Given an orbit P in M , choose a point $x \in P$ with $G_x = H$. Consider a linear slice [1, p. 171], $S_x \simeq E$ at x in M , where H acts orthogonally on an Euclidean space E . Let $V = (E^H)^\perp$ denote the orthogonal complement

of E^H with respect to an H -invariant inner product on E . Put L to be the unit sphere of V with respect to the associated H -invariant metric.

Since $L \simeq S^l$ canonically, where $l + 1 = \dim V$, we can put a smooth structure on L . Clearly $H \times L \rightarrow L$ is smooth, because H acts orthogonally on V . Since $\text{len } L < \text{len } M$, we have by the inductive hypothesis that L , with the H -orbit type filtration, is a smooth H -pseudomanifold.

Now by [1, p. 308] we have a diffeomorphism $\Gamma \simeq G \times_H E$ between smooth G -manifolds, where Γ is the tubular neighborhood of P corresponding to S_x .

Hence for a chart W of G/H at eH , we have a diffeomorphism of filtered spaces

$$\pi_0^{-1}(W) \cdot S_x \simeq W \times \{E^H \oplus (E^H)^\perp\} \simeq \mathfrak{R}^i \times cL,$$

for $i = \dim G/H + \dim E^H$, since $V - \{0\} \simeq L \times (0, 1)$ is a diffeomorphism between smooth H -manifolds, and the strata of M are smooth submanifolds [1, p. 309]. Therefore M is a smooth G -pseudomanifold.

§2. The embedding theorem

In this section we extend to smooth G -pseudomanifolds the equivariant embedding theorem of Mostow.

First we prove the following.

Lemma 2.1. *Let X be a smooth G -pseudomanifold and $\Gamma \simeq G \times_H S_x$ a tubular neighborhood of the orbit $G(x)$, corresponding to a conical slice S_x given in Definition 1.6 (C2). Then Γ with the restricted G -action is also a smooth G -pseudomanifold.*

PROOF. Let $S_x \simeq^\phi \mathfrak{R}^{i_0} \times cL$ be a conical slice at x in X , for a smooth H -pseudomanifold L . Then given $y = \phi^{-1}(p_1 \phi(y), [l, r]) \in S_x$, $r > 0$ and S_l a conical K -slice at l in L , where $K = G_y = H_y = H_l$, we have that

$$S_y \simeq^\phi \mathfrak{R}^{i_0} \times S_l \times (0, 1) \simeq^{1 \times \phi'} \mathfrak{R}^{i_0 + k_0 + 1} \times cQ,$$

is a conical K -slice at y in the H -space S_x , for some $k_0 \geq 0$. Here Q is a link of $H(l)$, which is a $n - i - k - 2$ -dimensional smooth K -pseudomanifold, where $k = k_0 + \dim H/K$. Therefore [1, p. 84], S_y is a conical K -slice at y in X .

$$\text{Let } \pi_0 : G \rightarrow G/H, \quad \pi_1 : H \rightarrow H/K, \quad \text{and} \quad \pi_2 : G \rightarrow G/K.$$

Consider local sections $\sigma_0 : W_0 \rightarrow G$ of π_0 , and $\sigma_1 : W_1 \rightarrow H$ of π_1 , where $W_0 = \{gH : g \in G^0\}$ and $W_1 = \{\sigma_0(gH)^{-1}gK : g \in G^0\}$, for some open neighborhood $G^0 \subset G$ of e , with $\sigma_0(eH) = \sigma_1(eK) = e$.

Then we have a local section $\sigma_2 : W_2 \rightarrow G$ of π_2 , given by

$$\sigma_2(gK) = \sigma_0(gH) \cdot \sigma_1(\sigma_0(gH)^{-1}gK), \quad \text{where } W_2 = \{gK : g \in G^0\}.$$

Since $G^0 = \sigma_0(W_0) \cdot \sigma_1(W_1) \cdot K$, we obtain the following diffeomorphism

$$W_2 \simeq \sigma_2(W_2) = \sigma_0(W_0) \cdot \sigma_1(W_1) \simeq \sigma_0(W_0) \times \sigma_1(W_1) \simeq W_0 \times W_1.$$

Now by the explicit formulation of φ (see p. 3), the following composition is a diffeomorphism

$$\begin{aligned} \sigma_2(W_2) \cdot S_y &\simeq^{\varphi} \sigma_0(W_0) \cdot \mathbb{R}^{i_0} \times (0, 1) \times \sigma_1(W_1) \cdot S_l \simeq^{1 \times \varphi'} \\ W_0 \times W_1 \times \mathbb{R}^{i_0+k_0+1} \times cQ &\simeq \sigma_2(W_2) \cdot \mathbb{R}^{i_0+k_0+1} \times cQ \simeq \mathbb{R}^{i+k+1} \times cQ. \end{aligned}$$

□

We now obtain the following result.

Theorem 2.2. *Let X be a compact, smooth G -pseudomanifold. Then there is a Euclidean space $\mathbb{R}^n$ with an orthogonal G -action, together with an equivariant smooth embedding $\theta : X \rightarrow \mathbb{R}^n$.*

PROOF. By induction on the length of X . For $\text{len } X = 0$ the statement follows from the smooth embedding theorem of Mostow. Assume that the statement is valid for G -pseudomanifolds of length strictly smaller than $\text{len } X$.

Let P be an orbit in X , with a conical slice S_x in X at $x \in P$ given by Definition 1.6 (C2). Hence, there is an H -equivariant homeomorphism $\phi : S_x \rightarrow D(\mathbb{R}^{i_0}, 1) \times cL$, with $G_x = H$, where L is a compact smooth H -pseudomanifold, and H acts trivially on $D(\mathbb{R}^{i_0}, 1)$ the open unit disk of $\mathbb{R}^{i_0}$. Assume that $L \neq \emptyset$.

Now if Γ is the tubular neighborhood of P corresponding to the slice S_x , then Γ is a smooth G -pseudomanifold by Lemma 2.1. Hence, since the map $\Phi : \Gamma \rightarrow G \times_H S_x$ is a G -equivariant homeomorphism, we can put on $G \times_H S_x$ a smooth G -pseudomanifold structure, such that Φ is a diffeomorphism.

Since $\text{len } L < \text{len } X$, by the inductive hypothesis there is a Euclidean space $\mathbb{R}^m$ with an orthogonal H -action, together with an equivariant smooth embedding $\theta_1 : L \rightarrow \mathbb{R}^m$.

Therefore there is an H -equivariant smooth embedding $\theta_2 : S_x \rightarrow V$, where $\theta_2 = (1 \times c\theta_1)\phi$ and $V = \mathbb{R}^{i_0} \oplus \mathbb{R} \oplus \mathbb{R}^m$, which has an orthogonal

H -action given by the sum of these representations, with H acting trivially on $\mathfrak{R}^{i_0}$ and $\mathfrak{R}$. Here $c\theta_1 : cL \rightarrow c\mathfrak{R}^m \subset \mathfrak{R} \oplus \mathfrak{R}^m$.

Let $D(V, r) = \{v \in V : \|v\| < r\}$, and $c(L, r) = L \times [0, r)/(l, 0) \sim (l', 0)$ for $0 < r \leq 1$. Using a suitable homothety, we may assume that there is an H -equivariant smooth embedding of S_x into $D(V, \sqrt{3})$.

By symmetry there is an H -equivariant smooth embedding of $S_x(r) = \phi^{-1}\{D(\mathfrak{R}^{i_0}, r) \times c(L, r)\}$ into $D(V, r\sqrt{3})$. It can easily be shown using an equivariant retraction, that $S_x(r)$ is a conical slice in X at $x \in P$.

Hence the following composition is a G -equivariant smooth embedding

$$\Gamma \xrightarrow{\Phi} G \times_H S_x \xrightarrow{[1 \times \theta_2]} G \times_H D(V, \sqrt{3}) \xrightarrow{[1 \times i]} G \times_H V.$$

For $L = \emptyset$ the above is trivially satisfied.

To conclude the proof, we shall give an equivariant smooth embedding of $G \times_H V$ into Euclidean space, as in [1] p. 315.

By [1] p. 24, there exists an orthogonal representation of G on some Euclidean space V_0 and a point $v_0 \in V_0$ with $G_{v_0} = H$. Now by [1] p. 18, the orthogonal representation of H on the Euclidean space V given above, may be extended to an orthogonal representation of G on some Euclidean space $V' \supset V$, which extends the H -action on V . Then, G acts orthogonally on $W = V_0 \oplus V'$ via the sum of these two representations (i.e. diagonally).

Consider the map $\alpha : G \times_H V \rightarrow V_0 \oplus V' = W$, defined by $\alpha[g, v] = g(v_0 + v)$. If $\alpha[g, v] = \alpha[g', v']$ then $g(v_0 + v) = g'(v_0 + v')$, so that $g^{-1}g'(v_0) = v_0$ and $g^{-1}g'(v') = v$. Thus $h = g^{-1}g' \in H$ and $h(v') = v$. Therefore $[g, v] = [gh, h^{-1}v] = [g', v']$ and hence α is injective. Since $G \times_H V$ has the differentiable structure induced from that of $G \times V$ and since the action map $G \times V \rightarrow W$ is smooth, it follows that α is smooth.

Now the isotropy group at $[e, v]$ is H_v , and this is also the isotropy group at $v_0 + v \in W$. Thus α takes the orbit of $[e, v]$ diffeomorphically onto $G(v_0 + v)$. The differential of α is thus injective on the tangent space to the orbit at $[e, v]$. However, the normal space to the orbit of $[e, v]$ is V and α maps this one-one affinely into W . Hence α_* is injective on the whole tangent space to $G \times_H V$ at $[e, v]$. By equivariance, α_* is everywhere injective, so that α is an injective immersion. Since α is obviously proper, it is a smooth embedding.

(It is only important that α be an embedding near the 0-section G/H .)

Hence the following map β is a G -equivariant smooth embedding

$$\Gamma \xrightarrow{\Phi} G \times_H S_x \xrightarrow{[1 \times \theta_2]} G \times_H D(V, \sqrt{3}) \xrightarrow{[1 \times i]} G \times_H V \xrightarrow{\alpha} W.$$

Now given $1 > s > t > 0$, let $f : \mathfrak{R} \rightarrow \mathfrak{R}$ be a smooth function such that

$$\begin{cases} f(r) = 1 & \text{for } r \leq t \\ f(r) \neq 0 & \text{for } r < s \\ f(r) = 0 & \text{for } r \geq s. \end{cases}$$

If $\rho : \Gamma \rightarrow [0, 1]$ is the smooth invariant function obtained from the ratio of cL , we can define a smooth equivariant map $\psi : \Gamma \rightarrow W$ by $y \mapsto f(\rho(y)) \cdot \beta(y)$ for $y \in \Gamma$. Since $\rho^{-1}([0, s])$ is closed [1, p. 34], ψ extends to a smooth equivariant map on X .

Also the smooth invariant function $\gamma : \Gamma \rightarrow \mathfrak{R}$, given by $y \mapsto f(\rho(y)) \cdot s/t$ for $y \in \Gamma$, extends to a smooth invariant function on X .

Thus for each orbit P in X we have an orthogonal representation of G on an Euclidean space W_x , and a smooth equivariant map $\psi_x : X \rightarrow W_x$, which is a smooth embedding on the tubular neighborhood Γ_x of P corresponding to the conical slice $S_x(t)$ at $x \in P$.

Additionally, we have a smooth invariant function $\gamma_x : X \rightarrow \mathfrak{R}$, which is nonzero exactly on Γ_x .

Since X is compact, it can be covered by finitely many tubular neighborhoods $\Gamma_{x_1}, \dots, \Gamma_{x_k}$. Let $\theta : X \rightarrow W_{x_1} \oplus \dots \oplus W_{x_k} \oplus \mathfrak{R}^k \simeq \mathfrak{R}^n$ be given as follows

$$\theta(x) = (\psi_{x_1}(x), \dots, \psi_{x_k}(x), \gamma_{x_1}(x), \dots, \gamma_{x_k}(x)) \quad \text{for } x \in X,$$

this map is clearly smooth and equivariant.

If $x, y \in \bigcup \Gamma_{x_p}$ and $\theta(x) = \theta(y)$, then we have that for some $p = 1, \dots, k$, $\gamma_{x_p}(x) = \gamma_{x_p}(y) \neq 0$, which implies that $x, y \in \Gamma_{x_p}$ and hence that $x = y$, since ψ_{x_p} is injective on Γ_{x_p} . Because ψ_{x_p} is a smooth embedding on Γ_{x_p} for all p , it follows that θ is a smooth embedding. $\square$

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