

Investigation of relative increments of distributions functions

By Z. I. SZABO (Debrecen)

Summary. The monotone properties of hazard rates and relative increments of probability distribution functions are investigated. The main results are formulated in Theorems 1 and 2 containing sufficient conditions under which the corresponding relative increment function and hazard rate increase (or decrease, or have two monotone phases). Our method enables us to avoid the inconvenient term $1 - F$ appearing in the expressions of the relative increment function, the hazard rate and our auxiliary function in Lemma 1. Instead, it deals with the relatively simple expression f/f' . The results have been applied to logistic, extreme value, Fisher's z -, Pareto of the third kind, Weibull, trigonometric and many other distributions.

Introduction

We shall use standard mathematical notations such as

$\mathbb{R}$ means the set of all real numbers;
iff means “if and only if”.

The relative increment function was introduced and used first by PORTER and DUDMAN (1960) [they called it the relative increment of decay or RID index], and was further used and investigated by ADLER and SZABO (1972, 1974, 1979a, 1979b, 1984) and SZABO (1976, 1989).

In this paper we investigate the *hazard rate* and *relative increment functions* of some (cumulative) distribution functions.

Let f be a (probability) density function. The corresponding distribution function is defined as usual:

$$F(x) = \int_{-\infty}^x f(t)dt.$$

Key words and phrases: distribution function, density function, hazard rate, relative increment function, (strictly) increasing/decreasing function.

By the *relative increment function* [briefly, RIF] of F we mean the fraction

$$h(x) = [F(x + a) - F(x)]/[1 - F(x)],$$

where a is a positive constant, and $F(x) < 1$ for all x .

Monotone properties of RIFs are important from the points of view of

- (a) statistics, probability theory;
- (b) applied statistics, e.g., in

(b.1) modelling bounded growth processes in biology, medicine and dental science [see ADLER and SZABO (1972, 1974, 1979a, 1979b, 1984), PORTER and DUDMAN (1960), and SZABO (1989)]

and in

(b.2) reliability and actuarial theories, where the probability that an individual, having survived to time x , will survive to time $x + a$ is $h(x)$; “death rate per unit time” in the time interval $[x, x + a]$ is $h(x)/a$, and the *hazard rate (failure rate or force of mortality)* is defined to be

$$\lim_{a \rightarrow 0} h(x)/a = f(x)/[1 - F(x)].$$

[See Sec. 7, Chap. 33. in Vol. 2 of JOHNSON and KOTZ (1970), or §5.34 and §5.38 of STUART and ORD (1987).]

In Sec. 7.2, Chap. 33 of JOHNSON and KOTZ (1970), some distributions are classified by their increasing/decreasing hazard rates.

We will need the following

Lemma 1. *Let F be a twice differentiable distribution function with $F(x) < 1$, $f(x) > 0$ for all x . We define the auxiliary function Ψ as follows:*

$$\Psi(x) := [F(x) - 1] \cdot f'(x)/f^2(x).$$

If $\Psi < (>)1$, then the function h , the RIF of F strictly increases (strictly decreases).

PROOF. Suppose $\Psi < 1$. Let $G(x) := 1 - F(x)$. Since $G > 0$, the RIF h can be written in the form $h(x) = 1 - G(x + a)/G(x)$. The function h strictly increases, iff

$$\begin{aligned} G(x + a)/G(x) &> G(x + 2a)/G(x + a), \quad \text{i.e., iff} \\ \ln G(x + a) &> [\ln G(x) + \ln G(x + 2a)]/2, \quad \text{i.e., iff} \end{aligned}$$

G is strictly log concave.

From the condition $\Psi(x) < 1$, we obtain

$$G(x) \cdot G''(x) < G'^2(x), \quad \text{i.e., } [\ln G(x)]'' < 0.$$

Thus G is strictly log concave. When $\Psi > 1$, the proof is the same. □

In a very similar way, one can prove the

Lemma 2. *If the conditions of Lemma 1 are fulfilled, then the following implications hold:*

the RIF h increases iff $\Psi \leq 1$;

the RIF h decreases iff $\Psi \geq 1$.

Remark 0.1. There is an immediate connection with the theory of reliability. By the Mathematical Preliminaries of BARLOW and PROSCHAN (1967) Sec. 1. p. 549, a distribution function F has IFR (increasing failure rate) iff $\ln[1 - F(x)]$ is concave down i.e., iff $\Psi(x) \leq 1$. Similarly, F has DFR (decreasing failure rate) iff $\ln[1 - F(x)]$ is concave up, i.e., $\Psi(x) \geq 1$.

Remark 0.2. Through the entire paper we investigate the auxiliary function Ψ . In order to get rid of the inconvenient term $(F - 1)$ in Ψ , we reduce all problems to simple formulae containing the fraction f/f' only.

The main results

Theorem 1. *Let f be a probability density function and F be the corresponding distribution function with the following properties.*

- (1) $\left\{ \begin{array}{l} I = (r, s) \subseteq \mathbb{R} \text{ is the possible largest finite or infinite open} \\ \text{interval in which } f > 0 \text{ (i.e., } I \text{ is the open support of } f); \\ r \text{ and } s \text{ may belong to the extended real line} \\ \mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}; \end{array} \right.$
- (2) *there exists an $m \in I$ at which f' is continuous and $f'(m) = 0$;*
- (3) *$f' > 0$ in (r, m) , and $f' < 0$ in (m, s)*
- (4) *f is twice differentiable in (m, s)*
- (5) *$(f/f')' = d/dx[f(x)/f'(x)] > 0$ in (m, s) .*

Then the corresponding continuous RIF h is either strictly increasing in I , or strictly increasing in (r, y) and strictly decreasing in (y, s) for some $y \in I$.

Moreover, if $\Psi(s^-) = \lim_{x \rightarrow s^-} \Psi(x) \in \mathbb{R}^$ exists, then*

- (a) *h strictly increases in I , if $\Psi(s^-) \leq 1$;*
- (b) *h strictly increases in (r, y) and strictly decreases in (y, s) for some y in I , if $\Psi(s^-) > 1$.*

PROF. It is sufficient to show that $\Psi < 1$ in (r, m) and, when Ψ reaches the value one (say, at x_0), then it strictly increases in (x_0, s) .

It follows from (1) that

$$(6) \quad 0 < F < 1 \quad \text{in } I.$$

By virtue of (1), (3) and (6), we have $\Psi = (F - 1) \cdot f'/f^2 < 0$ in (r, m) . Thus, by Lemma 1, the RIF h strictly increases in (r, m) . [If $m = s$, then h strictly increases in the entire interval I .]

By (2) and (4), f' is continuous in $[m, s)$, and so is Ψ . Thus, $\Psi < 1$ in $(r, m + p)$ for some $p > 0$.

We have two cases.

Case 1: $\Psi < 1$ in I . Then h strictly increases in I . In this case, if $\Psi(s^-)$ exists, it will not exceed unity: $\Psi(s^-) \leq 1$.

Case 2: $\Psi(x_0) \geq 1$ for some $x_0 \in (m, s)$. Then

$$(7) \quad [F(x_0) - 1] \cdot f'(x_0) \geq f^2(x_0).$$

The conditions (1) and (4) allow us to form the derivative

$$\Psi' = [f^2 f' + (F - 1) \cdot (f f'' - 2 f'^2)]/f^3$$

in (m, s) . We shall prove that $\Psi'(x_0) > 0$. By contraposition, we assume that $\Psi'(x_0) \leq 0$, i.e.,

$$(8) \quad [f^2 \cdot f' + (2f'^2 - f \cdot f'') \cdot (1 - F)]|_{x=x_0} \leq 0.$$

We replace $f^2(x_0)$ in the first term by $[F(x_0) - 1] \cdot f'(x_0)$. According to the relations (7) and $f'(x_0) < 0$, the left-hand side of (8) will decrease, and we get

$$(1 - F) \cdot (f'^2 - f \cdot f'')|_{x=x_0} \leq 0.$$

By virtue of (3) and (6), this is equivalent to

$$\begin{aligned} f'^2(x_0) &\leq f(x_0) \cdot f''(x_0), \quad \text{i.e.,} \\ 1 - f(x_0) \cdot f''(x_0)/f'^2(x_0) &= (f/f')'|_{x=x_0} \leq 0, \end{aligned}$$

which contradicts (5). Hence, $\Psi'(x_0) > 0$, and Ψ , which is continuous in (m, s) , strictly increases in some neighborhood of x_0 . It follows that, *once Ψ reaches the value 1 [say, at $y \in (m, s)$], it will strictly increase in (y, s)* . Thus, the corresponding RIF h strictly decreases in (y, s) . In this case, we have

$$1 < \Psi(s^-) \in \mathbb{R} \cup \{\infty\}. \quad \square$$

(Roughly speaking, the “main idea” of the proof is that Ψ *strictly increases at x if $\Psi(x) \geq 1$*).

Remark 1.1. If s is finite, one can consider the special case, when $m = s$, i.e. $(m, s) = \emptyset$. Then $\Psi < 0$ and h strictly increases in $(r, m) = I$ and, of course, no inequality (5) is required.

Remark 1.2. If r is finite, we can consider the case when $m = r$, i.e. $(r, m) = \emptyset$. Then it is enough to check the value of

$$\Psi(r^+) = \lim_{x \rightarrow r^+} \Psi(x) \quad \text{and} \quad \Psi(s^-),$$

provided the conditions (1), (3–5) are fulfilled.

If $\Psi(s^-) > 1$, then

- (a) if $\Psi(r^+) > 1$ or $[\Psi(r^+) = 1$ and $\Psi > 1$ in some right neighborhood of $r]$, then $\Psi > 1$ in the entire interval I since the “main idea” of the proof of Thm. 1 applies. Thus, by our Lemma 1, the RIF h strictly decreases in I ;
- (b) if $\Psi(r^+) < 1$ or $[\Psi(r^+) = 1$ and $\Psi < 1$ in some right neighborhood of $r]$, then there exists y in I such that $\Psi < 1$ in (r, y) and $\Psi > 1$ in (y, s) , since the “main idea” of the proof applies. So, the Lemma 1 gives that the RIF h strictly increases in (r, y) and strictly decreases in (y, s) .

If $\Psi(s^-) < 1$ or $[\Psi(s^-) = 1$ and $\Psi < 1$ in some left neighborhood of $s]$, then $\Psi < 1$ in I , and h strictly increases in I .

The event $[\Psi(s^-) = 1$ and $\Psi \geq 1$ in some left neighborhood of $s]$ is impossible because of the “main idea”.

Remark 1.3. If $f_\infty := \lim_{x \rightarrow \infty} x \cdot f(x) = 0$ then, by L'Hospital's rule, we have $\lim_{x \rightarrow \infty} \{[F(x) - 1]/[x \cdot f(x)]\} = \lim_{x \rightarrow \infty} [1 + x \cdot f'(x)/f(x)]^{-1}$.

Remark 1.4. If $\lim_{x \rightarrow s^-} f^2(x)/f'(x) = 0$, then L'Hospital's rule gives

$$\begin{aligned} \Psi(s^-) &= \lim_{x \rightarrow s^-} [F(x) - 1]/[f^2(x)/f'(x)] \\ &= \lim_{x \rightarrow s^-} f'^2(x)/[2f'^2(x) - f(x) \cdot f''(x)] = \lim_{x \rightarrow s^-} [1 + (f(x)/f'(x))']^{-1}. \end{aligned}$$

Remark 1.5. Since $(f/f')' = -(\ln f)''/[(\ln f)']^2$, then condition (5) can be formulated as follows:

$$(5') \quad (\ln f)'' < 0, \quad x \in (m, s).$$

We define the functions f and g be $\overset{0}{\sim}$ -equivalent (we write $f \overset{0}{\sim} g$), if $(\ln f(x))'' = (\ln g(x))''$. E.g., if f has the form

$$f(x) = c \cdot \exp(Ax + B) \cdot g(x),$$

then $(\ln f)' = (\ln c + Ax + B + \ln g)' = A + (\ln g)'$, and $f \overset{0}{\sim} g$. We denote $(\ln f)''$ by ℓ'' .

Remark 1.6. If the conditions $\lim_{x \rightarrow s^-} f^2(x)/f'(x) = 0$ and (5) [or (5')] are fulfilled for some density f , then

$$0 < [1 + (f(x)/f'(x))']^{-1} < 1$$

and, by the Remark 1.4, we get

$$\Psi(s^-) \in [0, 1].$$

Remark 1.7. The assertions of Theorem 1 do not remain true, if we replace the condition (5) by the weaker one

$$(f/f')' \geq 0, \quad x \in (m, s)$$

because, for the exponential distribution $f(x) = \lambda \cdot \exp(-\lambda \cdot (x-a))$, $x > a$, we have $f/f' = -\lambda^{-1}$ and $\Psi \equiv 1$. By Lemma 2, the RIF h increases and decreases at the same time, thus $h = \text{const.}$ [In other words, by the Remark 1.5, $f \stackrel{u}{\sim} g \equiv 1$, so $(\ln g)'' \equiv 0$ for each x .]

In addition, this feature of the exponential distribution is characteristic. [If $h(x) \equiv \text{const.}$, then $\Psi(x) \equiv 1$, i.e., $(G'^2 - G \cdot G'')/G'^2 = 0$, where $G := 1 - F(x)$. Hence $(G/G')' = 0$, $G'/G = c$, $\ln G = c \cdot (x-a)$ and $F = 1 - \exp(c \cdot (x-a))$. From the requirement $\lim_{x \rightarrow \infty} F(x) = 1$ it follows that $c = -\lambda$, $\lambda > 0$.]

Remark 1.8. The value of $\Psi(r^+)$ can be determined easily: $\Psi(r^+) = -\lim_{x \rightarrow r^+} f'(x)/f^2(x)$.

Let us apply our results to 11 distributions as follows.

Example 1. $F(x) = \sin(x)$; $I = (0, \pi/2)$; $(f/f')' = \sin^{-2} x > 0$ in I ; $m = 0 \notin I$.

Remark 1.4 applies:

$$\lim_{x \rightarrow \pi/2^-} (f^2/f') = \lim_{x \rightarrow \pi/2^-} (-\cos^2 x / \sin x) = 0,$$

thus

$$\Psi(\pi/2^-) = \lim_{x \rightarrow \pi/2^-} [1 + \sin^{-2} x]^{-1} = 1/2.$$

Remark 1.2 applies to give $\Psi < 1$ in I , and h strictly increases in I .

Example 2. $F(x) = 2 - \operatorname{ch} x$; $I = (\ln(2 - 3^{1/2}), 0)$; $m = r$; $(f/f')' = 4 \cdot e^{2x} \cdot (1 + e^{2x})^{-2} > 0$, everywhere.

Remark 1.4 applies:

$$\lim_{x \rightarrow 0^-} (f^2/f') = \lim_{x \rightarrow 0^-} (-\operatorname{sh}^2 x / \operatorname{ch} x) = - \lim_{x \rightarrow 0^-} \operatorname{sh}^2 x = 0,$$

thus

$$\Psi(0^-) = \lim_{x \rightarrow 0^-} [1 + 4 \cdot e^{2x} / (1 + e^{2x})^2]^{-1} = 1/2.$$

Remark 1.2 applies to give that the RIF h strictly increases in I .

Example 3. $F(x) = (1 - e^{-\lambda x})^k$, $\lambda > 0$, $k > 1$.

Conditions of Theorem 1 are fulfilled with $I = (0, \infty)$;

$$f'(x) = 0 \text{ iff } e^{\lambda \cdot x} = k, \text{ so } m = (\ln k)/\lambda \in I;$$

$$f'(x) > 0 \text{ iff } k > e^{\lambda \cdot x}, \text{ i.e. } x \in (0, m).$$

Similarly, $f'(x) < 0$ iff $x \in (m, \infty)$; now we have $f \stackrel{u}{\sim} g = (1 - e^{-\lambda x})^{k-1}$, so $\ell'' = -\lambda^2 \cdot (k-1)e^{\lambda x} \cdot (e^{\lambda x} - 1)^{-2} < 0$, $x \in I$.

Remark 1.6 applies:

$$\lim_{x \rightarrow \infty} (f^2/f') = k \cdot \lim_{x \rightarrow \infty} e^{-\lambda x} \cdot (1 - e^{-\lambda x})^k \cdot (k \cdot e^{-\lambda x} - 1)^{-1} = 0,$$

so $\Psi(\infty) \in [0, 1]$, and the RIF h strictly increases in I .

Example 4. $F(x) = 1 - \exp(-\lambda \cdot e^x)$, $\lambda > 0$.

Theorem 1 applies with $I = \mathbb{R}$; $m = -\ln \lambda (\in I)$;

$$f'(x) > 0 \text{ iff } 1 > \lambda \cdot e^x \text{ i.e. } x \in (-\infty, m).$$

Similarly, $f'(x) < 0$ iff $x \in (m, \infty)$.

Remark 1.5 gives $f \stackrel{u}{\sim} g = \exp(-\lambda \cdot e^x)$, and $\ell'' = (-\lambda \cdot e^x)'' = -\lambda \cdot e^x < 0$ in $\mathbb{R}$.

We apply Remark 1.6: $\lim_{x \rightarrow \infty} (f^2/f') = \lambda \cdot \lim_{x \rightarrow \infty} \exp(-\lambda \cdot e^x) \cdot (e^{-x} - \lambda)^{-1} = 0$, so we have $\Psi(\infty) \leq 1$, and the RIF h strictly increases in I . (Actually, $\Psi(x) \equiv 1 - e^{-x}/\lambda < 1$).

Example 5. $F(x) = (1 + e^{-x})^{-k}$, $k > 0$. [9, Chap. 12, Sec. 4.5].

Theorem 1 applies: $I = \mathbb{R}$; $m = \ln k (\in I)$;

$$f'(x) > 0 \text{ iff } k \cdot e^{-x} > 1, \text{ i.e. } x \in (-\infty, m).$$

Similarly, $f'(x) < 0$ iff $x \in (m, \infty)$; $(f/f')' = [(1 + e^x)/(k - e^x)]' = (k + 1) \cdot e^x \cdot (k - e^x)^{-2} > 0$, $x \in \mathbb{R} \setminus \{m\}$.

Remark 1.6 applies: $\lim_{x \rightarrow \infty} (f^2/f') = k \cdot \lim_{x \rightarrow \infty} (1 + e^{-x})^{-k} \cdot (k - e^x)^{-1} = 0$, thus $\Psi(\infty) \in [0, 1]$, and the RIF h strictly increases in $I = \mathbb{R}$.

(Actually, $\Psi(\infty) = \lim_{x \rightarrow \infty} [1 + (k + 1) \cdot e^x \cdot (k - e^x)^{-2}]^{-1} = 1$.)

Example 6. $F(x) = 2^{-k}(1 + \operatorname{th} x)^k$, $k > 0$. [9, Chap. 12, Sec. 4.5].

Theorem 1 applies: $I = \mathbb{R}$; $m = (\ln k)/2 \in I$; $f'(x) > 0$ iff $e^{2x} < k$, i.e. $x \in (-\infty, m)$. Similarly, $f'(x) < 0$ iff $x \in (m, \infty)$.

Remark 1.5 applies: $f \stackrel{u}{\sim} g = (1 + \operatorname{th} x)^{k-1}/(e^x + e^{-x})^2$, thus $\ell'' = 2(k+1) \cdot [(e^{2x} + 1)^{-1}]' = -4 \cdot (k+1) \cdot e^{2x} \cdot (e^{2x} + 1)^{-2} < 0$, $x \neq 0$.

Remark 1.6 applies: $\lim_{x \rightarrow \infty} (f^2/f') = k \cdot 2^{-k} \lim_{x \rightarrow \infty} [(1 + \operatorname{th} x)^k / (k - e^{2x})] = 0$, thus $\Psi(\infty) \in [0, 1]$, and the RIF h strictly increases in $\mathbb{R}$.

Example 7. Logistic distribution $F(x) = (1 + e^{-\lambda x})^{-1}$, $\lambda > 0$.

The conditions of Theorem 1 are fulfilled: $I = \mathbb{R}$; $m = 0 \in I$;

$f'(x) > 0$ iff $e^{-\lambda x} > 1$ iff $x < 0$ i.e. $x \in (-\infty, 0)$; $f'(x) < 0$ iff $x \in (0, \infty)$;

f'' exists in $(-\infty, 0) \cup (0, \infty)$.

Remark 1.5 applies, since $f \stackrel{u}{\sim} g = (1 + e^{-\lambda x})^{-2}$, and $\ell'' = 2\lambda \cdot [e^{-\lambda x} \cdot (1 + e^{-\lambda x})^{-1}]' = -2\lambda \cdot e^{\lambda x} \cdot (e^{\lambda x} + 1)^{-2} < 0$ if $x \in (0, \infty)$.

Remark 1.6 applies, since $\lim_{x \rightarrow \infty} (f^2/f') = \lim_{x \rightarrow \infty} (e^{-\lambda x} - e^{\lambda x})^{-1} = 0$, so $\Psi(\infty) \leq 1$, thus the RIF h strictly increases in I . (Actually, $\Psi(x) = 1 - \exp(-\lambda x) < 1$ in $\mathbb{R}$.)

On the other hand, the compression of the x -axis does not change asymptotic behavior of Ψ and monotonic properties of the RIF, so one can consider the logistic distribution as a special case of that in Example 5.

Example 8. Fisher's z -distribution

$f(x) = C \cdot e^{nx} \cdot (1 + k \cdot e^{2x})^{-\alpha}$, where $k := n/n'$, $\alpha := (n + n')/2 (> 0)$, $(0 <) C := 2 \cdot k^{n/2} \cdot \Gamma(\alpha) \cdot [\Gamma(n/2) \cdot \Gamma(n'/2)]^{-1}$ and n, n' are positive integers. Theorem 1 applies:

$I = \mathbb{R}$; $m = 0 \in I$;

$f'(x) > 0$ iff $e^{2x} < 1$ i.e. $x \in (-\infty, 0)$. Similarly, $f'(x) < 0$ iff $x \in (0, \infty)$.

Remark 1.5 applies to give

$f \stackrel{u}{\sim} g = (1 + k \cdot e^{2x})^{-\alpha}$, and

$$\ell'' = -2\alpha k \cdot [e^{2x}/(1 + k \cdot e^{2x})]' = -4\alpha k \cdot e^{2x} (1 + k \cdot e^{2x})^{-2} < 0, \\ x \in (0, \infty).$$

Remark 1.6 applies as well, since

$$\lim_{x \rightarrow \infty} (f^2/f') = C/n \cdot \lim_{x \rightarrow \infty} (e^{-2x} + k) \cdot e^{nx} \cdot (e^{-2x} - 1)^{-1} \cdot (1 + k \cdot e^{2x})^{-\alpha} \\ = -k \cdot C/n \cdot \lim_{x \rightarrow \infty} e^{nx} \cdot (1 + k \cdot e^{2x})^{-\alpha} = 0, \text{ thus } \Psi(\infty) \leq 1,$$

and the RIF h strictly increases in I . (Actually, it can be shown that $\Psi(\infty) = 1$.)

Example 9. Weibull distribution when $\alpha > 1$; $F(x) = 1 - \exp(-\lambda \cdot x^\alpha)$, $\lambda > 0$.

Theorem 1 applies: $I = (0, \infty)$; $f'(x) = 0$ iff $x^\alpha = (\alpha - 1)/(\lambda\alpha)$, so $m = [(\alpha - 1)/(\lambda\alpha)]^{1/\alpha} \in I$; $f'(x) > 0$ iff $\alpha - 1 > \lambda\alpha \cdot x^\alpha$, i.e. $x \in (0, m)$. Similarly, $f'(x) < 0$ iff $x \in (m, \infty)$.

Remark 1.5 applies: $f \stackrel{||}{\sim} g = x^{\alpha-1} \cdot \exp(-\lambda \cdot x^\alpha)$, and $\ell'' = (1 - \alpha) \cdot x^{-2} \cdot (1 + \lambda\alpha \cdot x^\alpha) < 0$.

Remark 1.6 applies:

$$\lim_{x \rightarrow \infty} (f^2/f') = \lambda\alpha \cdot \lim_{x \rightarrow \infty} \exp(-\lambda \cdot x^\alpha)/[(\alpha - 1) \cdot x^{-\alpha} - \lambda\alpha] = 0,$$

thus $\Psi(\infty) \in [0, 1]$, and the RIF h strictly increases in I .

(Actually, $\Psi(x) = 1 - (\alpha - 1)/(\lambda\alpha \cdot x^\alpha) < 1$ in I .)

Example 10. (Extreme value distribution)

$f(x) = \exp(-x - e^{-x})$. [11, §5.47, p. 192]

Theorem 1 applies: $I = \mathbb{R}$; $m = 0 \in I$;

Remark 1.5 applies: $f \stackrel{||}{\sim} g = \exp(-e^{-x})$, and $\ell'' = -e^{-x} < 0$.

Remark 1.6 applies: $\lim_{x \rightarrow \infty} f^2/f' = \lim_{x \rightarrow \infty} (e^{-x} - 1)^{-1} \cdot \exp(-x - e^{-x}) = 0$, we have $\Psi(\infty) \in [0, 1]$, and the RIF h strictly increases in I .

Example 11. $F(x) = 1 - 2 \cdot [c \cdot (1 + e^x)^k - c + 2]^{-1}$; $c, k > 0$ (Sec. 4.5, Chap. 12 in [9]).

Theorem 1 applies if $k = 1$ or ($k = 2$ and $0.0122 \leq c \leq 0.3125$).

Example 11.1. $k = 1$. $I = \mathbb{R}$; $m = \ln 2 - \ln c \in I$; $f'(x) > 0$ iff $2 > c \cdot e^x$ i.e. $x \in (-\infty, m)$; similarly, $f' < 0$ iff $x \in (m, \infty)$.

Remark 1.5 applies: $f \stackrel{||}{\sim} g = (2 + c \cdot e^x)^{-2}$, thus

$$\ell'' = 4 \cdot [(2 + c \cdot e^x)^{-1}]' = -4c \cdot e^x \cdot (2 + c \cdot e^x)^{-2} < 0.$$

Remark 1.6 applies: $\lim_{x \rightarrow \infty} (f^2/f') = 2c \cdot \lim_{x \rightarrow \infty} e^x \cdot (2 - c \cdot e^x)^{-1} \cdot (2 + c \cdot e^x)^{-1} = 0$, thus $\Psi(\infty) \in [0, 1]$ and the RIF h strictly increases in I .

Example 11.2. Let $k = 2$ and $0.0122 \leq c \leq 0.3125$. Theorem 1 applies.

(1): $I = \mathbb{R}$;

(2): $f'(x) = 0$ iff $t(x) = 0$, where $t(x) := T(e^x)$ and $T(y) := 2 + 2 \cdot (2 - c) \cdot y - 3c \cdot y^2 - 2c \cdot y^3$, $y = e^x > 0$. We have $T(2) = 10 - 32c \geq 0$, $T(\infty) = -\infty$. $T'(y)$ has the zeros y_1, y_2 , with

$$y_1 < 0 < y_2 = ([(8 - c)/(3c)]^{\frac{1}{2}} - 1)/2.$$

The point of inflexion of T is at $-\frac{1}{2}$, and $T(-\frac{1}{2}) = c/2 > 0$. It follows that T is concave down in $(-\frac{1}{2}, \infty)$ and has a unique positive zero y_0 in $(2, \infty)$. Furthermore, $m = \ln y_0 > \ln 2$ is the mode of f , since $f'(m) = T(y_0) = 0$.

(3): We have $t(x) > 0$ and $f'(x) > 0$ in $(-\infty, m)$, since $T(y) > 0$ for $y \in (0, y_0)$. Similarly, $f'(x) < 0$ in (m, ∞) , since $T < 0$ in (y_0, ∞) .

Remark 1.5 applies: $f \stackrel{||}{\sim} g = (1 + e^x) \cdot [v(x)]^{-2}$, where $v(x) := 2 + 2c \cdot e^x + c \cdot e^{2x}$. Hence,

$$\ell'' = [e^x/(1 + e^x) - 4c \cdot (e^x + e^{2x})/v(x)]' < 0$$

iff $e^x/(1 + e^x)^2 < 4c \cdot e^x \cdot (2 + 4 \cdot e^x + c \cdot e^{2x})/[v(x)]^2$, i.e. $0 < W$, where $W := 4 \cdot (2c - 1) + 24c \cdot e^x + 36c \cdot e^{2x} + 4c \cdot (4 + c) \cdot e^{3x} + 3c^2 \cdot e^{4x}$.

If we replace c by 0.0122 and e^x by 2, then W decreases since $e^x > e^m = y_0 > 2$, and we get

$$W > 4 \cdot (20c^2 + 82c - 1) > 4 \cdot 0.0034 > 0,$$

which means that (5') is fulfilled in (m, ∞) .

Remark 1.6 applies as well:

$$\lim_{x \rightarrow -\infty} (f^2/f') = 4c \cdot \lim_{x \rightarrow -\infty} e^x \cdot (1 + e^x)^2 \cdot [v(x) \cdot t(x)]^{-1} = 0,$$

thus $\Psi(\infty) \in [0, 1]$, and the RIF h strictly increases in I .

Theorem 2. *Let f be a density function with (1), (3–4), $m = r$ and*

$$(9) \quad (f/f')' < 0 \quad \text{in } (m, s).$$

Then r is finite, and

$$(10) \quad \text{if } \Psi(r^+) < 1 \text{ or}$$

$$[\Psi(r^+) = 1 \text{ and } \Psi < 1 \text{ in some right neighborhood of } r],$$

then $\Psi < 1$ in I , and the corresponding RIF strictly increases in I ;

$$(11) \quad \text{if } \Psi(r^+) > 1,$$

then

$$(11.1) \quad \begin{aligned} &\text{if } \Psi(s^-) \geq 1, \\ &\text{then } \Psi > 1 \text{ and the RIF strictly decreases in } I; \end{aligned}$$

$$(11.2) \quad \begin{aligned} &\text{if } \Psi(s^-) < 1, \\ &\text{then } \Psi > 1 \text{ in } (r, y) \text{ and } \Psi < 1 \text{ in } (y, s) \text{ for some } y \in I, \end{aligned}$$

thus the RIF strictly decreases first and, after reaching its local minimum, strictly increases.

PROOF. By (4), f' is continuous in $(m, s) = I$, and so is Ψ . Since $f' < 0$ in I , f decreases and the value of r must be finite.

Suppose (10) holds. Then $\Psi < 1$ in $(r, m + p)$ for some $p > 0$. Let $x_0 \in (m, m + p)$. Then $\Psi(x_0) < 1$, i.e.

$$(12) \quad [F(x_0) - 1] \cdot f'(x_0) < f^2(x_0).$$

We will prove that $\Psi'(x_0) < 0$. We assume that $\Psi'(x_0) \geq 0$, i.e.,

$$(13) \quad [f^2 \cdot f' + (2f'^2 - f \cdot f'') \cdot (1 - F)]|_{x=x_0} \geq 0$$

By virtue of (12), the left-hand side in (13) will increase, if we replace f^2 in the first term by $[F(x_0) - 1] \cdot f'(x_0)$:

$$(1 - F) \cdot (f'^2 - f \cdot f'')|_{x=x_0} \geq 0,$$

which is equivalent to $f'^2(x_0) \geq f(x_0) \cdot f''(x_0)$, i.e., $(1 - f \cdot f''/f'^2)|_{x=x_0} = (f/f')'|_{x=x_0} \geq 0$, which contradicts (9). Thus, $\Psi'(x_0) < 0$, i.e. Ψ strictly decreases at x_0 . Consequently, Ψ will be a strictly decreasing function in (m, s) , since it is continuous there.

Hence, $\Psi < 1$ in the entire interval I , and the corresponding RIF h strictly increases in I .

Roughly speaking, the main idea of the proof has been the following:
if $\Psi(x_0) < 1$, then Ψ strictly decreases at each $x \in [x_0, s)$.

From this it follows that, if (11) and (11.1) are fulfilled, then there is no $x_0 \in I$ with $\Psi(x_0) \leq 1$. Furthermore, if (11) and (11.2) hold, then there is a unique $y \in (r, s)$ with $\Psi(y) = 1$. Thus $\Psi > 1$ in (r, y) and $\Psi < 1$ in (y, s) . Then the corresponding RIF strictly decreases in (r, y) and, after taking its local minimum at y , strictly increases. $\square$

Remark 2.1. There is no density function f with (1–4) and (9) because, for some $v > 0$, we have $f'' < 0$ in $U = (m, m + v)$, so $f \cdot f''/f'^2 < 1$ and $(f/f')' > 0$ in U .

Remark 2.2. Similarly as in Remark 1.5, the condition (9) can be formulated as follows:

$$(9') \quad \ell'' := (\ln f)'' > 0 \quad \text{in } (m, s).$$

Let us check a few examples in which the Theorem 2 and Remark 2.2 apply.

Example 12. $F(x) = 1 + \operatorname{sh} x$, $I = (\ln(2^{\frac{1}{2}} - 1), 0)$, $m = r = \ln(2^{\frac{1}{2}} - 1)$;

$$\ell'' = (\ln(\operatorname{ch} x))'' = (\operatorname{th} x)' = \operatorname{sech}^2 x > 0 \text{ in } I;$$

(10): $\Psi(r^+) = \lim_{x \rightarrow r^+} \operatorname{th}^2 x = [\operatorname{th}(\ln(2^{\frac{1}{2}} - 1))]^2 = \frac{1}{2} < 1$, thus the RIF strictly increases in I . (Actually, $\Psi(x) = \operatorname{th}^2 x < 1$ in I .)

Example 13. $F(x) = 1 + \tan x$, $I = (-\pi/4, 0)$, $m = r = -\pi/4$;

$$\ell'' = -2 \cdot (\ln \cos x)'' = 2/\cos^2 x > 0 \text{ in } I;$$

(10): $\Psi(-\pi/4^+) = \lim_{x \rightarrow -\pi/4^+} 2 \sin^2 x = 1$ and $\Psi(x) = 2 \cdot \sin^2 x < 1$ in I , so the RIF strictly increases in I . (The inequality $\Psi(x) < 1$ itself implies that h strictly increases.)

Example 14. $F(x) = 1 - (\ln x)^{-\lambda}$, $\lambda > 0$, $I = (e, \infty)$, $m = e(= r)$;

$$\begin{aligned} f \stackrel{u}{\sim} g &= (\ln x)^{-\lambda-1}/x, \quad \ell'' = -[\ln x + (\lambda + 1) \cdot \ln(\ln x)]'' \\ &= [1 + (\lambda + 1) \cdot (1 + \ln x)/(\ln x)^2]/x^2 > 0, \quad x \in I; \end{aligned}$$

(11): $\Psi(e^+) = 1 + 2/\lambda > 1$;

(11.1): $\Psi(\infty) = \infty > 1$, and the RIF strictly decreases in I .

Example 15. $F(x) = 1 - \exp(-\lambda \cdot x^\alpha)$, $\lambda > 0$, $\alpha \in (0, 1)$, Weibull distribution.

$$I = (0, \infty); \quad m = 0(= r); \quad f \stackrel{u}{\sim} g = x^{\alpha-1} \cdot \exp(-\lambda \cdot x^\alpha),$$

$$\ell'' = [(\alpha - 1) \cdot \ln x - \lambda \cdot x^\alpha]'' = (1 - \alpha) \cdot (x^{-2} + \lambda \alpha \cdot x^{\alpha-2}) > 0, \quad x \in I;$$

(11): $\Psi(0^+) = \infty > 1$;

(11.1): $\Psi(\infty) = 1$, thus the RIF strictly decreases in I .

Example 16. $F(x) = 1 - a \cdot \exp(-bx) - c \cdot \exp(-dx)$; $a, b, c, d > 0$;

$$b \neq d; \quad a + c = 1, \quad I = (0, \infty), \quad m = 0(= r);$$

$$\begin{aligned} \ell'' &= (\ln[ab \cdot \exp(-bx) + cd \cdot \exp(-dx)])'' \\ &= abcd \cdot (b - d)^2 \cdot \exp(-bx - dx)/[ab \cdot \exp(-bx) \\ &\quad + cd \cdot \exp(-dx)]^2 > 0 \text{ for all } x \in \mathbb{R}; \end{aligned}$$

(11): $\Psi(0^+) = [A + B \cdot (b^2 + d^2)]/[A + B \cdot (2bd)] > 1$ where $A = a^2 b^2 + c^2 d^2$, $B = ac$, since $(b - d)^2 > 0$ and $b^2 + d^2 > 2bd$;

(11.1): $\Psi(\infty) = 1$, thus the RIF strictly decreases in I .

Example 17. $F(x) = 1 - k \cdot \exp(-bx) \cdot x^{-a}$; $a, b, k > 0$ (Pareto distribution of the third kind in Sec. 2, Chap. 19 of [9]). $I = (k, \infty)$; $m = k (= r)$;

$$f \stackrel{u}{\sim} g = (a + bx) \cdot x^{-a-1}, \quad \ell'' = [\ln(a + bx) - (a + 1) \cdot \ln x]'' \\ = a \cdot x^{-2} \cdot (a + bx)^{-2} \cdot [(a + 1) \cdot (a + 2bx) + b^2 x^2] > 0, \quad x \in I;$$

$$(11): \quad \Psi(k^+) = 1 + a/(a + bk)^2 > 1;$$

$$(11.1): \quad \Psi(\infty) = 1, \text{ and the RIF } h \text{ strictly decreases in } I.$$

Remark 2.3. By the Remark 0.1 we can say that the distributions in Examples 1–13 are IHR (increasing hazard rate), while the ones in Examples 14–17 are DHR (decreasing hazard rate) distributions.

Remark 2.4. We wish to emphasize that the method applied in this paper, particularly in Theorems 1 and 2, enables us to eliminate the inconvenient term

$$1 - F(x) = \int_x^\infty f(t)dt,$$

appearing in each of the relative increment functions, the hazard rate (or failure rate) and the auxiliary function Ψ . Instead, our method deals with the relatively simple expression f/f' .

Acknowledgement. I am grateful to Dr. MICHAEL WOODROOFE for his valuable remarks on strictly concave functions.

References

- [1] P. ADLER and Z. SZABO, The suitability of different mathematical models in describing cumulative caries prevalence curves of individual teeth, European Meeting of Statisticians, Budapest, *Proceedings in Colloquia Mathematica Soc. J. Bolyai* **9** (1972), 43–49.
- [2] P. ADLER and Z. SZABO, Die Analyse von drei Verteilungsfunktionen zur Beschreibung des Kumulativen Kariesbefalles im Spiegel des relativen Karieszuwachses, *Biom. Z.* **16** (1974), 217–232.
- [3] P. ADLER and Z. SZABO, Comparison of two mathematical descriptions of the cumulative caries prevalence, *Acta Med. Acad. Sci. Hung.* **36** (1979a), 211–222.
- [4] P. ADLER and Z. SZABO, Interpretation of a Cauchy distribution function describing cumulative caries prevalence curves, *Comm. Dent. Oral Epidem.* **7** (1979b), 82–86.
- [5] P. ADLER and Z. SZABO, Mathematical description and use in forecasting of the caries curve, *Comm. Dent. Oral Epidem.* **12** (1984), 96–102.
- [6] R. E. BARLOW and F. PROSCHAN, Exponential Life Test Procedures When The Distribution Has Monotone Failure Rate, *Journal of the American Statistical Association* **62** (1967), 548–560.
- [7] W. FELLER, An Introduction to Probability Theory and its Applications, Vol. I, II, Wiley, New York, 1971.

- [8] I. S. GRADSHTEYN and I. M. RYZHIK, Table of Integrals, Series and Products, *Academic Press, San Diego*, 1980.
- [9] N. L. JOHNSON and S. KOTZ, Distributions in Statistics. Continuous Univariate Distributions, Vol. 1, 2, *Houghton Mifflin, Boston – New York*, 1970.
- [10] D. R. PORTER and J. A. DUDMAN, Assessment of caries increments I. Constitution of the R.I.D. index, *J. Dent. Res.* **39** (1960), 1056–1061.
- [11] A. STUART and J. K. ORD, Kendall's Advanced Theory of Statistics, Vol. 1. Distribution Theory, *Griffin, London*, 1987.
- [12] Z. SZABO, Über den relativen Zuwachs von Verteilungsfunktionen, *Biom. Z.* **18** (1976), 33–44.
- [13] Z. SZABO, Relative Increments of Distribution Functions and Their Use in Biometrics, Joint Meeting of the American Statistical Association at Washington, *D.C. Proceedings*, Biopharmaceutical Section, 1989, pp. 212–216.

Z. I. SZABO
INST. MATHS. & INFORMATICS
KOSSUTH UNIV.
H-4010 DEBRECEN
HUNGARY

(Received August 16, 1995)