

Errata: Nullstellensatz theorems and radical classes

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We regret to have to announce that most of the results starting from Section 4 of the above paper (appeared in this journal in Vol. 47 (1995), 65–80) are not correct as stated. We are indebted to Dr BARRY GARDNER for pointing out the impossibility of our claim that the radical and semisimple classes arising in Example 5.3 of our paper were as stated: the semisimple class of the idempotent radical is of course not the class of zero rings. We subsequently discovered an error in the proof of Theorem 4.5: the assumption in line 1 of paragraph 3 that $g_i \in C_{F_Y}(L)$ is erroneous: one can only be sure that $g_i \in F_Y$. A more restrictive notion of “Nullstellensatz theorem” must be used, in order that Theorem 4.5 hold. However, a uniform modification of subsequent proofs makes possible the salvaging of the main results and many of the examples.

Given the extensive nature of the changes, we are prepared to answer queries from interested parties via e-mail, at the following address: stokes@prodigal.murdoch.edu.au. We now provide details of the changes.

The first change is to Definition 4.1, which becomes:

Definition 4.1. Suppose $\mathcal{F} \subseteq F_N$, and $M \in \mathcal{U}$ with $S \subseteq M$. Define

$$\mathcal{F}_M(S) = \{a : a \in M, \text{ there exist } f \in \mathcal{F} \text{ and } a_1, a_2, \dots \in M \text{ such that} \\ a_j = a \text{ for some } j > 0 \text{ and } f(a_1, a_2, \dots) \in (S)_M\}.$$

Definition 4.3 is unchanged, although in effect considerably strengthened by the change to Definition 4.1. Lemma 4.2 still holds, although its proof is modified in a natural way, to become:

PROOF. Suppose that $\mathcal{F}_M(\{0\}) = \{0\}$, that is, that for some $f \in F_N$ and for all $a_1, a_2, \dots \in M$, it is the case that $a_1 = a_2 = \dots = 0$ for all j whenever $f(a_1, a_2, \dots) = 0$. Suppose $h \in \mathcal{F}_R(\{0\})$; thus there exists $f \in \mathcal{F}$ and $h_1, h_2, \dots \in R$ such that $h = h_j$ for some $j > 0$ and $f(h_1, h_2, \dots) = 0$. But $h_i = g_i + \mathcal{IV}_M^X(H)$ for some $g_i \in F_X$. Hence $f(g_1 + \mathcal{IV}_M^X(H), g_2 + \mathcal{IV}_M^X(H), \dots) = 0 + \mathcal{IV}_M^X(H)$, that is, $f(g_1, g_2, \dots) \in \mathcal{IV}_M^X(H)$. Hence, for all $(a_1, a_2, \dots) \in \mathcal{V}_M^X(H)$, $f(g_1(a_1, a_2, \dots), g_2(a_1, a_2, \dots), \dots) = 0$, and so $g_i(a_1, a_2, \dots) = 0$ for each i , whence $g_i \in \mathcal{IV}_M^X(H)$ for each i . That is, $h = h_j = g_j + \mathcal{IV}_M^X(H) = 0 \in R$. $\square$

In the same way, the proofs of Lemma 4.4, Theorem 4.5 and Proposition 4.6 may be modified (and in the case of Theorem 4.5, the logic made sound) with no change to the statements of the results. Unfortunately, Example 4.10 is no longer an example (as far as we can see), although the others continue to be, with the explanations as they currently appear still valid.

Moving on to Section 5, Definition 5.1 is unchanged. However, the statement and proof of Theorem 5.2 are replaced by the following discussion:

In general, suppose M is semantically minimal in $\mathcal{U}$. It is obvious that $F'_X = F_X / \mathcal{IV}_M^X(\{0\})$ is the free algebra on the generators X in the variety generated by M . Let the universal class $\mathcal{U}'$ be the restriction of $\mathcal{U}$ to the variety $\mathcal{W}'$ generated by M . Then $\mathcal{U}' = Q(\mathcal{G}')$, where $\mathcal{G}' = \{F'_{X_1}, F'_{X_2}, \dots\}$. Moreover, in F'_X , $\mathcal{IV}_M^X(H) = (H)_{F_X}$, so M is semantically minimal in $\mathcal{U}'$. Thus, as above, M gives rise to the smallest possible radical class, and all elements of $\mathcal{U}'$ are semisimple. We now look at some examples where this happens.

Example 5.3 should no longer be entitled “The Idempotent Radical”: replace this with “Zero Algebras”. The proof that the given ring R is semantically minimal is unchanged, but the deduction from Theorem 5.2 of the existence of a Nullstellensatz, and thus of radical and semisimple classes of the stated form is not valid. Instead, the above comments replacing Theorem 5.2 and its proof can be invoked to show that R is an example of a $\mathbf{Q}$ -algebra which gives rise to the radical class consisting only of the zero algebra. Precisely the same comments apply to Examples 5.4 and 5.5: in each case, the proof that the given object is semantically minimal is unchanged, but the conclusion is modified in the same way as for Example 5.3. Neither Example 5.4 nor Example 5.5 needs a title change.

The first half of Section 6, up to the end of the proof of Proposition 6.2, must be replaced with the following.

Let $\mathcal{Y}$ be a universal class. For $\mathcal{H} \subseteq F_{\mathcal{N}}$, define $\mathcal{R}H(R) = \{a : a \in R, \text{ there exist } f \in \mathcal{H} \text{ and } b_2, b_3, \dots \in R \text{ for which } f(a, b_2, b_3, \dots) = 0\}$. In [3], we define $\mathcal{H} \subseteq F_{\mathcal{N}}$ to be $\mathcal{Y}$ -associating if, for all $M \in \mathcal{Y}$, $f, g \in \mathcal{F}$, $r, a_i \in M$ and $b_j \in I$ where I is some ideal of M such that $\mathcal{R}H(I) = I$, if $g(h(r, a_i), b_j) = 0$, then there exists $h \in \mathcal{F}$ and $c_k \in M$ such that $h(r, c_k) = 0$.

It was shown in [3] that if $\mathcal{H}$ is $\mathcal{Y}$ -associating, then $\mathcal{R}H = \{M : M \in \mathcal{Y}, M = \mathcal{R}H(M)\}$ is a radical class in $\mathcal{Y}$, and the following result was proved.

Proposition 6.1. *Let $\mathcal{H} \subseteq F_{\mathcal{N}}$. If $\mathcal{R}H(M) \triangleleft M$ for all $M \in \mathcal{Y}$ and whenever $f(r_1, r_2, r_3, \dots, r_n) = 0$ for some $r_1, r_2, \dots, r_n \in M, f \in F$, then all of $r_1, r_2, r_3, \dots, r_n$ are in $\mathcal{R}F'(R)$, then $\mathcal{R}H$ is a radical class if and only if $\mathcal{H}$ is $\mathcal{Y}$ -associating.*

Now for any $\mathcal{F} \subseteq F_{\mathcal{N}}$, define $\mathcal{F}' = \{f : f \in F_{\mathcal{N}}, \text{ there exists } g \in \mathcal{F} \text{ for which } f(x_1, x_2, \dots) = g(x_{\sigma(1)}, x_{\sigma(2)}, \dots) \text{ for some permutation } \sigma \text{ of } 1, 2, \dots\}$. Then clearly for all $M \in \mathcal{Y}$, $\mathcal{R}F'(M) = \mathcal{F}_M(\{0\})$.

Proposition 6.2. *Suppose M possesses a Nullstellensatz in $\mathcal{U}$ with family $\mathcal{F}$. Then $\mathcal{F}'$ is $\mathcal{U}$ -associating.*

PROOF. Let $\mathcal{C}$ be the radical operation on $\mathcal{U}$ induced by the Nullstellensatz for M . From Proposition 6.1 and Theorem 2.3 (ii), $\mathcal{R}F'(R) = \mathcal{F}_R(\{0\}) = C_R(\{0\}) = \mathcal{R}^{\mathcal{C}}(R) \triangleleft R$. The other condition in Proposition 6.1 is evidently satisfied, so $\mathcal{F}$ is $\mathcal{U}$ -associating. $\square$

The comments to follow are unchanged.

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