

On the convergence of the iteration methods to a common fixed point for a pair of mappings

By JUI-CHI HUANG (Taipei)

Abstract. Let T and I be two compatible self-mappings of a closed convex subset C of a normed space X satisfying $I(C) \supseteq (1-k)I(C) + kT(C)$ for all $k \in [0, 1]$ and $\|Tx - Ty\| \leq \alpha\|Ix - Iy\| + \beta \max[\|Tx - Ix\|, \|Ty - Iy\|] + \gamma \max[\|Ix - Iy\|, \|Tx - Ix\|, \|Ty - Iy\|]$, where $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$. Let $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ be real sequences satisfying (i) $0 \leq \alpha_n, \beta_n \leq 1$, (ii) $\liminf \alpha_n > 0$, and (iii) $\lim \beta_n < 1$. It is proved that if for some $x_0 \in C$, the sequence $\{x_n\}_{n=0}^\infty$ defined by $Iy_n = (1 - \beta_n)Ix_n + \beta_nTx_n$ and $Ix_{n+1} = (1 - \alpha_n)Ix_n + \alpha_nTy_n$ ($n \geq 0$) converges to a point z of C and if I is continuous at z then T and I have a unique common fixed point. Further if I is continuous at Tz then T and I have a unique common fixed point at which T is continuous. A similar theorem is proved for involutions of a pair of selfmaps.

Let T and I be two mappings of a metric space (X, d) into itself. SESSA [6] defined T and I to be weakly commuting if $d(TIx, ITx) \leq d(Tx, Ix)$ for any $x \in X$. Clearly two commuting mappings weakly commute, but two weakly commuting mappings in general do not commute. Refer to Example 1 in DIVICCARO et al. [1]. GERALD JUNGCK [3] defined T and I to be compatible mappings, if $d(Tx, Ix) \rightarrow 0$ implies $d(TIx, ITx) \rightarrow 0$. It can be seen that two weakly commuting mappings are compatible but the converse is not true. Examples supporting this fact can be found in [3].

In 1987, DIVICCARO et al. [1], established the following result:

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Theorem A. *Let T and I be two weakly commuting mappings of a closed, convex subset C of a Banach space X into itself satisfying the inequality*

$$(I) \quad \|Tx - Ty\|^p \leq a\|Ix - Iy\|^p + (1 - a) \max[\|Tx - Ix\|^p, \|Ty - Iy\|^p]$$

for all x, y in C where $0 < a < 1/2^{p-1}$ and $p \geq 1$. If I is linear, nonexpansive in C and such that $I(C)$ contains $T(C)$, then T and I have a unique common fixed point at which T is continuous.

Later, in 1991, ROY O. DAVIES [2], showed the following theorems:

Theorem B. *Let T and I be two self-mappings of a non-empty closed convex subset C of a Banach space X , satisfying the inequality*

$$(1) \quad \begin{aligned} \|Tx - Ty\| &\leq \alpha\|Ix - Iy\| + \beta \max[\|Tx - Ix\|, \|Ty - Iy\|] \\ &+ \gamma \max[\|Ix - Iy\|, \|Tx - Ix\|, \|Ty - Iy\|] \end{aligned}$$

for all x, y in C where $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$. Further, let I weakly commute with T and let I be linear and nonexpansive in C . If $I(C)$ contains $T(C)$, then the equations $x = Ty = Iy$ have a unique solution for $x \in C$, and x is a common fixed point of T and I , at which T is continuous.

Theorem C. *Condition (I) with $0 < a < 1$ and $p \geq 1$ implies (1) for a certain triple $\alpha, \beta, \gamma > 0$ with $\alpha + \beta + \gamma = 1$.*

Thus Theorem B not only implies Theorem A, but also implies that the condition $0 < a < 1/2^{p-1}$ can be relaxed to $0 < a < 1$.

Recently, H. K. PATHAK and R. GEORGE [5] established the following Theorem D which omits linearity and nonexpansiveness of the map I , and the proof of Theorem A is made under considerably weaker conditions on the mappings, i.e. replacing a weakly commuting pair of maps (T, I) with compatible maps, and using an iteration method of Mann type. Also the range of p has been extended to the case when $0 < p < 1$. He proved the following

Theorem D. *Let T and I be two compatible selfmaps of a closed convex bounded subset C of a normed space X such that $I(C) \supseteq (1 - k)I(C) + kT(C)$ where $0 < k < 1$ is fixed and satisfies (I) with $0 < a < 1$ and $p > 0$. If for some $x_0 \in X$, the sequence $\{x_n\}$ defined by*

$$Ix_{n+1} = (1 - k)Ix_n + kIx_n, \quad \forall n \geq 0$$

converges to a point z of C and if I is continuous at z then T and I have a unique common fixed point. Further if I is continuous at Tz then T and I have a unique common fixed point at which T is continuous.

Following the basic ideas of Theorem D of H. K. PATHAK and R. GEORGE [5] and drawing inspiration from Theorem B of ROY O. DAVIES [2], we shall use an iteration method of Ishikawa type to establish the following result:

Theorem 1. *Let T and I be two compatible self-mappings of a closed convex subset C of a normed space X such that*

$$(2) \quad I(C) \supseteq (1 - k)I(C) + kT(C)$$

for all $k \in [0, 1]$ and satisfying (1) with $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$.

Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying

- (i) $0 \leq \alpha_n, \beta_n \leq 1$,
- (ii) $\underline{\lim} \alpha_n > 0$, and
- (iii) $\overline{\lim} \beta_n < 1$.

If for some $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$(3) \quad Iy_n = (1 - \beta_n)Ix_n + \beta_nTx_n, \quad n \geq 0$$

$$(4) \quad Ix_{n+1} = (1 - \alpha_n)Ix_n + \alpha_nTy_n, \quad n \geq 0$$

converges to a point z of C and if I is continuous at z then T and I have a unique common fixed point. Further if I is continuous at Tz then T and I have a unique common fixed point at which T is continuous.

PROOF. Since $\underline{\lim} \alpha_n > 0$ and $\overline{\lim} \beta_n < 1$, there exist $a > 0$, $b < 1$, and an integer $N \geq 1$ such that $a \leq \alpha_n$ and $\beta_n \leq b$ for all $n \geq N$. Hence, for $n \geq N$, we have

$$\|Ix_{n+1} - Ix_n\| = \alpha_n\|Ty_n - Ix_n\| \geq a\|Ty_n - Ix_n\|.$$

By hypothesis, this implies that $\|Ty_n - Ix_n\| \rightarrow 0$. Since $\|Ty_n - Iz\| \leq \|Ty_n - Ix_n\| + \|Ix_n - Iz\|$, we have $\|Ty_n - Iz\| \rightarrow 0$. Observe that

$$(5) \quad \|Tx_n - Ix_n\| \leq \|Tx_n - Ty_n\| + \|Ty_n - Ix_n\|,$$

$$(6) \quad \begin{aligned} \|Iy_n - Ix_n\| &= \beta_n\|Tx_n - Ix_n\| \quad (\text{by (3)}) \\ &\leq \beta_n[\|Tx_n - Ty_n\| + \|Ty_n - Ix_n\|], \end{aligned}$$

and

$$(7) \quad \|Ty_n - Iy_n\| \leq (1 - \beta_n)\|Ty_n - Ix_n\| + \beta_n\|Ty_n - Tx_n\| \quad (\text{by(3)}).$$

From (1), we have

$$(8) \quad \begin{aligned} & \|Tx_n - Ty_n\| \leq \alpha\|Ix_n - Iy_n\| \\ & + \beta \max[\|Tx_n - Ix_n\|, \|Ty_n - Iy_n\|] \\ & + \gamma \max[\|Ix_n - Iy_n\|, \|Tx_n - Ix_n\|, \|Ty_n - Iy_n\|]. \end{aligned}$$

Let $c_n := \|Tx_n - Ty_n\|$ and $d_n := \|Ty_n - Ix_n\|$. Introducing (5), (6) and (7) into (8), we obtain

$$(9) \quad \begin{aligned} c_n & \leq \alpha\beta_n(c_n + d_n) \\ & + \beta \max\{c_n + d_n, (1 - \beta_n)d_n + \beta_n c_n\} \\ & + \gamma \max\{\beta_n(c_n + d_n), c_n + d_n, (1 - \beta_n)d_n + \beta_n c_n\}. \end{aligned}$$

From (9), we consider the following cases:

Case 1. For all $n \geq N$, we have

$$c_n \leq \alpha\beta_n(c_n + d_n) + \beta(c_n + d_n) + \gamma\beta_n(c_n + d_n).$$

Since $1 - \alpha\beta_n - \beta - \gamma\beta_n = (1 - \beta) - \beta_n(\alpha + \gamma) = (1 - \beta) - \beta_n(1 - \beta) = (1 - \beta)(1 - \beta_n) > (1 - \beta)(1 - b) > 0$, we have

$$(10) \quad c_n \leq \frac{1}{(1 - \beta)(1 - b)} d_n.$$

Case 2. For all $n \geq N$, we have

$$c_n \leq \alpha\beta_n(c_n + d_n) + \beta(c_n + d_n) + \gamma(c_n + d_n).$$

Now $1 - \alpha\beta_n - \beta - \gamma = \alpha - \alpha\beta_n = \alpha(1 - \beta_n) > \alpha(1 - b) > 0$. Thus

$$(11) \quad c_n \leq \frac{1}{\alpha(1 - b)} d_n.$$

Case 3. For all $n \geq N$, we have

$$c_n \leq \alpha\beta_n(c_n + d_n) + \beta(c_n + d_n) + \gamma[(1 - \beta_n)d_n + \beta_n c_n].$$

Now $1 - \alpha\beta_n - \beta - \gamma\beta_n > (1 - \beta)(1 - b) > 0$. Thus

$$(12) \quad c_n \leq \frac{1}{(1 - \beta)(1 - b)} d_n.$$

Case 4. For all $n \geq N$, we have

$$c_n \leq \alpha\beta_n(c_n + d_n) + \beta[(1 - \beta_n)d_n + \beta_n c_n] + \gamma\beta_n(c_n + d_n).$$

Now $1 - \alpha\beta_n - \beta\beta_n - \gamma\beta_n = 1 - \beta_n(\alpha + \beta + \gamma) = 1 - \beta_n > 1 - b > 0$. Thus

$$(13) \quad c_n \leq \frac{1}{1 - b} d_n.$$

Case 5. For all $n \geq N$, we have

$$c_n \leq \alpha\beta_n(c_n + d_n) + \beta[(1 - \beta_n)d_n + \beta_n c_n] + \gamma(c_n + d_n).$$

Now $1 - \alpha\beta_n - \beta\beta_n - \gamma = 1 - \gamma - \beta_n(\alpha + \beta) = 1 - \gamma - \beta_n(1 - \gamma) = (1 - \gamma)(1 - \beta_n) > (1 - \gamma)(1 - b) > 0$. Thus

$$(14) \quad c_n \leq \frac{1}{(1 - \gamma)(1 - b)} d_n.$$

Case 6. For all $n \geq N$, we have

$$c_n \leq \alpha\beta_n(c_n + d_n) + \beta[(1 - \beta_n)d_n + \beta_n c_n] + \gamma[(1 - \beta_n)d_n + \beta_n c_n].$$

Now $1 - \alpha\beta_n - \beta\beta_n - \gamma\beta_n = 1 - \beta_n(\alpha + \beta + \gamma) = 1 - \beta_n > 1 - b > 0$. Thus

$$(15) \quad c_n \leq \frac{1}{1 - b} d_n.$$

By (10)–(15), we obtain

$$(16) \quad c_n \leq \max \left\{ \frac{1}{(1 - \beta)(1 - b)}, \frac{1}{\alpha(1 - b)}, \frac{1}{(1 - \gamma)(1 - b)} \right\} d_n.$$

Now $d_n := \|Ty_n - Ix_n\| \rightarrow 0$. Thus $c_n := \|Tx_n - Ty_n\| \rightarrow 0$ by (16). Also, $\|Tx_n - Iz\| \leq \|Tx_n - Ty_n\| + \|Ty_n - Iz\|$, this implies that $\|Tx_n - Iz\| \rightarrow 0$,

and hence $\|Tx_n - Ix_n\| \rightarrow 0$. In order to prove that $Tz = Iz$ we remark that

$$\begin{aligned}
 & \|Iz - Tz\| \leq \|Iz - Tx_n\| + \|Tx_n - Tz\| \\
 (17) \quad & \leq \|Iz - Tx_n\| + \alpha\|Ix_n - Iz\| + \beta \max [\|Tx_n - Ix_n\|, \|Tz - Iz\|] \\
 & \quad + \gamma \max [\|Ix_n - Iz\|, \|Tx_n - Ix_n\|, \|Tz - Iz\|] \quad (\text{by(1)}).
 \end{aligned}$$

Taking the limit of (17) as $n \rightarrow \infty$ gives $\|Iz - Tz\| \leq (\beta + \gamma)\|Tz - Iz\|$, which implies that $Iz = Tz$.

From JUNGCK [4], T and I commute at the coincidence point, i.e., $TIz = ITz$. Hence, by using (1) we have

$$\begin{aligned}
 \|T^2z - Tz\| & \leq \alpha\|ITz - Iz\| + \beta \max [\|T^2z - ITz\|, \|Tz - Iz\|] \\
 & \quad + \gamma \max [\|ITz - Iz\|, \|T^2z - ITz\|, \|Tz - Iz\|] \\
 & = (\alpha + \gamma)\|T^2z - Tz\|,
 \end{aligned}$$

whence $T^2z = Tz$, i.e., Tz is a fixed point of T . Now $ITz = TIz = T^2z = Tz$, i.e. Tz is also a fixed point of I . To prove uniqueness, suppose that u is also a common fixed point of T and I . From (1) we have

$$\begin{aligned}
 \|u - Tz\| & = \|Tu - T^2z\| \\
 & \leq \alpha\|Iu - ITz\| + \beta \max [\|Tu - Iu\|, \|T^2z - ITz\|] \\
 & \quad + \gamma \max [\|Iu - ITz\|, \|Tu - Iu\|, \|T^2z - ITz\|] \\
 & = (\alpha + \gamma)\|u - Tz\|.
 \end{aligned}$$

Thus $u = Tz$.

Finally, let $\{z_n\}$ be a sequence of points of C , with limit Tz . Observe that

$$\begin{aligned}
 (18) \quad & \|Tz_n - Iz_n\| \leq \|Tz_n - ITz\| + \|ITz - Iz_n\| \\
 & = \|Tz_n - T^2z\| + \|ITz - Iz_n\|
 \end{aligned}$$

and (1) yields

$$\begin{aligned}
 (19) \quad & \|Tz_n - T^2z\| \leq \alpha\|Iz_n - ITz\| + \beta \max [\|Tz_n - Iz_n\|, \|T^2z - ITz\|] \\
 & \quad + \gamma \max [\|Iz_n - ITz\|, \|Tz_n - Iz_n\|, \|T^2z - ITz\|] \\
 & \leq \alpha\|Iz_n - ITz\| + (\beta + \gamma)[\|Tz_n - T^2z\| + \|ITz - Iz_n\|] \quad (\text{by(18)}).
 \end{aligned}$$

Thus $\|Tz_n - T^2z\| \leq \frac{1}{\alpha}\|Iz_n - ITz\|$. Since I is continuous at Tz , we have $\|Tz_n - T^2z\| \rightarrow 0$ and this means that T is continuous at Tz .

Remark 1. In our Theorem 1, assume that I is continuous at a point of C instead of nonexpansive in C as in Theorem B, and the weakly commuting pair of maps (T, I) can be replaced by compatible maps. Also, the hypothesis of the linearity of I is not necessary in our result.

Theorem 2. Let T and I be two compatible selfmappings of a closed convex subset C of a normed space X such that $I(C) \supseteq (1-k)I(C) + kT(C)$ where $0 < k < 1$ is fixed and satisfying (1) with $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$. If for some $x_0 \in C$, the sequence $\{x_n\}$ defined by

$$Ix_{n+1} = (1-k)Ix_n + kTx_n, \quad \forall n \geq 0$$

converges to a point z of C and if I is continuous at z then T and I have a unique common fixed point. Further if I is continuous at Tz then T and I have a unique common fixed point at which T is continuous.

PROOF. We have $\|Ix_{n+1} - Ix_n\| = k\|Tx_n - Ix_n\|$. Thus, by hypothesis, $\|Tx_n - Ix_n\| \rightarrow 0$, and hence $\|Tx_n - Iz\| \rightarrow 0$. Now, taking the limit of the following inequality as $n \rightarrow \infty$:

$$\begin{aligned} \|Iz - Tz\| &\leq \|Iz - Tx_n\| + \|Tx_n - Tz\| \\ &\leq \|Iz - Tx_n\| + \alpha\|Ix_n - Iz\| + \beta \max [\|Tx_n - Ix_n\|, \|Tz - Iz\|] \\ &\quad + \gamma \max [\|Ix_n - Iz\|, \|Tx_n - Ix_n\|, \|Tz - Iz\|] \end{aligned}$$

yields that $\|Iz - Tz\| \leq (\beta + \gamma)\|Tz - Iz\|$. Then $Iz = Tz$. The sequel of the proof is the same as that of Theorem 1.

Remark 2. By Theorem C, Theorem 2 generalizes Theorem D for $p \geq 1$, and we note that the boundedness of C in Theorem D is not a necessary condition in the case $p \geq 1$. Also Corollary 2 in [5] is a special case of Theorem 2.

Assuming I to be the identity map of X in Theorem 1, we have the following

Corollary 1. *Let T be a self-mapping of a closed convex subset C of a normed space X satisfying*

$$\begin{aligned} \|Tx - Ty\| &\leq \alpha\|x - y\| + \beta \max [\|Tx - x\|, \|Ty - y\|] \\ &\quad + \gamma \max [\|x - y\|, \|Tx - x\|, \|Ty - y\|] \end{aligned}$$

for all x, y in C , where $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$.

Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying

$$(i) \ 0 \leq \alpha_n, \ \beta_n \leq 1, \ (ii) \ \underline{\lim} \alpha_n > 0, \text{ and } (iii) \ \overline{\lim} \beta_n < 1.$$

If for some $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \quad n \geq 0$$

and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \quad n \geq 0$$

converges to a point z in C then T has a unique fixed point at which T is continuous.

Remark 3. The case $p \geq 1$ of Corollary 1 in [5] is a special case of Corollary 1 by Theorem C.

In following result, assume that T and I are involutions instead of assuming that I is continuous as in Theorem 1.

Theorem 3. *Let T, I be selfmaps of a closed convex subset C of a normed space X satisfying*

- (a) T and I are compatible,
- (b) $T^2 = I^2 :=$ the identity map,
- (c) T and I satisfy (1) with $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$,
- (d) $I(C) \supseteq (1 - k)I(C) + kT(C)$ for all $k \in [0, 1]$.

Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying

$$(i) \ 0 \leq \alpha_n, \ \beta_n \leq 1, \ (ii) \ \underline{\lim} \alpha_n > 0, \ (iii) \ \overline{\lim} \beta_n < 1.$$

If for some $x_0 \in C$, there is a sequence $\{x_n\}$ defined by

$$\begin{aligned} Iy_n &= (1 - \beta_n)Ix_n + \beta_n Tx_n, \quad n \geq 0 \\ Ix_{n+1} &= (1 - \alpha_n)Ix_n + \alpha_n Ty_n, \quad n \geq 0 \end{aligned}$$

and for which $\{Ix_n\}$ converges to a point u of C , then u is a unique common fixed point of T and I .

PROOF. As in the proof of Theorem 1, the conditions on $\{Ix_n\}$ imply that $\|Ty_n - Ix_n\| \rightarrow 0$. Also, $\|Ty_n - u\| \leq \|Ty_n - Ix_n\| + \|Ix_n - u\|$. Then $\|Ty_n - u\| \rightarrow 0$. By (5)–(16), we have $\|Tx_n - Ty_n\| \rightarrow 0$. Thus $\|Tx_n - u\| \rightarrow 0$, and hence $\|Tx_n - Ix_n\| \rightarrow 0$. In (c) set $x = x_n$, $y = Iu$ to get

$$\begin{aligned} \|Tx_n - TIu\| &\leq \alpha \|Ix_n - I^2u\| + \beta \max [\|Tx_n - Ix_n\|, \|TIu - I^2u\|] \\ &\quad + \gamma \max [\|Ix_n - I^2u\|, \|Tx_n - Ix_n\|, \|TIu - I^2u\|]. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ and using (b), one obtains $\|u - TIu\| \leq (\beta + \gamma)\|TIu - u\|$, which implies that $u = TIu$. Thus $Tu = T^2Iu = Iu$, and p is a coincidence point for T and I . From JUNGCK [4] T and I commute at coincidence points.

Using (c) with $x = u$, $y = Tu$,

$$\begin{aligned} \|Tu - T^2u\| &\leq \alpha \|Iu - ITu\| + \beta \max [\|Tu - Iu\|, \|T^2u - ITu\|] \\ &\quad + \gamma \max [\|Iu - ITu\|, \|Tu - Iu\|, \|T^2u - ITu\|], \end{aligned}$$

thus $\|Tu - u\| \leq (\alpha + \gamma)\|Tu - u\|$, and $u = Tu = Iu$. Uniqueness follows from (c).

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JUI-CHI HUANG
DEPARTMENT OF MATHEMATICS
TAMKANG UNIVERSITY TAMSUI
TAIPEI, TAIWAN 25137
REPUBLIC OF CHINA

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