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# On associative algebras which are sum of two almost commutative subalgebras

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**Abstract.** The following theorem is proved: if an associative algebra $A$ over an arbitrary field can be decomposed into a sum $A = B + C$ with almost commutative subalgebras $B$ and $C$ (an algebra is called here almost commutative if it has a commutative ideal of finite codimension) then the algebra $A$ possesses a nilpotent ideal $I$ such that the quotient algebra $A/I$ is almost commutative.

## 1. Introduction

In the paper of K.I. Beidar and A.V. Mikhalev [4] the following problem was stated: whether a sum $R = A+B$ of two associative $PI$-rings $A$ and $B$ is a $PI$-ring? There are positive answers to this question for some classes of rings $A$ and $B$ which are near to commutative [4], [5] (every sum of two commutative rings is a $PI$-ring [2]).

Any associative algebra over an arbitrary field which has a commutative ideal of finite codimension (we will call a such algebra almost commutative) is a $PI$-algebra and the question about the structure of a sum of two such algebras is of interest. In this paper, the following result is obtained: every sum of two almost commutative algebras contains a nilpotent ideal such that the quotient algebra on this ideal is almost commutative, in particular, every such sum is a $PI$-algebra. Similar question in group theory i.e. the question about structure of the product of two almost abelian

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(finite-by-abelian) groups is open although it was proved in some cases that this product is almost soluble (see [7], [6] and others).

All considered algebras and rings are associative, the ground field $F$ is arbitrary. The centre of an algebra (or a ring) $A$ is denoted by $Z(A)$. For $F$-subspaces $X$ and $Y$ of an algebra $A$, as usual, $[X,Y]=\{xy-yx \mid x\in X, y\in Y\}$; for a subset $S$ of $A$ and for a subalgebra $B$ of $A$ we will denote by $\mathrm{Ann}^l_B(S)$ and $\mathrm{Ann}^r_B(S)$ the left and corresponding the right annulator of $S$ in the subalgebra $B$.

The following statement is the main result of this paper:

**Theorem.** *Let $A$ be an associative algebra over an arbitrary field which is decomposable into a sum $A=B+C$, where $B$ and $C$ are almost commutative subalgebras of $A$. Then the algebra $A$ contains a nilpotent ideal $I$ such that the quotient algebra $A/I$ is almost commutative.*

Previously, we prove a series of lemmas, some these results can be of interest out of this work. In particular, the Proposition 2 which is used in the proof of the Theorem is an extension (for algebras over a field) of the result of O.H. Kegel about sum of two nilpotent associative rings [8].

**Lemma 1** (See for example [9]). *Let $A$ be an associative algebra over an arbitrary field and $B$ a subalgebra of $A$ with $\dim A/B<\infty$. Then $B$ contains an ideal $I$ of algebra $A$ such that $\dim A/I<\infty$.*

**Lemma 2.** *Let $I$ be an one-sided or two-sided commutative ideal of a ring $R$. Then $R$ has an ideal $J$ such that $J^2=0$ and $(I+J)/J\subseteq Z(R/J)$.*

Proof. Let $I$ be for example a right ideal of the ring $R$ and $i,i_1\in I$, $r\in R$ any elements. Then

$$i_1(ir-ri)=i(i_1r)-(i_1r)i=0$$

because $i_1r\in I$ and $[I,I]=0$. Hence $I[I,R]=0$. Let $T=\mathrm{Ann}^r_R(I)$. Clearly, $T$ is an ideal of the ring $R$ and $[I,R]\subseteq T$. For any element $t\in T$ it holds $(ir-ri)t=irt=0$ (because $rt\in T$, $IT=0$) and therefore $[I,R]T=0$. Now let $J=\mathrm{Ann}^l_T(T)$. It is obvious that $J$ is an ideal of the ring $R$ and $J^2=0$. As $[I,R]\subseteq J$, we have $(I+J)/J\subseteq Z(R/J)$. The case of the left ideal can be considered analogously.

**Lemma 3.** *Let $A$ be an associative algebra and $I$ an ideal of the algebra $A$. If $J$ is an ideal of the subalgebra $I$ then it holds:*

*(1) if subalgebra $J$ is nilpotent then $J$ lies in a nilpotent ideal $J_I$ of the algebra $A$ and $J_I \subseteq I$;*

*(2) if subalgebra $J$ is finite dimensional then $J$ lies in an ideal $J_I$ of the algebra $A$ such that $J_I \subseteq I$ and $J_I$ contains a nilpotent ideal $T$ of the algebra $A$ with $\dim J_I/T < \infty$.*

PROOF. (1) See for example [1, Lemma 1.1.5].

(2) Let $J_I$ be the smallest ideal of the algebra $A$ which contains $J$ and lies in the ideal $I$ of $A$. Since $J_I^3 \subseteq J$ (see [1, Lemma 1.1.5]), $J_I^3$ is a finite dimensional ideal of the algebra $A$. If $J_I^3 = 0$ then we set $T = J_I$ and the statement (2) is proved. Let $J_I^3 \neq 0$ and $T = \mathrm{Ann}^l_{J_I}(J_I^3)$. Clearly, $T$ is an ideal of the algebra $A$ and $(T \cap J_I^3)^2 = 0$. Further, $T/(T \cap J_I^3) \simeq T + J_I^3/J_I^3$ is a nilpotent algebra as a subalgebra of the nilpotent algebra $J_I/J_I^3$ and therefore the ideal $T$ is nilpotent. Since $\dim J_I^3 < \infty$, we have, clearly, $\dim J_I/T < \infty$. The statement (2) and the Lemma are proved.

**Lemma 4.** *Let $R$ be an associative ring and $I$ any commutative ideal of $R$. If the quotient ring $R/I$ is commutative or nilpotent then the ring $R$ contains some nilpotent ideal with the commutative quotient ring.*

PROOF. We may restrict ourselves by Lemma 2 only to the case $I \subseteq Z(R)$. First, let the quotient ring $R/I$ be commutative. For any elements $i \in I$, $r_1, r_2 \in R$ we have

$$ir_1r_2 - r_2ir_1 = 0 = i[r_1, r_2]$$

(because $ir_1 \in I \subseteq Z(R)$) and therefore $I[R, R] = 0$. Let $J$ denote the anulator of the ideal $I$ in $I$. Then $J$ is an ideal of the ring $R$ with $J^2 = 0$ and $[R, R] \subseteq J$ (because $[R, R] \subseteq I$). Thus the quotient ring $R/J$ is commutative and and the proof is complete in the case of commutative quotient ring $R/I$. Now let the quotient ring $R/I$ be nilpotent. If $(R/I)^2=0$ then this case follows from the above considered case. Let the statement of Lemma be true for an arbitrary ring $R$ with $(R/I)^n = 0$, $n \geq 2$, prove it for a ring $R$ with condition $(R/I)^{n+1} = 0$. Denote $N = R^2 + I$. Clearly, $N/I$ is an ideal of the quotient ring $R/I$ and $(N/I)^n = 0$. By inductive assumption the subring $N$ contains some nilpotent ideal $T$ such that the quotient ring $N/T$ is commutative. By Lemma 3 $T$ lies in some nilpotent ideal $S$ of the ring $R$ with $S \subseteq N$. Then the quotient ring $\overline{R} = R/S$ contains a commutative ideal $\overline{N} = N/S$ such that $(\overline{R}/\overline{N})^2 = 0$. As was proved above the ring $\overline{R}$ contains some nilpotent ideal $\overline{J} = J/S$ such that $\overline{R}/\overline{J}$ is commutative. It is obvious that $J$ is a nilpotent ideal of the ring $R$ and the quotient ring $R/J$ is commutative. The proof is complete.

**Lemma 5.** *Let $A$ be an almost commutative associative algebra and $I$ a commutative ideal of $A$ with $\dim A/I < \infty$. Then:*

*(1) $[A, A]I$ lies in some nilpotent ideal of the algebra $A$;*

*(2) for some nilpotent ideal $J$ the quotient algebra $A/J$ contains a finite dimensional ideal $T/J$ such that the quotient algebra $A/T$ is commutative.*

Proof. (1) If $I \subseteq Z(A)$ then we have for any elements $a_1, a_2 \in A$ and $i \in I$

$$(a_1a_2 - a_2a_1)i = (a_1i)a_2 - a_2(a_1i) = 0,$$

because $a_1i \in I \subseteq Z(A)$ and therefore $[A, A]I = 0$. Now if $I \nsubseteq Z(A)$ then going to the quotient algebra $A/J$ on some nilpotent ideal $J$ with $I \subseteq Z(A/J)$ (it exists by Lemma 2) we get $[A, A]I \subseteq J$.

(2) We can assume, without loss of generality, by Lemma 2 that $I \subseteq Z(R)$. Clearly, $T = \mathrm{Ann}_A(I)$ is an ideal of the algebra $A$ and by part 1 of this Lemma $T \supseteq [A, A]$. Let denote $J = T \cap I$. Obviously, $J^2 = 0$ and $T/J$ is a finite dimensional ideal of the algebra $A/J$. At that the quotient algebra $(A/J)/(T/J) \simeq A/T$ is commutative.

For convenience and shortness we introduce the following:

*Definition 1.* An associative algebra $A$ over an arbitrary field will be called an *NCF*-algebra if it contains a nilpotent ideal with almost commutative quotient algebra.

An ideal of an associative algebra will be called an *NCF*-ideal if it is an *NCF*-algebra.

In particular, every nilpotent, commutative and finite dimensional algebras are *NCF*-algebras by this definition.

**Proposition 1.** *The following statements hold:*

*(1) every subalgebra and every quotient algebra of an NCF-algebra are NCF-algebras;*

*(2) if $A$ and $B$ are NCF-algebras then the direct product $A \times B$ is an NCF-algebra;*

*(3) every extension of an NCF-algebra by other NCF-algebra is an NCF-algebra.*

PROOF. The statements (1) and (2) of the Proposition are obvious. Prove the statement (3), i.e. show that an algebra $A$ is an *NCF*-algebra if it contains an *NCF*-ideal $B$ such that $A/B$ is also an *NCF*-algebra. Consider some cases previously:

(a) The quotient algebra $A/B$ is finite dimensional.

Let $I$ be a nilpotent ideal of the *NCF*-algebra $B$ such that $B/I$ is almost commutative. By part 1 of Lemma 3 $I$ lies in some nilpotent ideal of the algebra $A$ and the latest lies in $B$. Then we can assume, without loss of generality, $I = 0$ i.e. $B$ is commutative. Since $\dim A/B < \infty$, the algebra $A$ is almost commutative by Lemma 1 and therefore it is an *NCF*-algebra.

(b) The ideal $B$ is finite dimensional.

The right annulator $C = \mathrm{Ann}^r_A(B)$ is an ideal of the algebra $A$, and since $C/(B \cap C) \simeq C + B/B$ is an *NCF*-algebra, the ideal $C$ is an *NCF*-algebra in view of equality $(B \cap C)^2 = 0$. It follows from the inequality $\dim A/C < \infty$ and part (a) of this proof that $A$ is an *NCF*-algebra.

(c) The ideal $B$ is commutative.

In the *NCF*-algebra $A/B$ there exists a nilpotent ideal $N/B$ such that quotient algebra $(A/B)/(N/B) \simeq A/N$ is almost commutative. Without loss of generality one can assume in view of Lemma 4 and part 1 of Lemma 3 that the ideal $N$ is commutative. Denote by $S/N$ any commutative ideal of finite codimension in algebra $A/N$. By Lemma 4 and part 1 of Lemma 3 we can assume also $S$ is commutative. Since $\dim A/S < \infty$, we obtain that $A$ is an *NCF*-algebra.

Now prove the statement (3) in general case. Let $N$ be any nilpotent ideal of the subalgebra $B$ such that the quotient algebra $B/N$ is almost commutative. By part 1 of Lemma 3 one can assume without loss of generality $N = 0$ i.e. the ideal $B$ is almost commutative. Analogously, by part 2 of Lemma 5 and part 1 of Lemma 3 we can consider the subalgebra $B$ has a finite dimensional ideal $T$ with commutative quotient algebra $B/T$. Similarly, one can assume the subalgebra $T$ of $A$ lies in some finite dimensional ideal $T_B$ of algebra $A$ such that $T_B \subseteq B$ and $B/T_B$ is commutative. Then $A/T_B$ is an *NCF*-algebra in view of part (c) of this proof. Since $\dim T_B < \infty$, the algebra $A$ is an *NCF*-algebra by part (b) of this proof. The proof is complete.

It follows from Lemma 1 and Proposition 1 the next statement:

**Corollary 1.** *If an associative algebra $A$ has an NCF-subalgebra $B$ and* $\dim A/B < \infty$ *then $A$ is an NCF-algebra.*

**Lemma 6** ([2], [3, Th. 2.2])**.** *Let $R$ be an associative ring which is decomposable into a sum $R = A + B$ of two commutative subrings $A$ and $B$. Then $R$ has an ideal $I$ with $I^2 = 0$ and commutative quotient ring $R/I$.*

**Lemma 7.** *Let $A$ be an associative algebra over an arbitrary field $F$, $B$ and $C$ commutative subalgebras of $A$ and let $I$ be an ideal of $A$ which lies in the $F$-subspace $B + C$. Then $I$ is an NCF-ideal.*

PROOF. Let $I_B = \{b \in B \mid$ there exists $i \in I$ of the form $i = b + c$, $c \in C\}$ i.e. $I_B$ is a projection of the ideal $I$ into subalgebra $B$. Analogously, define the projection $I_C$ of $I$ on subalgebra $C$. Obviously, it holds for elements $i_1, i_2 \in I$, $i_1 = b_1 + c_1$, $i_2 = b_2 + c_2$, where $b_i \in B$, $c_i \in C$, $i = 1, 2$ the equality

$$i_1 i_2 = (b_1 + c_1)(b_2 + c_2) = i_1 c_2 + c_1 i_2 + b_1 b_2 - c_1 c_2.$$

Thus $b_1 b_2 - c_1 c_2 \in I$, and hence $I_B$, $I_C$ are subalgebras of $B$ and corresponding $C$. It is easy to see that $I_B + I_C$ is a subalgebra of $A$, and since the subalgebras $I_B$ and $I_C$ are both commutative, $I_B + I_C$ is an *NCF*-algebra by Lemma 6. Then the ideal $I$ which lies in $I_B + I_C$ is an *NCF*-algebra. The proof is complete.

For convenience we give the following definition:

*Definition 2.* An associative algebra $A$ over an arbitrary field $F$ decomposable into a sum $A = B + C$ of two almost commutative subalgebras $B$ and $C$ will be called a minimal $BM$-counter-example if $A$ satisfies the following conditions:

(1) $A$ is not an *NCF*-algebra;

(2) the subalgebras $B$ and $C$ have commutative ideals $B_0$ and corresponding $C_0$ such that $\dim B/B_0 + \dim C/C_0 < \infty$ and the number $\dim A/(B_0 + C_0)$ is the smallest;

(3) the algebra $A$ has not nonzero ideals which lie in the $F$-subspace $B_0 + C_0$ from the condition (2).

**Lemma 8.** *Let $A = B + C$ be a minimal $BM$-counter-example. Then for every nonzero ideal $I$ of $A$ the quotient algebra $A/I$ is an NCF-algebra. Besides, the algebra $A$ has not nonzero NCF-ideals.*

Proof. Let $\dim A/(B_0 + C_0) = n$ where $B_0$ and $C_0$ are the commutative ideals of subalgebra $B$ and corresponding $C$ from Definition 2. If $n = 0$ then the algebra $A$ is a sum of two commutative subalgebras $B_0$ and $C_0$ and hence it is an *NCF*-algebra by Lemma 6. This contradicts to the choice of the algebra $A$ and therefore $n \geq 1$. Let $I$ be a nonzero ideal of the algebra $A$ such that $A/I$ is not an *NCF*-algebra. Denote

$$\overline{A} = A + I/I, \quad \overline{B} = B + I/I, \quad \overline{C} = C + I/I,$$

$$\overline{B_0} = B_0 + I/I, \quad \overline{C_0} = C_0 + I/I.$$

Let $m = \dim \overline{A}/(\overline{B}_0 + \overline{C}_0)$. By the Definition 2 it holds $I \nsubseteq B_0 + C_0$ and hence $m < n$. Denote by $\overline{T}$ the sum of all ideals of the algebra $\overline{A}$ which lie in the $F$-subspace $\overline{B}_0 + \overline{C}_0$. The ideal $\overline{T}$ is an *NCF*-algebra by Lemma 7 and therefore the quotient algebra $\overline{A}/\overline{T}$ is not an *NCF*-algebra in view of the choice of $A$ and Proposition 1.

Since the $F$-subspace $(\overline{B}_0 + \overline{C}_0)/\overline{T}$ does not contain nonzero ideals of the algebra $\overline{A}/\overline{T}$ and its codimension in $\overline{A}/\overline{T}$ is equal to $m$, $m < n$, it contradicts to the choice of $A$. The obtained contradiction proves that $A/I$ is an *NCF*-algebra.

Now let $J$ be a nonzero *NCF*-ideal of the algebra $A$. As has just been proved $A/J$ is an *NCF*-algebra, and then $A$ is an *NCF*-algebra by Proposition 1. The latest is impossible and hence $A$ has not nonzero *NCF*-ideals. The proof is complete.

**Lemma 9.** *Let $R$ be an associative ring which is decomposable into a sum $R = A + B$ of two subrings $A$ and $B$ and let $R_0$ be a subring of $R$ with $R_0 \supseteq B$. If $R_0$ contains an ideal $A_0$ of the subring $A$ then $R_0$ contains some ideal $J$ of the ring $R$ such that $J \supseteq A_0$.*

Proof. Consider the subring $J = A_0 + BA_0 + A_0B + BA_0B$ of the ring $R$. Clearly, $J \subseteq R_0$ and $A_0 \subseteq J$. As $A_0$ is an ideal of the subring $A$, $J$ obviously, is an ideal of the ring $R$.

**Lemma 10.** *Let $A = B+C$ be a minimal $BM$-counter-example where subalgebras $B$ and $C$ satisfy all conditions of the Definition 2 and let $A_1 = B + C_1$ ($C_1 \subseteq C$) be a subalgebra of $A$. If $A_1$ is not an NCF-algebra then for some NCF-ideal $J$ of subalgebra $A_1$ the quotient algebra $A_1/J$ is a minimal $BM$-counter-example.*

Proof. Let $B_0 \subseteq B$ and $C_0 \subseteq C$ be commutative ideals of the sublagebra $B$ and corresponding $C$ from the Definition 2 and let $n = \dim A/(B_0 + C_0)$. Denote by $C_1'$ the subalgebra in $C$ which is generated by $C_1$ and $C \cap B$. Clearly,

$$B + C_1 = B + C_1', \qquad C_1' \cap B = C \cap B,$$

and therefore one can assume, without loss of generality, that $C_1' = C_1$ and hence $C_1 \cap B = C \cap B$. It follows from the latest equality that $C_0 \cap A_1 = C_0 \cap C_1$. Indeed, let $x \in C_0 \cap A_1$. Then $x = c_0, c_0 \in C_0, x = b+c_1$ for some $c_1 \in C_1$, $b \in B$. Hence $b = c_0 - c_1 \in B \cap C = B \cap C_1$ and $x \in C_1$. But then $x \in C_0 \cap C_1$ and $C \cap A_1 \subseteq C_0 \cap C_1$, because the element $x$ has been chosen in any way. The inclusion $C_0 \cap C_1 \subseteq C_0 \cap A_1$ is obvious.

Since $A_1 \cap (B_0 + C_0) \supseteq B_0$, it holds $A_1 \cap (B_0 + C_0) = B_0 + (C_0 \cap A_1)$ and therefore as proved above

$$A_1 \cap (B_0 + C_0) = B_0 + (C_0 \cap C_1).$$

Now denote by $J$ the sum of all ideals of the algebra $A_1$ which lie in $B_0 + (C_0 \cap C_1)$. By Lemma 7 $J$ is an *NCF*-ideal of the algebra $A_1$ and $A_1/J$ is not an *NCF*-algebra (because $A_1$ is not an *NCF*-algebra). It is easy to see that $A/J$ is a minimal $BM$-counter-example (in particular, $\dim A_1/((B_0 + C_0) \cap A_1) = n$). The proof is complete.

**Lemma 11.** *If $I$ is an one-sided finite dimensional ideal of an associative algebra $A$ then $A$ has a nilpotent ideal $J$ such that $(I + J)/J$ lies in some finite dimensional (two-sided) ideal of the quotient algebra $A/J$.*

Proof. Let $I$ be for example a right ideal of the algebra $A$ and $S = \mathrm{Ann}_A^r(I)$. Clearly, $S$ is an ideal of $A$ and $\dim A/S < \infty$. Further, $T = \mathrm{Ann}_A^l(S)$ is also an ideal of $A$, $T \supseteq I$ and $J = T \cap S$ is an ideal of the algebra $A$ with $J^2 = 0$. It is easy to see that $\dim T/J < \infty$ and $I + J/J \subseteq T/J$. The case of a left ideal of $A$ can be considered analogously. The Lemma is proved.

**Lemma 12.** *Let $A$ be an associative algebra over an arbitrary field $F$ and $a$ is an element in $A$ such that* $\dim A/\operatorname{Ann}^r_A(a) < \infty$ $(\dim A/\operatorname{Ann}^l_A(a) < \infty)$. *Then the element $a$ belongs to a finite dimensional right (corresponding left) ideal of the algebra $A$.*

Proof. Denote by $C$ the right annulator $\operatorname{Ann}^r_A(a)$ and let $\dim A/C < \infty$. Choose a complete system of representatives $\{h_1, \dots, h_n\}$ of the congruence classes of $A$ by $C$. We will show that $F$-subspace
$I = \langle a, ah_1, \dots, ah_n\rangle$ is a right ideal of the algebra $A$. If $g$ is any element in $A$ then $g$ is of the form $g = c + \sum_{i=1}^n \alpha_i h_i$ where $c \in C$, $\alpha_i \in F$, $i = 1, \dots, n$. Hence

$$ag = a\left(c + \sum_{i=1}^n \alpha_i h_i\right) = \sum_{i=1}^n \alpha_i a h_i \in I,$$

because $ac = 0$. Further, denote $h_i g = c_i + \sum_{j=1}^n \beta_{ij} h_j$ where $c_i \in C$, $\beta_{ij} \in F$, $i, j = 1, \dots, n$. Then we obtain

$$(ah_i)g = a(h_i g) = a\left(c_i + \sum_{j=1}^n \beta_{ij} h_j\right) = \sum_{j=1}^n \beta_{ij} a h_j \in I.$$

Therefore $I$ is a right ideal of the algebra $A$, $a \in I$ and $\dim I < \infty$. Analogously, one can consider the case $\dim A/\operatorname{Ann}^l_A(a) < \infty$. The proof is complete.

Let $A$ be a nilpotent algebra, $A \neq 0$. The number $n = n(A)$ such that $A^n = 0$, $A^{n-1} \neq 0$ will be called the index of nilpotency of $A$ and denoted by $n(A)$. The index of nilpotency of the zero algebra we assume to be equal 1.

An associative algebra $A$ will be called almost nilpotent if it has a nilpotent ideal of finite codimension. By $\bar{n}(A)$ will be denoted the smallest nilpotency index of all nilpotent ideals of $A$ of finite codimension in $A$.

**Lemma 13.** *If $I$ is a right (left) almost nilpotent ideal of an algebra $A$ then $A$ has a nilpotent ideal $J$ such that $I + J/J$ is a finite dimensional right (corresponding left) ideal of $A$.*

Proof. Let $I$ be, for example, a right ideal. Let $B$ be any nilpotent ideal of the subalgebra $I$ with $\dim I/B < \infty$ such that $n(B) = \bar{n}(I)$. If $\bar{n}(I) = 1$ then $\dim I < \infty$, that is $B = 0$, and Lemma is proved. First consider the case $\dim I/\operatorname{Ann}_I^r(g) < \infty$ for every element $g \in I$. Let the statement of Lemma be true for algebras with $\bar{n}(I) < k$, prove it for algebras with $\bar{n}(I) = k$. Choose a complete system of representatives $\{g_1, \dots, g_m\}$ of the congruence classes of $I$ by $B$. Using Lemma 12 one can easy show that there exists a finite dimensional right ideal $N$ of the algebra $I$ with $\{g_1, \dots, g_m\} \subseteq N$. Obviously, $I = B + N$. Then $T = \operatorname{Ann}_B^r(N)$ is a nilpotent ideal of the subalgebra $I$ and $\dim I/T < \infty$. Analogously, $I_0 = \operatorname{Ann}_I^r(I)$ is a nilpotent right ideal of the algebra $A$ and $I_0 \supseteq T \cap B^{k-1}$. Then as is well known (see for example [1, Lemma 1.1.2]) $I_0$ lies in some nilpotent ideal $S$ of the algebra $A$. At that the quotient algebra $\overline{A} = A/S$ has the right almost nilpotent ideal $\overline{I} = I + S/S$. Since $T \cap B^{k-1} \subseteq S$ then the ideal $\overline{T} = T + S/S$ of the subalgebra $\overline{I}$ has the nilpotency index $\leq k-1$ and therefore $\bar{n}(\overline{I}) \leq k-1$. By inductive assumption there exists in $\overline{A}$ some nilpotent ideal $\overline{J} = J/S$ such that $\overline{I} + \overline{J}/\overline{J}$ is a finite dimensional right ideal of the algebra $\overline{A}/\overline{J}$. Then $J$ is a nilpotent ideal of the algebra $A$ and $I + J/J$ is a finite dimensional right ideal of the quotient algebra $A/J$.

Now let $I_1 = \{i \in I \mid \dim I/\operatorname{Ann}_I^r(i) < \infty\}$. It is easy to see that $I_1$ is a right ideal of the algebra $A$, $I_1 \cap B$ is a nilpotent ideal in $I_1$ and $\dim I_1/(I_1 \cap B) < \infty$. Besides, $B^{k-1} \subseteq I_1$, because $B^{k-1}B = 0$ and $\dim I/B < \infty$. As has just been proved there exists in $A$ a nilpotent ideal $U$ such that $I_1 + U/U$ is a finite dimensional right ideal of the algebra $A/U$. One can assume without loss of generality (by Lemma 11) that the algebra $A$ has a finite dimensional ideal $M$ such that $M \supseteq B^{k-1}$. By inductive assumption (induction on $\overline{n}(I)$)the quotient algebra $A/M$ contains a nilpotent ideal $V/M$ such that $(I + V/M)/(V/M)$ is a finite dimensional right ideal of the algebra $(A/M)/(V/M)$. Let $V_1 = \operatorname{Ann}_V^r(M)$. It is easy to see that $V_1$ is a nilpotent ideal of the algebra $A$ and $I + V_1/V_1$ is a finite dimensional right ideal of the algebra $A/V_1$.

The case of the left ideal $I$ can be considered analogously. The Lemma is proved.

The statements below follow from Lemmas 11 and 13.

**Corollary 2.** *Let $A$ be an associative algebra and $I$ a right (left) almost nilpotent ideal of the algebra $A$. Then $I$ lies in some almost nilpotent ideal of the algebra $A$.*

**Corollary 3.** *If an associative algebra $A$ has an almost nilpotent ideal $I$ with almost nilpotent quotient algebra $A/I$ then the algebra $A$ is almost nilpotent.*

PROOF. One can assume, without loss of generality, that $\dim I < \infty$ (in view of Lemma 3). Let $J/I$ be a nilpotent ideal of the quotient algebra $A/I$ such that $\dim A/J < \infty$ and let $C = \mathrm{Ann}^r_J(I)$. Obviously, $C$ is an ideal of the algebra $A$ and $\dim J/C < \infty$ (and hence $\dim A/C < \infty$). Since $(C \cap I)^2 = 0$ and $C/(C \cap I) \simeq C + I/I$ is a nilpotent algebra then the ideal $C$ is nilpotent. The proof is complete.

**Proposition 2.** *If an associative algebra $A$ over an arbitrary field is decomposable into a sum $A = B + C$ with almost nilpotent subalgebras $B$ and $C$ of $A$ then the algebra $A$ is almost nilpotent.*

PROOF. Let the statement of the Proposition be false. Choose among all counter-examples to the Proposition an algebra $A = B + C$ with the smallest sum $\bar{n}(B) + \bar{n}(C)$. Clearly, $\bar{n}(B) \geq 2$ and $\bar{n}(C) \geq 2$ (if, for example, $\bar{n}(B) = 1$ then $\dim B < \infty$ and therefore the algebra $A$ is almost nilpotent in contradiction to the our assumption). Denote by $B_0$ and $C_0$ some nilpotent ideals of the subalgebra $B$ and corresponding of the subalgebra $C$ such that $\dim B/B_0 + \dim C/C_0 < \infty$ and $n(B_0) = \bar{n}(B)$, $n(C_0) = \bar{n}(C)$. Let $I = B_0^{\bar{n}(B)-1}$. It is easy to see that $IB_0 = B_0 I = 0$ and $A_0 = B + IC$ is a subalgebra from $A$ of the form $A_0 = B + C_1$ where $C_1 = C \cap A_0$. Note that $B_0$ is a right nilpotent ideal of the subalgebra $A_0$. Then the right ideal $B_0$ lies as is well known in some (two-sided) nilpotent ideal $S$ of the subalgebra $A_0$. The almost nilpotent subalgebra $C_1 + S/S$ is of finite codimension in $A_0/S$ and by Lemma 1 the quotient algebra $A_0/S$ is almost nilpotent. Then the algebra $A_0$ is almost nilpotent.

It is eeasy to see that $I + IC$ is a right ideal of the algebra $A$, and since $I + IC \subseteq A_0$, the subalgebra $I + IC$ is almost nilpotent. Further, $I + IC$ lies by Corollary 2 in some almost nilpotent ideal $T$ of the algebra $A$. The quotient algebra $\overline{A} = A/T$ is decomposable into a sum $\overline{A} = \overline{B} + \overline{C}$ where $\overline{B} = B + T/T, \overline{C} = C + T/T$. Since $I \subseteq T$, we have $\bar{n}(\overline{B}) < \bar{n}(B)$ and therefore the quotient algebra $A/T$ is almost nilpotent by choice of the algebra $A$. In view of Corollary 3 the algebra $A$ is almost nilpotent. It contradicts to the choice of $A$ and the proof is complete.

**Lemma 14.** *Let $A$ be an associative algebra without nonzero NCF-ideals decomposable into a sum $A = B + C$ of subalgebras $B$ and $C$ which contain commutative ideals $B_0 \subseteq B$ and $C_0 \subseteq C$ such that $\dim B/B_0 + \dim C/C_0 < \infty$. If $A_1 = B + B_0C$ is an NCF-subalgebra of $A$ then the subalgebra $C_1 = C \cap A_1$ is almost nilpotent.*

Proof. Since $B_0$ is an ideal of subalgebra $B$, the subspaces $B_0C$ and $A_1$ are obvius subalgebras of $A$. Let $g = b + c$ be any element in the $F$-subspace $B_0C \cap (B_0 + C_0)$ where $b \in B_0, c \in C_0$. It is easy to see that $C_0c = cC_0$ is a two-sided ideal of the algebra $C$ and $cC_0$ lies in the subalgebra $A_1 = B + B_0C = B + C_1$ Then there exists by Lemma 9 an ideal $S$ of algebra $A$ such that $cC_0 \subseteq S$ and $S \subseteq A_1$. Let $A_1$ be an *NCF*-subalgebra. Then $S$ is an *NCF*-ideal of the algebra $A$ and by conditions of Lemma $S = 0$. Hence $cC_0 = C_0c = 0$, that is, $c \in \mathrm{Ann}_{C_0}(C_0)$. In view of choice of the element $g = b+c$ this means $(B_0 + B_0C) \cap C_0 \subseteq \mathrm{Ann}_{C_0}(C_0)$. Further, it is easy to see that $\dim C_1/(A_1 \cap C_0) < \infty$ because $\dim C/C_0 < \infty$, and since $\dim B/B_0 < \infty$, we have

$$\dim C_1/((B_0 + B_0C) \cap C_0) < \infty.$$

Obviously, $(\mathrm{Ann}_{C_0}(C_0))^2 = 0$, hence $((B_0 + B_0C) \cap C_0)^2 = 0$ and therefore $C_1$ is an almost nilpotent subalgebra of the algebra $C$. The proof is complete.

**Lemma 15.** *Let $A = B+C$ be a minimal BM-counter-example where subalgebras $B$ and $C$ satisfy conditions of Definition 2. Then both subalgebras $B$ and $C$ are not almost nilpotent.*

Proof. Let the statement of Lemma be false. Then there exist minimal $BM$-counter-examples of the form $A = B + C$ such that one of the subalgebras $B$ or $C$ is almost nilpotent (by Proposition 2 both subalgebras $B$ and $C$ can not be almost nilpotent simultaneously). Choose among all such counter-examples an algebra $A = B + C$ with almost nilpotent subalgebra $B$ which has the smallest number $n = \bar{n}(B)$. Let $B_0'$ and $C_0$ be commutative ideals of finite codimension of the algebra $B$ and corresponding $C$ which satisfy conditions of Definition 2. Take a nilpotent ideal $N$ of subalgebra $B$ with $\dim B/N < \infty$ and $n(N) = \bar{n}(B)$ and set $B_0 = B_0' \cap N$. Obviously, $B_0$ is a commutative nilpotent ideal of finite codimension in $B$ and $n(B_0) = \bar{n}(B)$. It is easy to see that $A_1 = B + B_0C$ is a subalgebra of $A$. Show that $A_1$ is an $NCF$-algebra. Denote $I = (B_0)^{\bar{n}(B)-1}$. Then $I$ is a right nilpotent ideal of the subalgebra $A_1$ and hence $I$ lies in certain nilpotent ideal $S$ of $A_1$. The quotient algebra $A_1/S$ is decomposable into a sum

$$A_1/S = (B + S)/S + (B_0C + S)/S$$

and $B_0^{\bar{n}(B)-2} + S/S$ is a right nilpotent ideal in $A_1/S$. Therefore $B_0^{\bar{n}(B)-2} + S/S$ lies in some nilpotent ideal $S_1/S$ of the algebra $A_1/S$. Repeating this considering one can show that $B_0$ lies in some nilpotent ideal $T$ of the algebra $A_1$. Since $\dim B/B_0 < \infty$, the quotient algebra $A_1/T$ is an $NCF$-algebra by Corollary 1 and hence $A_1$ is an $NCF$-algebra. Obviously, $A_1 = B + C_1$ where $C_1 = C \cap A_1$. The subalgebra $C_1$ is almost nilpotent (see Lemma 14) and therefore the subalgebra $A_1$ is almost nilpotent by Proposition 2 as a sum of two almost nilpotent subalgebras $B$ and $C_1$. Then the right ideal $B_0 + B_0C$ of the algebra $A$ is almost nilpotent and lies by Corollary 2 in some almost nilpotent ideal $T_1$ of the algebra $A$. But $T_1 = 0$ by Lemma 8 and hence $B_0 = 0$. It follows from this $\dim A/C < \infty$ and $A$ is an $NCF$-algebra in view of Corollary 1. This contradicts to the choice of algebra $A$. The proof is complete.

**Lemma 16.** *Let $A = B + C$ be a minimal $BM$-counter-example, let $B_0$ and $C_0$ be commutative ideals of the subalgebras $B$ and corresponding $C$ from Definition 2. Then $B + B_0C$ and $B + CB_0$ are subalgebras of $A$ and at least one of these subalgebras is not an $NCF$-algebra.*

Proof. It is easy to see that $A_1 = B + B_0C$ and $A_2 = B + CB_0$ are subalgebras of $A$ because $B_0$ is an ideal of the subalgebra $B$. One can immediately check up that $B_0C$, $CB_0$ and $A_0 = B + B_0C + CB_0 + CB_0^2C$ are also subalgebras of the algebra $A$. Suppose the Lemma is false and both subalgebras $A_1$ and $A_2$ are $NCF$-algebras. Obviously, it holds $A_1 = B + C_1$, $A_2 = B + C_2$ where $C_1 = C \cap A_1$ and $C_2 = C \cap A_2$. Denote $N_i = C_i \cap \operatorname{Ann}_{C_0}(C_0)$, $i = 1, 2$. Repeating the considerations from the proof of Lemma 14 one can show that $\dim C_i/N_i < \infty$, $i = 1, 2$. Obviously, $N_i$ is an ideal of $C_i, N_i^2 = 0$, $i = 1, 2$. It is easy to see that $A_0 = B + C_3$ where $C_3 = C \cap A_0$. Show that $C_3$ is almost nilpotent. Since $A_0 = A_1 + A_2 + A_1A_2$, we have $A_0 = B + C_1 + C_2 + C_1C_2$. It follows from this equality that $C_3 = C_1 + C_2 + C_1C_2$. Really, the inclusion $C_1 + C_2 + C_1C_2 \subseteq C_3$ is obvious. Now let $c \in C_3 = C \cap A_0$ be any element. Then $c = b + x_1 + y_1 + \sum_{i=2}^{k} x_iy_i$ where $x_i \in C_1$, $y_i \in C_2$, $i = 1, \dots, k$.

Then $b \in C$ and hence $b \in C \cap B$. Since $C \cap B \subseteq C_1$ then $c \in C_1 + C_2 + C_1C_2$ and therefore $C_3 \subseteq C_1 + C_2 + C_1C_2$, because the element $c$ was chosen in any way. Thus $C_3 = C_1 + C_2 + C_1C_2$.

Choose a complete system of representatives $\{x_1, \dots, x_m\}$ of the congruence classes of $C_1$ by $N_1$ and analogous system $\{y_1, \dots, y_n\}$ of $C_2$ by $N_2$. Then

$$N_3 = N_1 + N_2 + \sum_{i=1}^{m} x_i N_2 + \sum_{j=1}^{n} y_j N_1 \subseteq \mathrm{Ann}_{C_0}(C_0),$$

and obviously $\dim C_3/N_3 < \infty$. Since $N_3^2 = 0$, the subalgebra $C_3$ of $A_0$ is almost nilpotent.

Show that $A_0 = B + C_3$ is not an *NCF*-subalgebra. Really, let $A_0$ be conversely an *NCF*-algebra. Then $J = B_0^2 + CB_0^2 + B_0^2C + CB_0^2C$ is an ideal of the algebra $A$ which lies in $A_0$ and hence $J$ is an *NCF*-ideal. By the conditions of the present Lemma and by Lemma 8 $J = 0$ and hence $B_0^2 = 0$. As a sum of two almost nilpotent subalgebras $B$ and $C_3$ the subalgebra $A_0$ is almost nilpotent by Proposition 2. But then $B_0 + B_0C (\subseteq A_0)$ is an almost nilpotent right ideal of the algebra $A$. By Corollary 2 $B_0 + B_0C$ lies in some almost nilpotent ideal of the algebra $A$. In view of conditions of this Lemma and Lemma 8 the latest ideal equals zero and hence $B_0 = 0$. Then obviously $\dim A/C < \infty$ and $A$ is an *NCF*-algebra by Corollary 1. This contradicts to the conditions of Lemma and hence $A_0 = B + C_3$ is not an *NCF*-algebra. By Lemma 10 the quotient algebra $A_0/J_0$ is a minimal $BM$-counter-example for a some *NCF*-ideal $J_0$. On other hand $A_0/J_0 = (B + J_0)/J_0 + (C_3 + J_0)/J_0$ where the subalgebra $C_3 + J_0/J_0$ is almost nilpotent and therefore $A_0/J_0$ is not an $BM$-counter-example by Lemma 15. The obtained contradiction proves the statement of Lemma.

Proof of the Theorem. Let the statement of the Theorem be false. Choose among all counter-examples to the Theorem a such associative algebra $A = B + C$ over a field $F$ which is not *NCF*-algebra and its $F$-subspace $B_0 + C_0$ is of the smallest codimension in $A$ where $B_0$ and $C_0$ are commutative ideals of subalgebras $B$ and corresponding $C$. Denote by $J_0$ the sum of all ideals of the algebra $A$ which lie in $B_0 + C_0$. Then $J_0$ is an *NCF*-ideal by Lemma 7 and $A/J_0$ is obviously a $BM$-counter-example. Thus one can assume, without loss of generality, that $J = 0$ and $A$ is a minimal $BM$-counter-example.

Note that $A_1 = B + B_0C$ and $A_2 = B + CB_0$ are subalgebras of $A$, and by Lemma 16 at least one of these subalgebras is not an *NCF*-algebra. Let $A_1$, for example, be not an *NCF*-algebra. Then the quotient algebra $A_1/J_1$ for some *NCF*-ideal $J_1$ of $A_1$ is a minimal $BM$-counter-example by Lemma 10. Denote $I_1 = \mathrm{Ann}_{B_0}(B_0)$. Obviously, $I_1$ is a right nilpotent ideal of the subalgebra $A_1 = B+B_0C$. Then $I_1$ lies in some nilpotent ideal $S_1$ of this algebra, and since the quotient algebra $A_1/J_1$ has not nonzero *NCF*-ideals (see Lemma 8), $S_1 \subseteq J_1$ and hence $I_1 \subseteq J_1$. We have $[B_0, B] \subseteq B_0$ ($B_0$ is a commutative ideal in $B$) and repeating the consideration from the proof of Lemma 2 one can show that $[B_0, B] \subseteq \mathrm{Ann}_{B_0}(B_0) = I_1$. But then $A_1/J_1$ is a sum of almost commutative subalgebra $C_1 + J_1/J_1$ and finite dimensional over its center subalgebra $B + J_1/J_1$. Therefore we can assume, without loss of generality, that in the initial minimal $BM$-counter-example $A = B+C$ holds the inclusion $B_0 \subseteq Z(B)$ (otherwise we can replace $A$ by $A_1/J_1$).

Now consider the right ideal $D_0 = \mathrm{Ann}_B(B_0)$ of the subalgebra $A_1 = B + B_0C$ (which is not an *NCF*-algebra by our choice). It is easy to see that $T_0 = D_0 + A_1D_0$ is an ideal of the algebra $A_1$ and $T_0 \subseteq \mathrm{Ann}^l_{A_1}(B_0)$. Further, denote $I_0 = \mathrm{Ann}^r_{A_1}(T_0)$. Obviously, $I_0$ is an ideal of the subalgebra $A_1$, $I_0 \supseteq B_0$ and $(I_0 \cap T_0)^2 = 0$. The quotient algebra $A_1/(I_0 \cap T_0)$ is not an *NCF*-algebra, because in the contrary case the algebra $A_1$ were also an *NCF*-algebra (in view of nilpotency of the ideal $I_0 \cap T_0$). The latest is impossible. The quotient algebra $A_1/I_0$ contains an almost commutative subalgebra $C_1 + I_0/I_0$ of finite codimension in $A_1/I_0$ (because $I_0 \supseteq B_0$) and therefore by Corollary 1 $A_1/I_0$ is an *NCF*-algebra. Then the quotient algebra $\overline{A}_1 = A_1/T_0$ is not an *NCF*-algebra, because in the contrary case the algebra $A_1/(I_0 \cap T_0)$ were also an *NCF*-algebra in view of embedding $A_1/(I_0 \cap T_0)$ into the product $(A_1/I_0) \times (A_1/T_0)$ and by Proposition 1. Note that $[B, B] \subseteq D_0 \subseteq T_0$. Really, we have for any elements $b_1, b_2 \in B$ and $b_0 \in B_0$

$$(b_1b_2 - b_2b_1)b_0 = b_1b_2b_0 - b_2b_1b_0 = b_1(b_2b_0) - (b_2b_0)b_1 = 0$$

because $b_2b_0 \in B_0 \subseteq Z(B)$. Hence the quotient algebra $\overline{A}_1 = A_1/T_0$ is a sum of the commutative subalgebra $\overline{B} = B + T_0/T_0$ and almost commutative subalgebra $\overline{C}_1 = C_1 + T_0/T_0$ where $C_1 = C \cap A_1$. It easy to see that certain quotient algebra $\overline{A}_1/\overline{S}_1$ is a minimal $BM$-counter-example for some *NCF*-ideal $\overline{S}_1$ from $\overline{A}_1$. We can assume, without loss of generality, that the subalgebra $B$ in original $BM$-counter-example $A = B + C$ is commutative. Repeating the above considerations in respect to one of the subalgebras $C + C_0B$ or $C + BC_0$ we can show that there exist minimal $BM$-counter-examples of the form $A = B + C$ with commutative subalgebras $B$ and $C$. It is impossible in view of [2] (see Lemma 6). The obtained contradiction proves the Theorem.

*Remark.* Let $A = B + C$ be the associative algebra from the main theorem and $B_0 \subseteq B$, $C_0 \subseteq C$ be some commutative ideals of $B$ and respectively $C$ of finite codimensions $p = \dim B/B_0$, $q = \dim C/C_0$. By this theorem the algebra $A$ contains a nilpotent ideal $I$ with almost commutative quotient algebra $A/I$. Let $K/I$ be any commutative ideal of $A/I$ of finite codimension. Then using the main theorem and Lemma 6 one can show that there exist two functions $f(x, y)$ and $g(x, y)$ such that the nilpotency index $n(I) \leq f(p, q)$ and $\dim A/K \leq g(p, q)$.

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