

On the power values of polynomials

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Abstract. In this paper we give a new, generalized version of a result of Brindza, Evertse and Györy, concerning superelliptic equations.

Let $f(x) \in \mathbb{Z}[x]$ be a polynomial of degree n and b be a nonzero integer. For effective upper bounds obtained by Baker's method for the exponent z in the equation

$$(1) \quad f(x) = by^z, \quad x, y, z \in \mathbb{Z} \text{ with } |y| > 1, \quad z > 1$$

we refer to [T], [ST], [Tu1], [Tu2], [ShT], [B1], [BEGy], [Bu].

For a polynomial P let $M(P)$ denote the Mahler height of it (cf. [M]). The purpose of this paper, which is related to a recent observation of BRINDZA on the number of the solutions of a generalization of the Ramanujan–Nagell equation [B3], is to derive a bound for z which is polynomial in $M(f)$. For brevity write $M = M(f)$.

Theorem. *If f has at least two distinct zeros, then*

$$z < cM^{3n} \log^3 |2b|,$$

where c is an effectively computable constant depending only on n .

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Remarks. If f is an irreducible monic and $b = 1$ then this inequality was proved by BRINDZA, GYÖRY and EVERTSE with different constants (see [BEGy], Th. 4). Moreover, if $n > 2$ and f is irreducible then a profound result of GYÖRY (cf. [Gy1] or [Gy2]) makes it possible to substitute cM^{3n} by an effective constant depending only on the discriminant of f .

1. Auxiliary results

To prove our Theorem, we need two lemmas. In what follows, for any non-zero algebraic number α , $h(\alpha)$ and $H(\alpha)$ denotes the logarithmic height and the classical (ordinary) height of α , respectively.

Lemma 1. *Let $\mathbb{K}$ be an algebraic number field of degree n and denote by R and r the regulator and the unit rank of $\mathbb{K}$, respectively. There exists a fundamental system of units $\varepsilon_1, \dots, \varepsilon_r$ for $\mathbb{K}$ so that*

$$h(\varepsilon_i) \leq c^* R, \quad i = 1, \dots, r$$

where c^* is an effectively computable constant depending only on n .

PROOF. This statement is a consequence of Lemma 1 in [BGy]. For other versions of this result cf. [B2] or [H]. $\square$

Lemma 2. *Let $\alpha_1, \dots, \alpha_n$ be nonzero algebraic numbers and let $A_1, \dots, A_n$ be positive real numbers with $A_i \geq \max\{H(\alpha_i), e\}$ for $i = 1, \dots, n$. Furthermore, let $b_1, \dots, b_n$ be rational integers with $\alpha_1^{b_1} \dots \alpha_n^{b_n} \neq 1$ and suppose that B is a positive real number satisfying $B \geq \max_{i=1, \dots, n} |b_i|$ and $B \geq e$. Now we have*

$$|\alpha_1^{b_1} \dots \alpha_n^{b_n} - 1| \geq B^{-c' \log A_1 \dots \log A_n},$$

where c' is an effectively computable constant depending only on n and on the degree of $\mathbb{Q}(\alpha_1, \dots, \alpha_n)$ over $\mathbb{Q}$.

PROOF. This is Theorem 1.2 in [PW]. $\square$

2. Proof of the Theorem

We have two cases to distinguish.

First we assume that f has an irreducible factor $P \in \mathbb{Z}[x]$ of degree $t \geq 2$. Let α be a zero of P , moreover, let R , h , D and r be the regulator, class number, discriminant and unit rank of the field $\mathbb{K} = \mathbb{Q}(\alpha)$, respectively. In the sequel, $c_1, c_2, \dots$ will denote effectively computable positive constants depending only on n . The well-known inequalities

$$hR \leq \sqrt{|D|} (\log |D|)^{n-1}, \quad (\text{cf. e.g. [L]})$$

and

$$|D| \leq n^n M(P)^{2n-2} \leq n^n M^{2n-2} \quad (\text{cf. [M]})$$

imply

$$(2) \quad hR < c_1 M^n.$$

Let a denote the leading coefficient of f and $\beta_1, \dots, \beta_n$ be the zeros of $g(x) = a^{n-1} f(\frac{x}{a})$. Set

$$\Delta(g) = \prod_{\beta_i \neq \beta_j} (\beta_i - \beta_j)^2,$$

and write g in the form $g(x) = P_1^{k_1}(x)P_2(x)$ where P_1 and P_2 are relatively prime polynomials in $\mathbb{Z}[x]$ and P_1 is an irreducible monic of degree t ; (actually $P_1(x) = a^t P(\frac{x}{a})$). Let $\beta_1, \dots, \beta_t$ be the zeros of P_1 and (x, y) be an arbitrary, however, fixed solution to (1). The g.c.d. of the principal ideals $\langle ax - \beta_1 \rangle$ and $\langle g(ax)(ax - \beta_1)^{-k_1} \rangle$ divides $\Delta^n(g)$, therefore, there are integral ideals A, B, C in $\mathbb{K}$ so that

$$(3) \quad A \langle ax - \beta_1 \rangle = BC^w \quad \text{where } w = \frac{z}{(z, k_1)},$$

furthermore,

$$\max\{N_{\mathbb{K}/\mathbb{Q}}(A), N_{\mathbb{K}/\mathbb{Q}}(B)\} \leq |a \cdot b \cdot \Delta(g)|^{n^2}.$$

Hence, by a well-known inequality (cf. for example [Gy3], Lemma 3) and by (2), the ideals A^h and B^h have generators α and β , respectively, with

$$\max\{\overline{|\alpha|}, \overline{|\beta|}\} \leq \exp(c_2 M^{n-1} (\log M)^n \log |2b|).$$

The relation (3) can be written as

$$\alpha(ax - \beta_1)^h = \varepsilon\beta\gamma^w$$

where γ is a generator of C^h and ε is a unit. Let $\varepsilon_1, \dots, \varepsilon_r$ be a fundamental system of units for $\mathbb{K}$ satisfying Lemma 1. Then we can express ε as $\varepsilon = \rho\varepsilon_1^{l_1} \dots \varepsilon_r^{l_r}$ where ρ is a root of unity and we may assume that $\max_{1 \leq i \leq r} |l_i| < w$ (the remaining factors, if any, are incorporated in γ).

If $|ax| \leq M(g) + 1$ then

$$2^z \leq |y|^z \leq (2M(g) + 1)^n$$

and the Theorem is proved. Otherwise, $|ax| > M(g) + 1$ and $|ax - \beta_i| > 1$, $i = 1, \dots, n$ implies

$$\begin{aligned} |ax - \beta_i| &\leq |a^{n-1}by^z|, \quad i = 1, \dots, n, \\ |a^{n-1}by^z|^h &\geq \max_{1 \leq i \leq t} |ax - \beta_i|^h \geq |\varepsilon_1|^{-nw} \dots |\varepsilon_r|^{-nw} |\alpha|^{-n} |\beta|^{-n} |\gamma|^w \end{aligned}$$

and

$$|\gamma| \leq |a^{n-1}b|^{\frac{h}{w}} |y|^{nh} |\alpha|^{\frac{n}{w}} |\beta|^{\frac{n}{w}} \prod_{i=1}^r |\varepsilon_i|^n.$$

If $w < nh$ then by $0.056 < R$ (cf. [Z]) we obtain $w < 20nhR$ and

$$z < c_3 M^{n-1} (\log(2M))^{n-1}.$$

In case of $w \geq nh$

$$|\gamma| \leq M |b|^{\frac{1}{n}} |y|^{nh} |\alpha| |\beta| \prod_{i=1}^r |\varepsilon_i|^n,$$

and we get

$$\log H \left(\frac{\gamma}{\gamma(2)} \right) \leq c_4 \log |2b| M^{n-1} (\log(2M))^n \log |y|.$$

We may assume that $|ax| \geq \frac{1}{2} |y|^{\frac{z}{n}}$. Indeed, otherwise $\max_{1 \leq i \leq n} |ax - \beta_i| \geq |y|^{\frac{z}{n}}$ yields

$$|ax| \geq |y|^{\frac{z}{n}} - M(g)$$

and the Theorem is proved. Supposing

$$\frac{|\beta_i - \beta_j|}{|ax - \beta_i|} \geq \frac{|\beta_2 - \beta_1|}{|ax - \beta_2|}, \quad 1 \leq i, j \leq t, \quad i \neq j$$

we have

$$\prod_{\substack{1 \leq i, j \leq t \\ \beta_i \neq \beta_j}} \frac{|\beta_i - \beta_j|}{|ax - \beta_i|} \leq \frac{|\Delta(g)| \cdot 2^n}{|y|^z}.$$

Then

$$\frac{|\beta_2 - \beta_1|}{|ax - \beta_2|} \leq |y|^{-\frac{z}{4}},$$

or else we can derive a bound for z better than stated in the Theorem. Avoiding the trivial case $\left(\frac{ax - \beta_1}{ax - \beta_2}\right)^h = 1$, whenever $\frac{1}{2}|y|^{\frac{z}{n}} \leq |\Delta(g)|^{n^2}$ we obtain

$$\log \left| \left(\frac{ax - \beta_1}{ax - \beta_2} \right)^h - 1 \right| \leq \log \left(h \left| \frac{ax - \beta_1}{ax - \beta_2} - 1 \right| \right) \leq -\frac{z}{8} \log |y|.$$

Finally, Lemma 2 yields

$$\begin{aligned} 0 \neq \left| \left(\frac{ax - \beta_1}{ax - \beta_2} \right)^h - 1 \right| &= \left| \left(\frac{\varepsilon_1}{\varepsilon_1^{(2)}} \right)^{l_1 h} \cdots \left(\frac{\varepsilon_r}{\varepsilon_r^{(2)}} \right)^{l_r h} \frac{\beta/\alpha}{\beta^{(2)}/\alpha^{(2)}} \left(\frac{\gamma}{\gamma^{(2)}} \right)^{wh} - 1 \right| \\ &\geq \exp \left(-c_5 \log |2b| M^{3n-3} (\log |2M|)^{3n-1} \log |y| \log w \right) \end{aligned}$$

and the comparison of the upper and lower bounds completes the proof (in the first case).

In the easier second case all the zeros of g are integral. Let k_i denote the multiplicities of β_i , $i = 1, 2$.

Repeating the argument one can have

$$u_i(ax - \beta_i) = v_i y_i^w$$

where $w = \frac{z}{(a, k_1 k_2)}$ and $u_i, v_i, y_i \in \mathbb{Z}$, $|y_i| > 1$, $i = 1, 2$.

To derive a bound for w from the equation

$$Ay_1^w - By_2^w = C$$

($A = u_2v_1$, $B = u_1v_2$, $C = u_1u_2(\beta_2 - \beta_1)$) one can apply Lemma 2 again, and we have

$$\frac{z}{\log z} \leq c_6 \log M \log |2b|,$$

and the Theorem is proved. $\square$

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