

On orthogonally additive functions

By KAROL BARON (Katowice) and PETER VOLKMANN (Karlsruhe)

Daróczy Zoltánnak és Kátai Imrének 60. születésnapjuk alkalmából

Abstract. Under appropriate conditions we decompose orthogonally additive functions into an additive and a quadratic part.

In this paper, G means an abelian group, and V is a vector space over a field Λ , where $\text{char } \Lambda \neq 2$. We also consider some binary relation $\perp$ in V , which we call orthogonality. In fact, the literature offers a lot of possibilities for $\perp$, all of them reflecting some properties of the ordinary orthogonality, which stems from an inner product. For our purposes we only need two properties of $\perp$, which are valid in many of the existing orthogonality spaces $(V, \perp)$; we quote them below as (O), (P).

A function $f : V \rightarrow G$ is called *orthogonally additive* (cf. the survey article by PAGANONI and RÄTZ [6]), if

$$(1) \quad f(x + y) = f(x) + f(y) \quad (x, y \in V; x \perp y),$$

it is called *additive*, if

$$f(x + y) = f(x) + f(y) \quad (x, y \in V),$$

Mathematics Subject Classification: 39B52, 46C99.

Key words and phrases: additive functions, orthogonality spaces, orthogonally additive functions, quadratic functions.

and it is called *quadratic*, if

$$(2) \quad f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad (x, y \in V).$$

Now we are able to formulate (O) and (P):

(O) $0 \perp 0$, and from $x \perp y$ the relations $-x \perp -y$, $\frac{x}{2} \perp \frac{y}{2}$ follow.

(P) If an orthogonally additive function from V to G is odd, then it is additive; if it is even, then it is quadratic.

Orthogonality spaces fulfilling (O), (P) (the abelian group G in (P) being arbitrary) can be found in the papers by RÄTZ [7]–[9], RÄTZ and SZABÓ [10], SZABÓ [12]–[19]. Very often (P) is true because the space V under consideration has the following property (which trivially implies (P)):

(Q) Orthogonally additive functions from V to G are additive.

For example, if V is a real normed space which is not an inner product space, and if $\perp$ is the orthogonality in the sense of Birkhoff and James, then (Q) holds. The proof of this fact has been a longer story, as can be realized from the series of papers by SUNDARESAN [11], GUDDER and STRAWTHER [4], LAWRENCE [5], RÄTZ [7], and SZABÓ [12]; cf. SZABÓ [16]. It will be clear in a moment that in the present paper (Q) is not of interest. So we are rather concerned with orthogonality spaces V satisfying (P) but not (Q); the papers of RÄTZ and SZABÓ [10] and of SZABÓ [14] are good references for this.

Under the assumptions (O), (P) we show in Theorem 1 that every orthogonally additive function $f : V \rightarrow G$ has the form

$$(3) \quad f(x) = a(x) + q(x) \quad (x \in V),$$

a being additive and q being quadratic. This theorem holds without further assumptions on the abelian group G : The case of a 2-torsion-free group has already been treated by RÄTZ and SZABÓ [10], whereas the case of an inner product space V (again no restriction upon G) can be found in [2]. Let us also mention that, with exception of Remark 2, the vector space V always can be replaced by an abelian group V , which is uniquely divisible by two; concerning the orthogonality $\perp$ in V , nothing has to be changed.

Let us start with two lemmas where the relation $\perp$ in V is not needed.

Lemma 1. *Let $f : V \rightarrow G$ satisfy $f(0) = 0$, and suppose (3) to hold, a being an additive function and q a quadratic one. Then a, q are uniquely determined, viz.*

$$(4) \quad a(x) = f\left(\frac{x}{2}\right) - f\left(-\frac{x}{2}\right) \quad (x \in V),$$

$$(5) \quad q(x) = 2 \left[f\left(\frac{x}{2}\right) + f\left(-\frac{x}{2}\right) \right] \quad (x \in V).$$

PROOF. Since $f(0) = a(0) = 0$, we get from (3) that $q(0) = 0$. So $x = 0$ in

$$(6) \quad q(x+y) + q(x-y) = 2q(x) + 2q(y) \quad (x, y \in V)$$

implies that q is an even function. Hence we get from (3) the relation

$$(7) \quad f(-x) = -a(x) + q(x) \quad (x \in V).$$

Subtracting this from (3), replacing x by $\frac{x}{2}$, and using the additivity of a gives (4). Now, addition of (3), (7) and multiplication by 2 yields

$$4q(x) = 2(f(x) + f(-x)) \quad (x \in V).$$

When using (6) with $y = x$, we can replace $4q(x)$ by $q(2x)$, and finally we replace x by $\frac{x}{2}$ to get (5).

Remark 1. From (3), (4), (5) it follows (after replacing x by $2x$) that

$$f(2x) = 3f(x) + f(-x) \quad (x \in V).$$

Lemma 2. *Let $f : V \rightarrow G$ be a function such that $f(0) = 0$. Then f is a solution of (2) if and only if*

$$(8) \quad f(x) = b(x, x) \quad (x \in V)$$

for some biadditive, symmetric $b : V \times V \rightarrow G$. The function b is uniquely determined by f , viz.

$$(9) \quad b(x, y) = f\left(\frac{x+y}{2}\right) - f\left(\frac{x-y}{2}\right) \quad (x, y \in V).$$

PROOF. Suppose first that $f : V \rightarrow G$ satisfies (8) with some biadditive, symmetric $b : V \times V \rightarrow G$. Then (9) is easily established:

$$\begin{aligned} f\left(\frac{x+y}{2}\right) - f\left(\frac{x-y}{2}\right) &= b\left(\frac{x}{2} + \frac{y}{2}, \frac{x}{2} + \frac{y}{2}\right) - b\left(\frac{x}{2} - \frac{y}{2}, \frac{x}{2} - \frac{y}{2}\right) \\ &= 4b\left(\frac{x}{2}, \frac{y}{2}\right) = b(x, y). \end{aligned}$$

Now let $f : V \rightarrow G$ be a solution of (2), such that $f(0) = 0$. Then f is even (put $x = 0$ in (2)). In this step of the proof we define the function $b : V \times V \rightarrow G$ by (9), and we show its biadditivity (by a routine argument; cf. [1], pp. 419, 420): (2) implies

$$(10) \quad b(z_1 + z_2, y) + b(z_1 - z_2, y) = 2b(z_1, y),$$

whence $b(2z_1, y) = 2b(z_1, y)$. Replacing the right hand side of (10) by this and then setting $z_1 = \frac{x_1+x_2}{2}$, $z_2 = \frac{x_1-x_2}{2}$ gives

$$b(x_1, y) + b(x_2, y) = b(x_1 + x_2, y).$$

The rest of the proof (of this lemma) is easy.

Lemma 3. *Suppose (O), (P) to hold, and let $f : V \rightarrow G$ be an orthogonally additive function satisfying $2f = 0$. Then $f = 0$.*

PROOF. From (1) and (O) we get $f(0) = 0$. Now define $g : V \rightarrow G$ by

$$g(x) = f(x) - f(-x) \quad (x \in V).$$

This function is odd, and using (O), we obtain that it is orthogonally additive. Then by (P) it is additive, in particular

$$g(2x) = 2g(x) = 2f(x) - 2f(-x) = 0 \quad (x \in V),$$

whence $g = 0$. So f is an even function, and applying (P) once more gives (2). Now $y = x$ in this formula yields $f(2x) = 4f(x) = 0$ ($x \in V$), whence $f = 0$.

Theorem 1. Suppose (O), (P) to hold, and let $f : V \rightarrow G$ be given. Then f is orthogonally additive, if and only if the following condition is fulfilled:

$$(11) \quad f(x) = a(x) + b(x, x) \quad (x \in V),$$

where $a : V \rightarrow G$ is additive, $b : V \times V \rightarrow G$ biadditive and symmetric, and

$$(12) \quad b(x, y) = 0 \quad (x, y \in V; x \perp y).$$

Moreover, in this case the functions a , b , and $q(x) = b(x, x)$ ($x \in V$) are uniquely determined; they are given by (4),

$$b(x, y) = 2 \left[f\left(\frac{x+y}{4}\right) + f\left(\frac{-x-y}{4}\right) - f\left(\frac{x-y}{4}\right) - f\left(\frac{-x+y}{4}\right) \right] \\ (x, y \in V),$$

and (5), respectively.

PROOF. It is easy to see that (11) defines an orthogonally additive function f , provided a , b are as in the above condition. So, let now f be an orthogonally additive function, i.e., a solution of (1). Define $a, q_0 : V \rightarrow G$ by

$$a(x) = f\left(\frac{x}{2}\right) - f\left(-\frac{x}{2}\right), \quad q_0(x) = f\left(\frac{x}{2}\right) + f\left(-\frac{x}{2}\right) \quad (x \in V).$$

Obviously a is odd, q_0 is even and, due to (O), both functions are orthogonally additive and $q_0(0) = 0$. Applying (P) we obtain additivity of a and quadraticity of q_0 . Lemma 2 shows the existence of a biadditive, symmetric function $b_0 : V \times V \rightarrow G$ such that

$$q_0(x) = b_0(x, x) \quad (x \in V).$$

Since q_0 is orthogonally additive, we get

$$2b_0(x, y) = 0 \quad (x, y \in V; x \perp y).$$

Consequently the function $b : V \times V \rightarrow G$ defined by

$$b(x, y) = 2b_0(x, y) \quad (x, y \in V)$$

is biadditive, symmetric and satisfies (12). On the other hand,

$$2f(x) = a(2x) + q_0(2x) = 2(a(x) + 2q_0(x)) \quad (x \in V),$$

i.e., the orthogonally additive function $f_0 = f - a - 2q_0$ satisfies $2f_0 = 0$. By Lemma 3 we get $f_0 = 0$, hence

$$f(x) = a(x) + 2q_0(x) = a(x) + b(x, x) \quad (x \in V).$$

This shows that f really fulfils the condition of the theorem. From Lemmas 1, 2 we get the uniqueness of a , b , q , as well as the formulas for their representation.

Remark 2. Assume Λ is a euclidean ordered field (cf. [3]), $\dim V \geq 2$, and $\varphi : V \times V \rightarrow \Lambda$ is bilinear, symmetric, positive definite. According to RÄTZ [8] the orthogonality $\perp$ defined by

$$x \perp y \Leftrightarrow \varphi(x, y) = 0$$

satisfies (O), (P), and every even orthogonally additive function $q : V \rightarrow G$ has the form $q(x) = g(\varphi(x, x))$ ($x \in V$) with an additive $g : \Lambda \rightarrow G$. Together with Theorem 1 it follows that every orthogonally additive function $f : V \rightarrow G$ has the form

$$(13) \quad f(x) = a(x) + g(\varphi(x, x)) \quad (x \in V)$$

with additive functions $a : V \rightarrow G$, $g : \Lambda \rightarrow G$. Moreover,

$$a(x) = f\left(\frac{x}{2}\right) - f\left(-\frac{x}{2}\right) \quad (x \in V),$$

$$g(\lambda) = f\left(\frac{u\sqrt{\lambda}}{\sqrt{2\varphi(u, u)}}\right) + f\left(-\frac{u\sqrt{\lambda}}{\sqrt{2\varphi(u, u)}}\right) \quad (\lambda \in \Lambda, \lambda \geq 0)$$

with an arbitrary (but fixed) $u \in V \setminus \{0\}$. When choosing $\Lambda = \mathbb{R}$, we get a real inner product space V , and (13) can be read as $f(x) = a(x) + g(\|x\|^2)$; this was the result obtained in [2].

Acknowledgment. The authors were supported by the Polish State Committee for Scientific Research (Grant No. 2 P03A 033 11). The second author also gratefully acknowledges the hospitality of the Silesian University at Katowice during his visit in 1997.

References

- [1] A. ALEXIEWICZ, Analiza funkcjonalna, *Państwowe Wydawnictwo Naukowe, Warszawa*, 1969.
- [2] K. BARON and J. RÄTZ, On orthogonally additive mappings on inner product spaces, *Bull. Polish Acad. Sci. Math.* **43** (1995), 187–189.
- [3] W. BENZ, Vorlesungen über Geometrie der Algebren, *Springer, Berlin*, 1973.
- [4] S. GUDDER and D. STRAWTHER, Orthogonally additive and orthogonally increasing functions on vector, *Pacific J. Math.* **58** (1975), 427–436.
- [5] J. LAWRENCE, Orthogonality and additive functions on normed linear spaces, *Colloq. Math.* **49** (1985), 253–255.
- [6] L. PAGANONI and J. RÄTZ, Conditional functional equations and orthogonal additivity, *Aequationes Math.* **50** (1995), 135–142.
- [7] J. RÄTZ, On orthogonally additive mappings, *ibid.* **28** (1985), 35–49.
- [8] J. RÄTZ, On orthogonally additive mappings II, *These Publ.* **35** (1988), 241–249.
- [9] J. RÄTZ, On orthogonally additive mappings III, *Abh. Math. Sem. Univ. Hamburg* **59** (1989), 23–33.
- [10] J. RÄTZ and GY. SZABÓ, On orthogonally additive mappings IV, *Aequationes Math.* **38** (1989), 73–85.
- [11] K. SUNDARESAN, Orthogonality and nonlinear functionals on Banach spaces, *Proc. Amer. Math. Soc.* **34** (1972), 187–190.
- [12] GY. SZABÓ, On mappings, orthogonally additive in the Birkhoff–James sense, *Aequationes Math.* **30** (1986), 93–105.
- [13] GY. SZABÓ, On orthogonality spaces admitting nontrivial even orthogonally additive mappings, *Acta Math. Hungar.* **56** (1990), 177–187.
- [14] GY. SZABÓ, ϕ -orthogonally additive mappings. I, *ibid.* **58** (1991), 101–111.
- [15] GY. SZABÓ, A characterization of generalized inner product spaces (φ -orthogonally additive mappings, III), *Aequationes Math.* **42** (1991), 225–238.
- [16] GY. SZABÓ, Continuous orthogonality spaces, *These Publ.* **38** (1991), 311–322.
- [17] GY. SZABÓ, ϕ -orthogonally additive mappings. II, *Acta Math. Hungar.* **59** (1992), 1–10.
- [18] GY. SZABÓ, A conditional Cauchy equation on normed spaces, *These Publ.* **42** (1993), 265–271.
- [19] GY. SZABÓ, Isosceles orthogonally additive mappings on inner product spaces, *ibid.* **46** (1995), 373–384.

KAROL BARON
 INSTYTUT MATEMATYKI
 UNIwersYTET ŚLĄSKI
 40-007 KATOWICE
 POLAND

PETER VOLKMANN
 MATHEMATISCHES INSTITUT I
 UNIVERSITÄT KARLSRUHE
 76128 KARLSRUHE
 GERMANY

(Received September 24, 1997; revised February 27, 1998)