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On the prime power divisors of the iterates of the Euler-φ function

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Abstract. In this paper we prove that for each fixed $k \geq 1$,

$$\Delta(\varphi_k(n)) := \Omega(\varphi_k(n)) - \omega(\varphi_k(n)) = (1 + o(1))\frac{1}{k}(\log\log x)^k(\log\log\log\log x)$$

holds for almost all n, and that

$$\lim_{x\to\infty} \frac{1}{x}\left\{n \leq x \;\Big|\; \frac{\Delta(\varphi(n)) - s(x)}{\sqrt{\log\log x}(\log\log\log\log x)} < z\right\} = \Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-t^2/2}dt,$$

where $s(x) = (\log\log x)(\log\log\log\log x) + c_1\log\log x + o(\log\log x)$.

1. Introduction

Let $\varphi_k(n) = \varphi(\varphi_{k-1}(n))$ $(\varphi_0(n) = n,\ \varphi_1(n) = \varphi(n))$ be the k-fold iterate of the Euler totient function. Let P be the set of primes, and the letters p, q, π, Q with and without suffixes denote prime numbers. Φ is the Gaussian distribution function. $\omega(n)$ counts the number of distinct prime factors of n, $\Omega(n)$ is the number of prime divisors of n counted with multiplicity. Let $\Delta(n) = \Omega(n) - \omega(n)$. As usual $p^r \parallel n$ means that $p^r \mid n$ but $p^{r+1} \nmid n$.

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For the variable x let $x_1 = \log x$, $x_r = \log x_{r-1}$ $(r = 2, 3, \dots)$.

In our recent paper [1] we proved that for each fixed k

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x \;\middle|\; \frac{\omega(\varphi(n)) - a_k x_2^{k+1}}{b_k x_2^{k+1/2}} < z \right\} = \Phi(z) \tag{1.1}$$

$$\lim \frac{1}{\pi(x)} \# \left\{ p \le x \;\middle|\; \frac{\omega(\varphi(p-1)) - a_k x_2^{k+1}}{b_k x_2^{k+1/2}} < z \right\} = \Phi(z) \tag{1.2}$$

where $a_k = \frac{1}{(k+1)!}$, $b_k = \frac{1}{k!\sqrt{2k+1}}$.

We are interested in the distribution of $\Delta(\varphi_k(n))$.

Theorem 1. *For ech fixed $k \geqq 1$, for all but $o(x)$ integers $n \leqq x$,*

$$\Delta(\varphi_k(n)) = a_{k-1}(1 + o(1)) x_2^k x_4 \tag{1.3}$$

holds.

Remark 1. Theorem 1 implies that (1.1) remains valid if we change ω by Ω. For an arbitrary additive function f and an interval $I \subseteq [1, \infty)$ let

$$f(n \mid I) = \sum_{\substack{q^\beta \| n \\ q \in I}} f(q^\beta),$$

and let $f_z(n) = f(n \mid [0, z])$.

We shall prove

Theorem 2. *Let*

$$s(x) := \sum_{p \le x} \frac{\Omega_{x_2^2}(p-1)}{p} + \pi(x_2) + \sum_{x_2 \le q \le x_2^2} (1 - e^{-x_2/q}) - \omega x_2,$$

where

$$\omega = \int_1^\infty \frac{\exp(-\xi)}{\xi^2} d\xi.$$

Then

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x \;\middle|\; \frac{\Delta(\varphi(n)) - s(x)}{\sqrt{x_2} x_4} < z \right\} = \Phi(z).$$

Remark 2. One can prove that $s(x) = x_2 x_4 + c_1 x_2 + o(x_2)$.

2. Lemmata

Let

$$s(x, D, \ell) := \sum_{\substack{p \le x \\ p \equiv \ell \pmod D}} 1/p.$$

Lemma 1. *Uniformly in $1 \le D \le x$ we have*

$$s(x, D, 1) \le \frac{c x_2}{\varphi(D)}. \tag{2.1}$$

Proof. From sieve theorems we know that the number of primes $p \equiv 1 \pmod D$ in the interval $[aD, 2aD]$ is less than $c_1 \frac{aD}{\log a} \frac{1}{\varphi(D)}$. Applying this for $a = 2, 2^2, 2^3, \dots$ one gets that

$$s(x, D, 1) \le \sum_{\substack{p < 2D \\ p \equiv 1 \pmod D}} \frac{1}{p} + \frac{c_1}{\varphi(D)} \sum_{j=1}^{j_0} \frac{1}{2^j D} \frac{2jD}{\log 2^j},$$

where j_0 is the largest integer for which $2^{j_0} D \le x$. The first sum is at most $\frac{1}{D+1}$, the second sum is less than $\frac{c_1}{\varphi(D)} \frac{1}{\log 2} \sum_{j=1}^{j_0} \frac{1}{j} < \frac{c_2}{\varphi(D)} \log j_0 < \frac{c_3}{\varphi(D)} x_2$. Hence (2.1) is immediate. $\square$

Let $U_k(x; D) := \#\{n \le x,\ D \mid \varphi_k(n)\}$.

Lemma 2. *Fo every $k \ge 0$, $r \ge 0$ there exist numerical values $c(k, r) \le c(k, r+1)$ for which*

$$U_k(x; D) \le C(k, \Omega(D)) \frac{x x_2^{k\Omega(D)}}{D}$$

whenever $e^e < x$, $1 \le D \le x$.

Proof. The assertion is true for $k = 0$. $C(0, r) = 1$ is a suitable choice. Assume that it is true for $k-1$ instead of k. Let $D = p_1^{a_1} \dots p_r^{a_r}$, and the prime decomposition of $\varphi_{k-1}(n)$ let $\prod \pi^{\delta_\pi}$. Then

$$\varphi_k(n) = \prod \pi^{(\delta_\pi - 1)} \prod (\pi - 1) = E_1 E_2.$$

Let $p_j^{e_j} \parallel E_1$, $b_j = \min(e_j, a_j)$, $D_1 = \prod_{j=1}^r p_j^{b_j}$, $D_2 = \prod_{j=1}^r p_j^{a_j - b_j}$.

Assume now that $D \mid \varphi_k(n)$. Then, either $D_2 = 1$, or $D_2 > 1$ and there is a product partition of D_2 as $\xi_1 \dots \xi_s$, $\xi_i > 1$ $(i = 1, \dots, s)$ for which $\xi_\ell \mid \pi_\ell - 1$, $\pi_\ell \mid \varphi_{k-1}(n)$ $(\ell = 1, \dots, s)$. It is clear that $s \le \sum_{j=1}^r (a_j - b_j)$. For a fixed choice of $\xi_1, \dots, \xi_s$ let $T = T_s$ run over those numbers $\pi_1, \dots, \pi_s \le x$ for which $\xi_\ell \mid \pi_\ell - 1$ $(\ell = 1, \dots, s)$. Then

$$U_k(x; D) \le \sum_{D_1 D_2 = D} \sum_s \sum_{\substack{\xi_1, \dots, \xi_s \\ \xi_1, \dots, \xi_s = D_2}} \sum_T U_{k-1}(x, D_1 T).$$

Since $\Omega(D_1 T) = \Omega(D_1) + s \le \Omega(D_1) + \Omega(D_2) = \Omega(D)$, from our hypothesis we obtain that

$$U_k(x; D) \le C(k-1, \Omega(D)) x x_2^{(k-1)\Omega(D)} \sum_{D_1 D_2 = D} \sum_s \sum_{\xi_1, \dots, \xi_s} \sum_T \frac{1}{D_1 T}.$$

From Lemma 1 we deduce that the sum $\sum \frac{1}{T}$ is less than $c^s x_2^s / \varphi(\xi_1) \cdots \varphi(\xi_s)$. Furthermore $\varphi(\xi_j) \ge \frac{1}{2} \xi_j$. Since the number of solutions of $D = D_1 D_2$ and that of the number of the product partition of D_2 is bounded by a function of $\Omega(D)$, and $s \le \Omega(D)$, we obtain that the assertion is true for k.

The proof is complete. $\square$

As a consequence we have

Lemma 3. *Let $K_x \to \infty$ arbitrarily slowly, $\omega_0 = K_x$, $\omega_j = x_2^{2j}$ $(j = 1, 2, \dots)$. Let $k \ge 0$ be fixed. Then the number M of that integers $n \le x$ for which there exists at least one prime $p \ge \omega_k$ such that $p^2 \mid \varphi_k(n)$ is $o(x)$.*

Proof. It is enough to observe that

$$M \le \sum_{p \ge \omega_k} U_k(x, p^2), \qquad \sum_{p \ge \omega_k} \frac{1}{p^2} \ll \frac{1}{\omega_k x_3}, \qquad \square$$

Lemma 4. *For $e^e \le y \le x$ we have*

$$\sum_{p \le x} (\Omega_y(p-1) - \log\log y)^2 \ll \frac{x}{\log x} \log\log y,$$

and

$$\sum_{p \le x} \frac{1}{p} \left| \Omega_y(p-1) - \log\log y \right| \ll x_2 (\log\log y).$$

Proof. The first, Turán–Kubilius type inequality can be proved by squaring out, applying the Siegel–Walfisz theorem. The second inequality is an easy consequence of the first one. $\square$

3. Proof of Theorem 1

Let the prime decomposition of $\varphi_{k-1}(n)$ be $\prod \pi^{\delta_\pi}$, $y = x_2^{2k}$. From Lemma 3 we obtain that $\Delta_y(\varphi_k(n)) = \Delta(\varphi_k(n))$ for all but $o(x)$ integers $n \leq x$. Furthermore,

$$\Omega_y(\varphi_k(n)) = \sum_{\substack{\pi \leq y \\ \pi \mid \varphi_{k-1}(n)}} (\delta_\pi - 1) + \sum_{\pi \mid \varphi_{k-1}(n)} \Omega_y(\pi - 1).$$

Let $\eta(n)$ denote the second sum. The first sum is $\Delta_y(\varphi_{k-1}(n))$. Thus we have

$$\Delta_y(\varphi_k(n)) = \Delta_y(\varphi_{k-1}(n)) + \eta(n) - \omega_y(\varphi_k(n)).$$

Since

$$\sum_{n \leq x} \omega_y(\varphi_k(n)) = \sum_{p \leq y} U_{k-1}(x;p),$$

from Lemma 2 and from the obvious inequality $U_{k-1}(x;p) \leq x$ we obtain that the right hand side is less than

$$x\pi(x_2^k) + C(k-1,1)xx_2^k \sum_{x_2^k \leq p \leq y} 1/p \ll xx_2^k.$$

Consequently

$$\frac{1}{x}\#\{n \leq x \mid \omega_y(\varphi_k(n)) \geq x_2^k x_5\} \to 0 \qquad (x \to \infty).$$

Let $\widetilde{\eta}(n) = (\log\log y)\omega(\varphi_{k-1}(n))$. Then

$$|\eta(n) - \widetilde{\eta}(n)| \leq \sum_{\pi \mid \varphi_{k-1}(n)} |\Omega_y(\pi - 1) - \log\log y|,$$

and so by Lemma 2, and Lemma 4

$$\sum_{n\le x}|\eta(n)-\widetilde{\eta}(n)|\le\sum_{\pi\le x}|\Omega_y(\pi-1)-\log\log y|U_{k-1}(x;\pi)$$

$$\ll xx_2^{k-1}\sum_{\pi\le x}|\Omega_y(\pi-1)-\log\log y|\frac{1}{\pi}\ll xx_2^k(\log\log y)^{1/2},$$

Thus

$$\frac{1}{x}\#\left\{n\le x\;\middle|\;|\eta(n)-\widetilde{\eta}(n)|>x_2^kx_4^{3/4}\right\}\to 0\qquad(x\to\infty).$$

Consequently,

$$\Delta(\varphi_k(n))=\Delta_y(\varphi_{k-1}(n))+(\log\log y)\omega(\varphi_{k-1}(n))+O\left(x_2^kx_4^{3/4}\right)$$

for all but $o(x)$ integers $n\le x$.

Since $\omega(\varphi_{k-1}(n))=a_{k-1}x_2^k+O\left(x_2^{k-1/2}x_5\right)$ (see (1.1)) for almost all n, and by induction on k we may assume that $\Delta_y(\varphi_{k-1}(n))=O\left(x_2^{k-1}x_4\right)$, therefore

$$\Delta(\varphi_k(n))=a_{k-1}x_2^kx_4+O\left(x_2^kx_4^{3/4}\right)$$

for almost all n. The proof is complete.

4. Further lemmata

Let $V=\exp(\exp(\sqrt{x_2}\,))$, $Y=\exp(x_1e^{-\sqrt{x_2}}\,)$, $u=\frac{x_1}{\log Y}$, $\beta=\log\frac{\log Y}{\log V}=x_2-2\sqrt{x_2}$, $J_1=\left[\frac{x_2}{x_3},x_2\right]$, $J_2=\left[x_2,x_2^2\right]$, $J=J_1\cup J_2$, $L=[V,Y]$.

Let

$$\psi_0(n):=\prod_{\substack{p<V\\p|n}}(p-1),\quad \psi_1(n):=\prod_{\substack{p|n\\p\in L}}(p-1),\quad \psi_2(n):=\prod_{\substack{p>Y\\p|n}}(p-1),$$

$\psi(n):=\psi_0(n)\psi_1(n)\psi_2(n)$, $T(n)=\prod_{p|n}p^{\alpha_p(n)-1}$ where $\alpha_p(n)$ is defined as the exponent for which $p^{\alpha_p(n)}\parallel n$.

Let

$$\tau(k):=\prod_{\substack{\pi\in L\\ \pi\equiv 1\pmod k}}(1-1/\pi).$$

From the prime number theorem for arithmetical progressions one can get easily that

$$\tau(k)=\exp(-\beta/\varphi(k))\left(1+O\left(\frac{1}{\varphi(k)\log V}\right)\right), \tag{4.1}$$

uniformly as $k\in\left[\sqrt{x_2},x_2^4\right]$, say.

Assume from now on that $q,q_1,q_2\in J$.

Let

$$\varrho(q_1,q_2):=\frac{\tau(q_1)\tau(q_2)}{\tau(q_1q_2)}.$$

Since $\tau(q_1q_2)=1+O\left(\frac{x_2}{q_1q_2}\right)=1+O\left(\frac{x_3^2}{x_2}\right)$, therefore

$$\varrho(q_1,q_2)=\tau(q_1)\tau(q_2)\left(1+O\left(\frac{x_2}{q_1q_2}\right)\right). \tag{4.2}$$

Let

$$\mathcal{A}(q):=\sum_{\substack{n\le x\\ \psi_1(n)\not\equiv 0 \pmod q}}1, \tag{4.3}$$

$$\mathcal{B}(q_1,q_2):=\sum_{\substack{n\le x\\ \psi_1(n)\equiv 0 \pmod{q_1}\\ \psi_1(n)\not\equiv 0 \pmod{q_2}}}1, \tag{4.4}$$

$$\mathcal{C}(q_1,q_2):=\sum_{\substack{n\le x\\ \psi_1(n)\not\equiv 0 \pmod{q_j}\\ j=1,2}}1. \tag{4.5}$$

As an immediate consequence of Theorem 2.5 in Halberstam–Richert [7], we obtain that

$$\mathcal{A}(q)=x\tau(q)(1+O(e^{-u})), \tag{4.6}$$

$$\mathcal{C}(q_1,q_2)=x\tau(q_1)\tau(q_2)\left(1+O\left(\frac{x_2}{q_1q_2}\right)\right). \tag{4.7}$$

Since $\mathcal{B}(q_1,q_2)+\mathcal{C}(q_1,q_2)=\mathcal{A}(q_2)$, therefore

$$\mathcal{B}(q_1,q_2)=x\tau(q_2)(1-\tau(q_1))+O\left(\frac{xx_2}{q_1q_2}\right). \tag{4.8}$$

Lemma 5. *Let*

$$K_1 := \sum_{n\le x} \left[\omega(\psi_1(n)|J_2) - A_x\right]^2, \tag{4.9}$$

where

$$A_x = \sum_{q\in J_2} (1-\tau(q)). \tag{4.10}$$

Then

$$K_1 \ll xx_2. \tag{4.11}$$

Proof. Let $S(x){:=}\sum_{n\le x}\omega(\psi_1(n)|J_2)$; $F(x){:=}\sum_{n\le x}\omega^2(\psi_1(n)|J_2)$. Then

$$\begin{aligned} S(x) &= \sum_{q\in J_2}(x-\mathcal{A}(q)) + O\left(x_2^2\right) = xA_x + O\Bigg(xe^{-u}\sum_{q\in J_2}\tau(q)\Bigg) \\ &= xA_x + O(x), \end{aligned}$$

since $\tau(q)\le 1$ and $\sum\tau(q)\le e^u$.

Furthermore, $F(x) = S(x) + \sum_1$, where

$$\textstyle\sum_1 = \displaystyle\sum_{\substack{q_1\ne q_2\\ q_1,q_2\in J_2}} \ \sum_{\substack{\psi_1(n)\equiv O \pmod{q_j}\\ j=1,2}} 1. \tag{4.12}$$

The inner sum of the right hand side of (4.12) equals to

$$[x] - \mathcal{B}(q_1,q_2) - \mathcal{B}(q_2,q_1) - \mathcal{C}(q_1,q_2).$$

Thus

$$\begin{aligned} \textstyle\sum_1 &= [x]\sum_{q_1\ne q_2} 1 - x\sum_{q\ne q_2}\tau(q_2)(1-\tau(q_2)) + \tau(q_1)(1-\tau(q_2)) \\ &\quad - x\sum_{q_1\ne q_2}\tau(q_1)\tau(q_2) + O(xx_2) = x\sum_{q_1\ne q_2}(1-\tau(q_1))(1-\tau(q_2)) + O(xx_2) \\ &= xA_x^2 + O(xx_2) - x\sum_{q\in J_2}(1-\tau(q))^2. \end{aligned}$$

Since

$$K_1 = F(x) - 2A_x S(x) + A_x^2[x],$$

we have

$$K_1 = xA_x^2 + O(xx_2) + xA_x + O(x) - 2A_x(xA_x + O(x)) \\ + O(xA_x) - x\sum_{q\in J_2}(1-\tau(q))^2 \ll xA_x,$$

and finally, from

$$A_x \ll \sum \beta/q_1 \ll x_2,$$

we obtain that

$$K_1 \ll xx_2.$$

This completes the proof of Lemma 5.

Let

$$\mathcal{B}_x = \sum_{q\in J_1} \tau(q). \tag{4.13}$$

From (4.1) we obtain that

$$\mathcal{B}_x = \sum_{q\in J_1} e^{-\beta/q-1}\left(1 + O\left(\frac{1}{q(\log V)^c}\right)\right) \\ = \sum_{q\in J_1} e^{-\beta/q}\left(1 + O\left(\frac{\beta}{q^2}\right) + O\left(\frac{1}{q(\log V)^c}\right)\right).$$

To estimate the main term

$$T := \sum_{q\in J_1} e^{-\beta/q},$$

we shall use the prime number theorem in the form

$$\Delta(z) := \pi(z) - \ell iz \ll ze^{-c_1\sqrt{x_3}} \quad \text{for } z \in J_1.$$

Thus

$$T = \int_{J_1} e^{-\beta/\eta}\frac{d\eta}{\log\eta} + \int_{J_1} e^{-\beta/\eta} d\Delta(\eta) = T_1 + T_2.$$

By partial intergation we have

$$T_2 = O\left(x_2 e^{-c_1\sqrt{x_3}}\right) + \int |\Delta(\eta)|\,|(e^{\beta/\eta})'|d\eta.$$

The integral on the right hand side is bounded by

$$\ll e^{-c_1\sqrt{x_3}} \int \eta \left(e^{-\beta/\eta}\right)' d\eta = O\left(x_2 e^{-c_1\sqrt{x_3}}\right) + e^{-c_1\sqrt{x_3}} \int_{J_1} e^{-\beta/\eta} d\eta.$$

Since $\frac{1}{\log \eta} = \frac{1}{x_3} + O\left(\frac{x_4}{x_3}\right)$ in $\eta \in J_1$, therefore

$$T_1 = \frac{1}{x_3}\left(1 + O\left(\frac{x_4}{x_3}\right)\right) \int_{J_1} e^{-\beta/\eta} d\eta.$$

Introducing the new variable $\xi = \beta/\eta$ we have

$$\int_{J_1} e^{-\beta/\eta} d\eta = \beta \int_{\beta/x_2}^{\beta x_3/x_2} \frac{\exp(-\xi)}{\xi^2} d\xi. \tag{4.14}$$

Let

$$\omega := \int_1^\infty \frac{\exp(-\xi)}{\xi^2} d\xi. \tag{4.15}$$

The integral on the right hand side of (4.14) equals to

$$\omega + O\left(\frac{1}{\sqrt{x_2}}\right) + O\left(\frac{\exp\left(-\frac{\beta x_3}{x_2}\right)}{x_3^2}\right) = \omega + O\left(\frac{1}{\sqrt{x_2}}\right),$$

consequently

$$T_1 = \beta\omega + O(\sqrt{x_2}),$$

and

$$T_2 = O\left(x_2 e^{-x_1\sqrt{x_3}}\right).$$

One can prove similarly that

$$\sum_{q\in J_1} e^{-\beta/q}\left(O\left(\frac{\beta}{q^2}\right) + O\left(\frac{1}{q(\log V)^c}\right)\right) = O(1).$$

Consequently

$$\mathcal{B}_x = \omega\beta + O(\sqrt{x_2}) = \omega x_2 + O(\sqrt{x_2}). \tag{4.16}$$

Let $h(n)$ be the number of those primes q in J_1 for which $q \nmid \psi_1(n)$.

Lemma 6. *Let*

$$K_2 := \sum_{n\le x}(h(n) - \mathcal{B}_x)^2, \tag{4.17}$$

Then

$$K_2 = O(xx_2). \tag{4.18}$$

Proof. From (4.6), (4.7) we infer that

$$U(x) := \sum_{n\le x} h(n) = \sum_{q\in J_1} \mathcal{A}(q) = x(1+O(e^{-u}))\mathcal{B}_x. \tag{4.19}$$

$$V(x) := \sum_{n\le x} h^2(n) = U(x) + \sum\nolimits_2, \tag{4.20}$$

where

$$\begin{aligned} \sum\nolimits_2 &= \sum_{n\le x} h(n)(h(n)-1) = \sum_{q_1,q_2\in J_1,\, q_1\ne q_2} \mathcal{C}(q_1,q_2) \\ &= x(1+O(e^{-u})) \sum_{q_1\ne q_2,\, q_1,q_2\in J_1} \varrho(q_1,q_2) \qquad (4.21) \\ &= x(1+O(e^{-u})) \sum_{q_1,q_2\in J_1,\, q_1\ne q_2} \tau(q_1)\tau(q_2)\left(1+O\left(\frac{\beta}{q_1q_2}\right)\right). \end{aligned}$$

Consequently

$$\sum\nolimits_2 = x\left(\mathcal{B}_x^2 - \sum_{q\in J_1}\tau^2(q) + O(e^{-u}\mathcal{B}_x^2) + O\left(x_2\left(\sum_{q\in J_1}\frac{\tau(q)}{q}\right)^2\right)\right). \tag{4.22}$$

Since

$$K_2 = U(x) + \sum\nolimits_2 - 2\mathcal{B}_x U(x) + [x]\mathcal{B}_x^2,$$

therefore, by (4.19), (4.20), (4.22) we get that

$$K_2 = O(x\mathcal{B}_x) + O(e^{-u}x\mathcal{B}_x^2) + O\Bigg(x\bigg(\sum_{q\in J_1}\tau^2(q) + x_2\bigg(\sum_{q\in J_1}\frac{\tau(q)}{q}\bigg)^2\bigg)\Bigg).$$

From (4.16), $\mathcal{B}_x \ll x_2$, furthermore

$$\sum_{q\in J_1}\tau^2(q) \ll \frac{x_2}{x_3}, \qquad \sum_{q\in J_1}\frac{\tau(q)}{q} \ll \sum_{q\in J_1}\frac{1}{q} = o(1),$$

thus $K_2 \ll xx_2$.

The proof of the lemma is finished. □

Let H be the number of primes in J_1. Then, by Lemma 6

$$\sum_{n\le x}(\omega(\psi_1(n)|J_1) - (H - \mathcal{B}_x))^2 \ll xx_2,$$

consequently

$$\frac{1}{x}\#\Big\{n \le x,\ |\omega(\psi_1(n)|J_1) - (H - \mathcal{B}_x)| > x_5\sqrt{x_2}\Big\} \ll \frac{1}{x_5^2}. \tag{4.23}$$

Similarly, from Lemma 5,

$$\frac{1}{x}\#\Big\{n \le x,\ |\omega(\psi_1(n)|J_2) - A_x| > x_5\sqrt{x_2}\Big\} \ll \frac{1}{x_5}. \tag{4.24}$$

5. The distribution of $\Omega_{x_2^2}(\psi(n))$

Let

$$A(x) = \sum_{p\le x}\frac{\Omega_{x_2^2}(p-1)}{p}, \qquad \mathcal{B}^2(x) = \sum_{p\le x}\frac{\Omega_{x_2^2}^2(p-1)}{p}. \tag{5.1}$$

We observe that

$$f(n) := \Omega_{x_2^2}(\psi(n)) = \sum_{p|n}\Omega_{x_2^2}(p-1) = \sum_{p|n} f(p)$$

is a strongly additive function depending on the parameter x. To prove that

$$\lim_{x\to\infty} \frac{1}{x} \#\Big\{n \le x,\ \Omega_{x_2^2}(\psi(n)) - A(x) \le z\mathcal{B}(x)\Big\} = \Phi(z), \tag{5.2}$$

we can apply Theorem 12.15 in Elliott [8], and the Berry–Esseen theorem (see Lemma 1.48 in [8]). The conditions of these theorems are satisfied. Indeed, by using the prime number theorem for arithmetical progressions and Lemma 1, we can deduce easily that

$$A(x) = x_2x_4 + O(x_2), \qquad \mathcal{B}^2(x) = (1+o(1))x_2x_4^2, \tag{5.3}$$

and that

$$\sum_{p\le x} \frac{\Omega^3_{x_2^2}(p-1)}{p} \ll x_2x_4^3. \tag{5.4}$$

(5.3), (5.4) imply that

$$\sum_{\substack{x^\epsilon < p \le x \\ |f(p)|>\lambda\mathcal{B}(x)}} \frac{1}{p} \to 0$$

for arbitrary positive constants ϵ and λ, thus the condition of Theorem 12.15 holds. Since the left hand side of (5.4) is $o(\mathcal{B}^3(x))$, therefore the remainder term in the Berry–Esseen theorem is $o(1)$, thus (5.2) holds.

6. Proof of Theorem 2

It is clear that for $\ell = 1, 2$:

$$\begin{aligned} &\omega(\psi_1(n)|J_\ell) \le \omega(\psi(n)|J_\ell) \\ &\le \omega(\psi_1(n)|J_\ell) + \omega(\psi_0(n))|J_\ell) + \omega(T(n)|J_\ell). \end{aligned} \tag{6.1}$$

Let

$$\textstyle\sum_0 = \sum_{n\le x} \omega(\psi_0(n)|J), \quad \sum_1 = \sum_{n\le x} \omega(T(n)|J), \quad \sum_2 = \sum_{n\le x} \omega(\psi_2(n)|J).$$

First we estimate $\sum_1$. We have

$$\sum_1 \ll \sum_{q\in J} \frac{x}{q^2} \ll x.$$

Similarly,

$$\sum_2 \ll \sum_{q\in J} \sum_{\substack{n\le x \\ \psi_2(n)\equiv O \pmod q}} 1 \ll \sum_{q\in J} \sum_{\substack{p\equiv 1 \pmod q \\ Y<p\le x}} x/p$$

$$\ll x \log \frac{\log x}{\log Y} \sum_{q\in J} 1/q \ll x\sqrt{x_2}.$$

Finally

$$\sum_0 \ll \sum_{q\in J} \sum_{\substack{n\le x \\ \psi_0(n)\equiv O \pmod q}} 1 \ll x \sum_{q\in J} \sum_{\substack{q|p-1 \\ p\le V}} 1/p$$

$$\ll x\sqrt{x_2} \sum_{q\in J} \frac{1}{q} = O(x\sqrt{x_2}\,).$$

We deduced that

$$\sum_0 + \sum_1 + \sum_2 = O(x\sqrt{x_2}\,).$$

Consequently

$$\frac{1}{x}\#\Big\{n\le x \mid \big|\omega(\psi(n)|J) - \omega(\psi_1(n)|J)\big| > x_5\sqrt{x_2}\Big\} \to 0. \tag{6.2}$$

We can observe that

$$\frac{1}{x}\#\Big\{n\le x \mid |\Delta(\varphi(n) - \Delta(\psi(n))| \ge K_x\Big\} \to 0,$$

if K_x is an arbitrary function which tends to infinity. Let $K_x = x_5$. Therefore it is enough to prove the theorem for $\psi(n)$ instead of $\varphi(n)$. From Lemma 3 we obtain that the asymptotic density of the integers $n \le x$ for which $p^2 \mid \psi(n)$ for some $p > x_2^2$ is zero. Thus $\Delta(\psi(n)) = \Delta_{x_2^2}(\psi(n))$ for all but $o(x)$ integers up to x. By sieve theorems one can deduce that the number of the integers $n \le x$ for which there is a prime $q \le x_2/x_3$ such that $q \nmid \psi(n)$ is less than

$$cx \sum_{q<x_2/x_3} \prod_{\substack{\pi \equiv 1 \pmod q \\ \pi < x}} \left(1 - \frac{1}{\pi}\right) \le cx \sum_{q<x_2/x_3} \exp\left(-\frac{x_2}{q-1}\right) = o(x).$$

Thus, removing no more than $o(x)$ integers $n \le x$, for the others

$$\omega_{x_2/x_3}(\psi(n)) = \pi(x_2/x_3). \tag{6.3}$$

Hence, and from (5.2) we obtain that

$$\omega_{x_2^2}(\psi(n)) = \pi(x_2/x_3) + \omega(\psi_1(n)|J) + O(x_5\sqrt{x_2}\,)$$

for all but $o(x_2)$ integers $n \le x$. By using (4.23) and (4.24) we get that

$$\omega_{x_2^2}(\psi(n)) = \pi(x_2) + A_x - \mathcal{B}_x + O(x_5\sqrt{x_2}\,) \tag{6.4}$$

for all but $o(x)$ integers $n \le x$.

By (5.2) we have that

$$\frac{1}{x}\#\left\{n \le x \;\Big|\; \frac{\Delta(\varphi(n)) - A(x) - \pi(x_2) - A_x + \mathcal{B}_x}{\sqrt{x_2 x_4}} < z\right\} \to \Phi(z) \quad \text{as } x \to \infty.$$

From (4.1), (4.10) we can deduce easily that

$$A_x = \sum_{x_2 \le q \le x_2^2} \left(1 - \exp\left(-\frac{x_2}{q}\right)\right) + O(\sqrt{x_2}\,).$$

Then, by (4.16) our theorem follows.

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References

[1] N. L. Bassily, I. Kátai and M. Wijsmuller, Number of prime divisors of $\varphi_k(n)$ where φ_k is the k-fold iterate of φ, *Journal of Number Theory* **65** (1997), 226–239.

[2] P. Erdős and C. Pomerance, On the normal number of prime factors of $\varphi(n)$, *Rocky Mountain Journal* **15** (1985), 343–352.

[3] M. Ram Marty and V. Kummar Marty, Analogue of the Erdős–Kac theorem for Fourier coefficients of modular forms, *Indian J. Pure Appl. Math.* **15** (1984), 1090–1101.

[4] I. Kátai, On the number of prime factors of $\varphi(\varphi(n))$, *Acta Math. Hung.* **58** (1–2) (1991), 211–225.

[5] I. kátai, Distribution of $\omega(\sigma(p+1))$, *Ann. Univ. Sci. Budapest, Sectio Math.* **34** (1991), 217–225.

[6] P. Erdős, A. Granville, C. Pomerance and C. Spiro, On the normal behaviour of the iterates of some arithmetic functions, Analytic Number Theory, Proc. Conf. in honor of Paul T. Bateman (B. C. Berndt et al., eds.), *Birkhäuser Boston*, 1990, 165–204.

[7] H. Halberstam and H. Richert, Sieve methods, *Academic Press*, *London, New York*, 1974.

[8] P. D. T. A. Elliott, Probabilistic Number Theory, *Springer Verlag*, *New York*, 1980.

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