

On some tangency relation of sets

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Abstract. In this paper the problem of the tangency relation $T_l(a, b, k, p)$ of sets of the classes $A_{p,k}^*$ having the Darboux property in a generalized metric space (E, l) is considered. Some conditions for this tangency relation will be given here.

Introduction

Let E be an arbitrary non-empty set and let l be a non-negative real function defined on the Cartesian product $E_0 \times E_0$ of the family E_0 of all non-empty subsets of the set E .

Let l_0 be a function defined by the formula

$$(1) \quad l_0(x, y) = l(\{x\}, \{y\}) \quad \text{for } x, y \in E.$$

If we put some conditions on the function l , then the function l_0 defined by (1) will be a metric of the set E . For this reason the pair (E, l) can be treated as a certain generalization of a metric space and we shall call it (see [9]) generalized metric space. Using (1) we may define in the space (E, l) , similarly as in a metric space, the following notions: the sphere $S_l(p, r)$ and the ball $K_l(p, r)$ with centre at the point p and radius r .

Let $S_l(p, r)_u$ denote the so-called u -neighbourhood of the sphere $S_l(p, r)$ in the space (E, l) defined by the formula:

$$(2) \quad S_l(p, r)_u = \begin{cases} \bigcup_{q \in S_l(p, r)} K_l(q, u) & \text{for } u > 0 \\ S_l(p, r) & \text{for } u = 0. \end{cases}$$

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Let k be a fixed positive real number and let a, b be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$(3) \quad a(r) \xrightarrow{r \rightarrow 0^+} 0 \quad \text{and} \quad b(r) \xrightarrow{r \rightarrow 0^+} 0.$$

We say that the pair (A, B) of the sets $A, B \in E_0$ is (a, b) -clustered at the point p of the space (E, l) , if 0 is a cluster point of the set of all real numbers $r > 0$ such that $A \cap S_l(p, r)_{a(r)} \neq \emptyset$ and $B \cap S_l(p, r)_{b(r)} \neq \emptyset$.

Let (see [9])

$$(4) \quad T_l(a, b, k, p) = \left\{ (A, B) : A, B \in E_0, \text{ the pair } (A, B) \right. \\ \left. \text{is } (a, b)\text{-clustered at the point } p \text{ of the space } (E, l) \right. \\ \left. \text{and } \frac{1}{r^k} l(A \cap S_l(p, r)_{a(r)}, B \cap S_l(p, r)_{b(r)}) \xrightarrow{r \rightarrow 0^+} 0 \right\}.$$

If $(A, B) \in T_l(a, b, k, p)$, then we say that the set $A \in E_0$ is (a, b) -tangent of order k to the set $B \in E_0$ at the point p of the space (E, l) .

We call $T_l(a, b, k, p)$ defined by (4) the (a, b) -tangency relation of order k at the point $p \in E$ or shortly: the tangency relation of sets in the generalized metric space (E, l) .

Let ρ be an arbitrary metric of the set E . By A' we shall denote the set of all cluster points of the set $A \in E_0$, and let

$$(5) \quad \rho(x, A) = \inf \{ \rho(x, y) : y \in A \} \quad \text{for } x \in E.$$

Let us put (see [4])

$$(6) \quad A_{p,k}^* = \left\{ A \in E_0 : p \in A' \text{ and there exists a number } \lambda > 0 \right. \\ \left. \text{such that } \limsup_{[A,p;k] \ni (x,y) \rightarrow (p,p)} \frac{\rho(x, y) - \lambda \rho(x, A)}{\rho^k(p, x)} \leq 0 \right\},$$

where

$$(7) \quad [A, p; k] = \{ (x, y) : x \in E, y \in A \\ \text{and } \rho(x, A) < \rho^k(p, x) = \rho^k(p, y) \}.$$

We shall denote by $d_\rho A$ the diameter of the set $A \in E_0$, and by $\rho(A, B)$ the distance of sets $A, B \in E_0$ in the metric space (E, ρ) .

We say that the set $A \in E_0$ has the Darboux property at the point p of the metric space (E, ρ) , written $A \in D_p(E, \rho)$ (see [4]), if there exists a number $\tau > 0$ such that the set $A \cap S_\rho(p, r)$ is non-empty for $r \in (0, \tau)$.

Let f be any subadditive increasing real function defined for all real non-negative numbers, such that $f(0) = 0$. By $\overline{F}_f$ we denote the class of all functions l fulfilling the conditions:

$$\begin{aligned} 1^\circ \quad & l : E_0 \times E_0 \longrightarrow \langle 0, \infty \rangle, \\ 2^\circ \quad & f(\rho(A, B)) \leq l(A, B) \leq f(d_\rho(A \cup B)) \quad \text{for } A, B \in E_0. \end{aligned}$$

Now

$$f(\rho(x, y)) = f(\rho(\{x\}, \{y\})) \leq l(\{x\}, \{y\}) \leq f(d_\rho(\{x\} \cup \{y\})) = f(\rho(x, y));$$

from here and from (1) it follows that

$$(8) \quad l_0(x, y) = l(\{x\}, \{y\}) = f(\rho(x, y)) \quad \text{for } l \in F_f \text{ and } x, y \in E.$$

It is easy to check that the function l_0 defined by (8) is a metric of the set E . From this and from Lemma 1 of the paper [4] it follows that setting $d_l = d_{l_0}$ we get

$$(9) \quad \frac{1}{r^k} d_l(A \cap S_l(p, r)_{a(r)}) \xrightarrow{r \rightarrow 0^+} 0,$$

if $l \in \overline{F}_f$, $A \in A_{p,k}^* \cap D_p(E, l)$ and

$$(10) \quad \frac{a(r)}{r^k} \xrightarrow{r \rightarrow 0^+} 0.$$

In this paper the problem of the tangency relation $T_l(a, b, k, p)$ of sets of the classes $A_{p,k}^*$ having the Darboux property in the generalized metric space (E, l) for $l \in \overline{F}_f$ is considered. Some conditions for this relation have been given here.

1. Two lemmas

Let l_1, l_2 be arbitrary functions belonging to the class $l \in \overline{F}_f$. We put by definition

$$(11) \quad (l_1 + l_2)(A, B) = l_1(A, B) + l_2(A, B) \quad \text{for } A, B \in E_0.$$

Let $S_l(p, r)_u$ be a u -neighbourhood of the sphere $S_l(p, r)$ in the space (E, l) defined by the formula (2).

Lemma 1. *If the functions $l_1, l_2 \in \overline{F}_f$, then*

$$(12) \quad S_{l_1+l_2}(p, r)_u = S_l(p, r/2)_{u/2} \quad \text{for } l \in \overline{F}_f.$$

PROOF. From the equalities (8) and (11), for $l \in \overline{F}_f$ we have

$$\begin{aligned} S_{l_1+l_2}(p, r) &= \{x \in E : (l_1 + l_2)(\{p\}, \{x\}) = r\} \\ &= \{x \in E : l_1(\{p\}, \{x\}) + l_2(\{p\}, \{x\}) = r\} \\ &= \{x \in E : 2l_0(p, x) = r\} \\ &= \{x \in E : l(\{p\}, \{x\}) = r/2\} = S_l(p, r/2). \end{aligned}$$

Therefore if $l_1, l_2 \in \overline{F}_f$ then

$$(13) \quad S_{l_1+l_2}(p, r) = S_l(p, r/2) \quad \text{for } l \in \overline{F}_f.$$

Analogously

$$(14) \quad K_{l_1+l_2}(p, r) = K_l(p, r/2) \quad \text{for } l \in \overline{F}_f.$$

From the definition (2) of the u -neighbourhood of the sphere $S_l(p, r)$, and from (13) and (14) we obtain

$$S_{l_1+l_2}(p, r)_u = S_l(p, r/2)_{u/2},$$

which yields the statement (12) of this lemma.

An immediate consequence of (12) is the following equality

$$(15) \quad S_{l_1+\dots+l_n}(p, r)_u = S_l(p, r/n)_{u/n} \quad \text{for } l, l_1, \dots, l_n \in \overline{F}_f. \quad \square$$

Lemma 2. *If $l \in \overline{F}_f$ and the function a fulfils the condition (10), then*

$$(16) \quad \frac{1}{r^k} d_l(A \cap S_l(p, r/n)_{a(r)/n}) \xrightarrow[r \rightarrow 0^+]{} 0$$

for $n \in \mathbb{N}$ and $A \in A_{p,k}^* \cap D_p(E, l)$.

PROOF. From the assumption (10) it follows that

$$(17) \quad \frac{1}{t^k} a(nt) \xrightarrow[t \rightarrow 0^+]{} 0.$$

Hence and from (9) for $A \in A_{p,k}^* \cap D_p(E, l)$ it results

$$(18) \quad \frac{1}{t^k} d_l(A \cap S_l(p, t)_{a(nt)}) \xrightarrow[t \rightarrow 0^+]{} 0.$$

Putting $r = nt$, from (18) we get

$$(19) \quad \frac{1}{r^k} d_l(A \cap S_l(p, r/n)_{a(r)}) \xrightarrow[r \rightarrow 0^+]{} 0.$$

Since $A \cap S_l(p, r/n)_{a(r)/n} \subset A \cap S_l(p, r/n)_{a(r)}$, we have

$$0 \leq d_l(A \cap S_l(p, r/n)_{a(r)/n}) \leq d_l(A \cap S_l(p, r/n)_{a(r)}).$$

Hence and from (19) we get the statement (16) of this lemma. $\square$

2. The main result

From (16) it follows in particular that

$$(20) \quad \frac{1}{r^k} d_l(A \cap S_l(p, r/2)_{a(r)/2}) \xrightarrow[r \rightarrow 0^+]{} 0,$$

if the function a fulfils the condition (10), $l \in \overline{F}_f$ and $A \in A_{p,k}^* \cap D_p(E, l)$.

From the definition

$$(21) \quad \begin{aligned} & (A, B) \in (T_{l_1}(a, b, k, p) \cup T_{l_2}(a, b, k, p)) \\ \iff & [(A, B) \in (T_{l_1}(a, b, k, p) \vee (A, B) \in T_{l_2}(a, b, k, p))], \end{aligned}$$

for $A, B \in E_0$ and $l_1, l_2 \in \overline{F}_f$.

Theorem 1. *If the non-decreasing functions a, b fulfil the condition*

$$(22) \quad \frac{a(r)}{r^k} \xrightarrow[r \rightarrow 0^+]{} 0 \quad \text{and} \quad \frac{b(r)}{r^k} \xrightarrow[r \rightarrow 0^+]{} 0,$$

and $l_1, l_2 \in \overline{F}_f$, then $(A, B) \in T_{l_1}(a, b, k, p) \cup T_{l_2}(a, b, k, p)$ iff $(A, B) \in T_{l_1+l_2}(a, b, k, p)$ for $A, B \in A_{p,k}^* \cap D_p(E, l)$, where $l \in \overline{F}_f$.

PROOF. We assume that $(A, B) \in T_{l_1}(a, b, k, p) \cup T_{l_2}(a, b, k, p)$. From here and from (21) for instance it follows that $(A, B) \in T_{l_1}(a, b, k, p)$ for $A, B \in A_{p,k}^* \cap D_p(E, l)$. Hence and from the fact that (see (8))

$$(23) \quad S_{l_1}(p, r) = S_{l_2}(p, r) = S_l(p, r) \quad \text{for } l, l_1, l_2 \in \overline{F}_f,$$

it follows

$$(24) \quad \frac{1}{t^k} l_1(A \cap S_l(p, t)_{a(t)}, B \cap S_l(p, t)_{b(t)}) \xrightarrow{t \rightarrow 0^+} 0.$$

Hence, from the fact that $l_1, l_2 \in \overline{F}_f$, from the properties of the function f and from the equality

$$(25) \quad f(d_\rho A) = d_l A \quad \text{for } A \in E_0,$$

we obtain

$$\begin{aligned} & (l_1 + l_2)(A \cap S_{l_1+l_2}(p, r)_{a(r)}, B \cap S_{l_1+l_2}(p, r)_{b(r)}) \\ &= l_1(A \cap S_{l_1+l_2}(p, r)_{a(r)}, B \cap S_{l_1+l_2}(p, r)_{b(r)}) \\ & \quad + l_2(A \cap S_{l_1+l_2}(p, r)_{a(r)}, B \cap S_{l_1+l_2}(p, r)_{b(r)}) \\ &= l_1(A \cap S_l(p, r/2)_{a(r)/2}, B \cap S_l(p, r/2)_{b(r)/2}) \\ & \quad + l_2(A \cap S_l(p, r/2)_{a(r)/2}, B \cap S_l(p, r/2)_{b(r)/2}) \\ &\leq 2f(d_\rho((A \cap S_l(p, r/2)_{a(r)/2}) \cup (B \cap S_l(p, r/2)_{b(r)/2}))) \\ &\leq 2f(d_\rho((A \cap S_l(p, r/2)_{a(r)/2}))) + 2f(d_\rho((B \cap S_l(p, r/2)_{b(r)/2}))) \\ & \quad + 2f(\rho(A \cap S_l(p, r/2)_{a(r)/2}, B \cap S_l(p, r/2)_{b(r)/2})) \\ &\leq 2d_l(A \cap S_l(p, r/2)_{a(r)/2}) + 2d_l(B \cap S_l(p, r/2)_{b(r)/2}) \\ & \quad + 2l_1(A \cap S_l(p, r/2)_{a(r)/2}, B \cap S_l(p, r/2)_{b(r)/2}), \end{aligned}$$

whence

$$\begin{aligned} & \frac{1}{r^k} (l_1 + l_2)(A \cap S_{l_1+l_2}(p, r)_{a(r)}, B \cap S_{l_1+l_2}(p, r)_{b(r)}) \\ (26) \quad & \leq \frac{2}{r^k} d_l(A \cap S_l(p, r/2)_{a(r)/2}) + \frac{2}{r^k} d_l(B \cap S_l(p, r/2)_{b(r)/2}) \\ & \quad + \frac{2}{r^k} l_1(A \cap S_l(p, r/2)_{a(r)/2}, B \cap S_l(p, r/2)_{b(r)/2}). \end{aligned}$$

From here, from (20) and (24), and from the theorems on the compatibility of the tangency relations of sets of the classes $A_{p,k}^*$ (see Theorem 2 and Theorem 3 of the paper [4]) we have

$$(27) \quad \frac{1}{r^k} (l_1 + l_2) (A \cap S_{l_1+l_2}(p, r)_{a(r)}, B \cap S_{l_1+l_2}(p, r)_{b(r)}) \xrightarrow{r \rightarrow 0^+} 0.$$

From the fact that $A, B \in D_p(E, l)$ for $l \in \overline{F}_f$, it follows that there exists a number $\tau > 0$ such that $A \cap S_l(p, r) \neq \emptyset$ and $B \cap S_l(p, r) \neq \emptyset$ for $r \in (0, \tau)$. Because $S_{l_1+l_2}(p, r) = S_l(p, r/2)$ and $r/2 \in (0, \tau/2) \subset (0, \tau)$, then $A \cap S_{l_1+l_2}(p, r) \neq \emptyset$ and $B \cap S_{l_1+l_2}(p, r) \neq \emptyset$, and so $A, B \in D_p(E, l_1+l_2)$. Hence it follows that the pair of sets (A, B) is (a, b) -clustered at the point p of the space (E, l_1+l_2) . From here and from (27) it follows that $(A, B) \in T_{l_1+l_2}(a, b, k, p)$.

Now we assume that $(A, B) \in T_{l_1+l_2}(a, b, k, p)$ for $A, B \in A_{p,k}^* \cap D_p(E, l)$. Hence we have

$$\frac{1}{r^k} (l_1 + l_2) (A \cap S_{l_1+l_2}(p, r)_{a(r)}, B \cap S_{l_1+l_2}(p, r)_{b(r)}) \xrightarrow{r \rightarrow 0^+} 0.$$

From here and from (11) it follows that

$$(28) \quad \frac{1}{r^k} l_1 (A \cap S_{l_1+l_2}(p, r)_{a(r)}, B \cap S_{l_1+l_2}(p, r)_{b(r)}) \xrightarrow{r \rightarrow 0^+} 0.$$

Hence and from Lemma 1 we get

$$\frac{1}{r^k} l_1 (A \cap S_l(p, r/2)_{a(r)/2}, B \cap S_l(p, r/2)_{b(r)/2}) \xrightarrow{r \rightarrow 0^+} 0.$$

From here and from the fact that $l_1 \in \overline{F}_f$

$$(29) \quad \frac{1}{r^k} f(\rho(A \cap S_l(p, r/2)_{a(r)/2}, B \cap S_l(p, r/2)_{b(r)/2})) \xrightarrow{r \rightarrow 0^+} 0.$$

Because for arbitrary sets $A, B \in E_0$

$$\begin{aligned} 0 &\leq \rho(A \cap S_l(p, r/2)_{a(r)}, B \cap S_l(p, r/2)_{b(r)}) \\ &\leq \rho(A \cap S_l(p, r/2)_{a(r)/2}, B \cap S_l(p, r/2)_{b(r)/2}), \end{aligned}$$

from here and from (29) it follows that

$$(30) \quad \frac{1}{r^k} f(\rho(A \cap S_l(p, r/2)_{a(r)}, B \cap S_l(p, r/2)_{b(r)})) \xrightarrow{r \rightarrow 0^+} 0.$$

Hence, from the properties of the function f and from the Theorem 3 of the paper [4] we get

$$(31) \quad \frac{1}{r^k} f(d_\rho((A \cap S_l(p, r/2)_{a(r)}) \cup (B \cap S_l(p, r/2)_{b(r)}))) \xrightarrow{r \rightarrow 0^+} 0.$$

Because for $A, B \in E_0$

$$\begin{aligned} 0 &\leq f(d_\rho((A \cap S_l(p, r/2)_{a(r/2)}) \cup (B \cap S_l(p, r/2)_{b(r/2)}))) \\ &\leq f(d_\rho((A \cap S_l(p, r/2)_{a(r)}) \cup (B \cap S_l(p, r/2)_{b(r)}))), \end{aligned}$$

from here and from (31) it follows that

$$(32) \quad \frac{1}{r^k} f(d_\rho((A \cap S_l(p, r/2)_{a(r/2)}) \cup (B \cap S_l(p, r/2)_{b(r/2)}))) \xrightarrow{r \rightarrow 0^+} 0.$$

Hence and from the fact that $l_1 \in \overline{F}_f$ we obtain

$$\frac{1}{r^k} l_1(A \cap S_l(p, r/2)_{a(r/2)}, B \cap S_l(p, r/2)_{b(r/2)}) \xrightarrow{r \rightarrow 0^+} 0,$$

i.e.

$$(33) \quad \frac{1}{t^k} l_1(A \cap S_l(p, t)_{a(t)}, B \cap S_l(p, t)_{b(t)}) \xrightarrow{t \rightarrow 0^+} 0.$$

From the assumption $A, B \in D_p(E, l)$ for $l \in \overline{F}_f$ and from (23) it follows that $A, B \in D_p(E, l_1)$ for $l_1 \in \overline{F}_f$. Hence we get that the pair of sets (A, B) is (a, b) -clustered at the point p of the space (E, l_1) . From here and from the condition (33) it follows that $(A, B) \in T_{l_1}(a, b, k, p)$, and this means that $(A, B) \in T_{l_1}(a, b, k, p) \cup T_{l_2}(a, b, k, p)$. This ends the proof of the theorem. $\square$

Let $A, B \in E_0$ and $l_1, l_2, \dots, l_n \in \overline{F}_f$; from the definition

$$(34) \quad (A, B) \in \bigcup_{i=1}^n T_{l_i}(a, b, k, p) \iff (A, B) \in T_{l_j}(a, b, k, p)$$

for an $j \in \{1, 2, \dots, n\}$.

Hence and from Theorem 1 we get

Corollary 1. *If the non-decreasing functions a, b fulfil the condition (22) and $l_1, l_2, \dots, l_n \in \overline{F}_f$, then $(A, B) \in \bigcup_{i=1}^n T_{l_i}(a, b, k, p)$ if and only if $(A, B) \in T_{l_1+\dots+l_n}(a, b, k, p)$, for $A, B \in A_{p,k}^* \cap D_p(E, l)$, where $l \in \overline{F}_f$.*

References

- [1] A. CHĄDZYŃSKA, On some classes of sets related to the symmetry of the tangency relation in a metric space, *Ann. Soc. Math. Polon., Comm. Math.* **16** (1972), 219–228.
- [2] S. GOŁĄB and Z. MOSZNER, Sur le contact des courbes dans les espaces metriques généraux, *Colloq. Math.* **10** (1963), 105–311.
- [3] J. GROCHULSKI, T. KONIK and M. TKACZ, On the tangency of sets in metric spaces, *Ann. Polon. Math.* **38** (1980), 121–131.
- [4] T. KONIK, On the compatibility of the tangency relations of sets of the classes $A_{p,k}^*$ in generalized metric spaces, *Demonstratio Math.* **19** (1986), 203–220.
- [5] T. KONIK, On the equivalence of the tangency relation of sets, *Ann. Soc. Math. Polon., Comm. Math.* **31** (1991), 73–78.
- [6] T. KONIK, On the tangency of sets in generalized metric spaces for certain functions of the class F_ρ^* , *Matematički Vesnik (Belgrade)* **43** (1991), 1–10.
- [7] T. KONIK, On the tangency of sets of the class $\tilde{M}_{p,k}$, *Publ. Math. Debrecen* **43** (3–4) (1993), 329–336.
- [8] T. KONIK, On the reflexivity symmetry and transitivity of the tangency relations of sets of the class $\tilde{M}_{p,k}$, *Journal of Geometry (Basel)* **52** (1995), 142–151.
- [9] W. WALISZEWSKI, On the tangency of sets in generalized metric spaces, *Ann. Polon. Math.* **28** (1973), 275–284.

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