

Star-Menger and related spaces

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Abstract. In this paper we introduce and study some notions related to the classical concepts of being a Menger space or a Rothberger space.

1. Introduction and definitions

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a topological space X . Then the symbol $S_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each n , $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$. The symbol $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is an element of $\mathcal{B}$ (see [7], [12]).

We are going now to introduce new selection hypotheses similar to the previous ones. As usual, for a subset A of a space X and a collection $\mathcal{P}$ of subsets of X , $\text{St}(A, \mathcal{P})$ denotes the star of A with respect to $\mathcal{P}$, that is the set $\cup\{P \in \mathcal{P} : A \cap P \neq \emptyset\}$; for $A = \{x\}$, $x \in X$, we write $\text{St}(x, \mathcal{P})$ instead of $\text{St}(\{x\}, \mathcal{P})$. We assume that all spaces are Hausdorff.

1.1. Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a space X . Then:

(a) The symbol $S_1^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each n , $U_n \in \mathcal{U}_n$ and $\{\text{St}(U_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$;

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(b) The symbol $S_{\text{fin}}^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and $\bigcup_{n \in \mathbb{N}} \{\text{St}(V, \mathcal{U}_n) : V \in \mathcal{V}_n\} \in \mathcal{B}$;

(c) By $U_{\text{fin}}^*(\mathcal{A}, \mathcal{B})$ we denote the selection hypothesis: for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of members of $\mathcal{A}$ there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every n , $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{\text{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$ or there is some $n \in \mathbb{N}$ such that $\text{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n) = X$.

1.2. Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a space X and let $\mathcal{K}$ be a family of subsets of X . Then we say that X belongs to the class $SS_{\mathcal{K}}^*(\mathcal{A}, \mathcal{B})$ if X satisfies the following selection hypothesis: for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(K_n : n \in \mathbb{N})$ of elements of $\mathcal{K}$ such that $\{\text{St}(K_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

When $\mathcal{K}$ is the collection of all one-point [resp., finite, compact] subspaces of X we write $SS_1^*(\mathcal{A}, \mathcal{B})$ [resp., $SS_{\text{fin}}^*(\mathcal{A}, \mathcal{B})$, $SS_{\text{comp}}^*(\mathcal{A}, \mathcal{B})$] instead of $SS_{\mathcal{K}}^*(\mathcal{A}, \mathcal{B})$.

1.3. Remark. The following games are naturally corresponded to the selection hypotheses introduced above.

(1) For $S_1^*(\mathcal{A}, \mathcal{B})$ we have the game $G_1^*(\mathcal{A}, \mathcal{B})$ played (on a space X) as follows: Two players, ONE and TWO, play an inning per positive integer. In the n -th inning ONE chooses an $\mathcal{U}_n \in \mathcal{A}$, to which TWO responds by choosing a $U_n \in \mathcal{U}_n$. The play $\mathcal{U}_1, U_1; \dots; \mathcal{U}_n, U_n; \dots$ is won by TWO if $\{\text{St}(U_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$; otherwise, ONE wins;

(2) The game $G_{\text{fin}}^*(\mathcal{A}, \mathcal{B})$ is played similarly, except that in the n -th inning TWO chooses a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$. The play $\mathcal{U}_1, \mathcal{V}_1; \dots; \mathcal{U}_n, \mathcal{V}_n; \dots$ is won by TWO if $\bigcup_{n \in \mathbb{N}} \{\text{St}(V, \mathcal{U}_n) : V \in \mathcal{V}_n\} \in \mathcal{B}$; otherwise, ONE wins;

(3) The game $SG_1^*(\mathcal{A}, \mathcal{B})$ is played in the following way: in the n -th inning ONE chooses some $\mathcal{U}_n \in \mathcal{A}$ and TWO responds by choosing a point $x_n \in X$. The play $\mathcal{U}_1, x_1; \dots; \mathcal{U}_n, x_n; \dots$ is won by TWO if $\{\text{St}(x_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ belongs to $\mathcal{B}$; otherwise, ONE wins;

(4) The game $SG_{\text{fin}}^*(\mathcal{A}, \mathcal{B})$ is played similarly, except that in the n -th inning TWO chooses a finite subset F_n of X . The play $\mathcal{U}_1, F_1; \dots; \mathcal{U}_n, F_n; \dots$ is won by TWO if $\{\text{St}(F_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a member of $\mathcal{B}$; otherwise, ONE wins;

(5) The game $\text{SG}_{\text{comp}}^*(\mathcal{A}, \mathcal{B})$ is played as the previous game, but in the n -th inning TWO chooses a compact subset K_n of X . TWO wins if $\{\text{St}(K_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, the play is won by ONE. $\square$

In this paper $\mathcal{A}$ and $\mathcal{B}$ will be collections of topologically significant open covers of a space X :

$\mathcal{O}$ – the collection of all open covers of X ;

Ω – the collection of ω -covers of X . An open cover $\mathcal{U}$ of X is an ω -cover [4] if X does not belong to $\mathcal{U}$ and every finite subset of X is contained in a member of $\mathcal{U}$;

Γ – the collection of γ -covers of X . An open cover $\mathcal{U}$ of X is a γ -cover [4] if it is infinite and for every $x \in X$ the set $\{U \in \mathcal{U} : x \notin U\}$ is finite.

Recall that a space X is said to have the *Menger property* [9], [5], [6] (resp. the *Rothberger property* [11]) if the selection hypothesis $\text{S}_{\text{fin}}(\mathcal{O}, \mathcal{O})$ (resp. $\text{S}_1(\mathcal{O}, \mathcal{O})$) is true for X (see also [10], [7], [12]).

Following this terminology we introduce the following definition.

1.4. Definition. A space X is said to have: (1) the *star-Rothberger property*, (2) the *star-Menger property*, (3) the *strongly star-Rothberger property*, (4) the *strongly star-Menger property*, (5) *star-K-Menger property* if it satisfies the selection hypothesis: (1') $\text{S}_1^*(\mathcal{O}, \mathcal{O})$, (2') $\text{S}_{\text{fin}}^*(\mathcal{O}, \mathcal{O})$, (3') $\text{SS}_1^*(\mathcal{O}, \mathcal{O})$, (4') $\text{SS}_{\text{fin}}^*(\mathcal{O}, \mathcal{O})$, (5') $\text{SS}_{\text{comp}}^*(\mathcal{O}, \mathcal{O})$.

2. Relations between star covering properties

We give first the following diagram which illustrates relationships between here defined properties and some other star covering properties, whose definitions can be found in [2]. Most of the implications follow almost directly from the definitions; we give a simple one in Proposition 2.1. Recall that a space X is said to be *strongly starcompact* [*strongly star-Lindelöf*, *star-L-Lindelöf*] if for every open cover $\mathcal{U}$ of X there is a finite [countable, Lindelöf] subset A of X such that $\text{St}(A, \mathcal{U}) = X$. X is *starcompact* [*star-Lindelöf*] if for every open cover $\mathcal{U}$ of X there exists a finite [countable] $\mathcal{V} \subset \mathcal{U}$ such that $\text{St}(\cup \mathcal{V}, \mathcal{U}) = X$.

We will also give some examples and assertions in order to compare the properties (and their combinations) from this diagram and to clarify it.

Observe a simple fact that any property from the diagram is an invariant of continuous mappings and is inherited by closed-and-open subspaces.

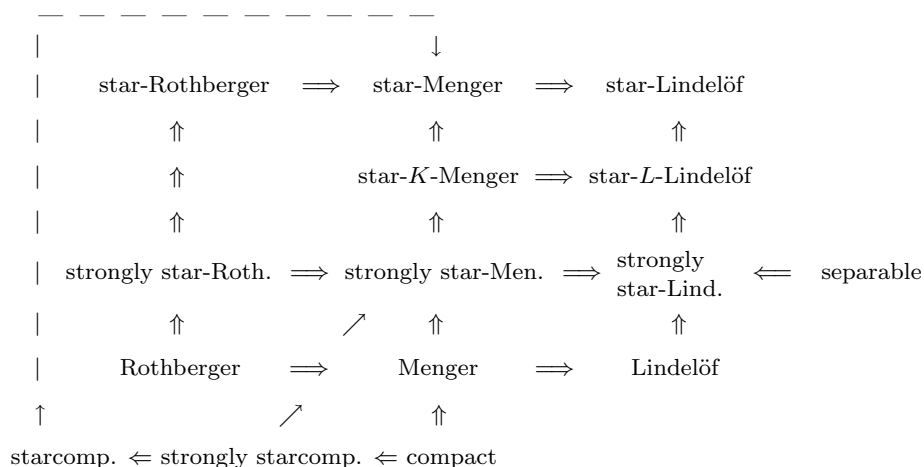


Diagram 1

2.1. Proposition. *Every (strongly)star-Menger space X is (strongly)star-Lindelöf.*

PROOF. Consider only the case when X is a star-Menger space. Let $\mathcal{U}$ be an open cover of X . Then, by definition, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every n , $\mathcal{V}_n$ is a finite subset of $\mathcal{U}$ and $\bigcup_{n \in \mathbb{N}} \text{St}(\cup \mathcal{V}_n, \mathcal{U}) = X$. Then $\mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is a countable subfamily of $\mathcal{U}$ satisfying $\text{St}(\cup \mathcal{V}, \mathcal{U}) = X$, i.e. X is a star-Lindelöf space. $\square$

2.2. (Matveev [8]) *Example.* Let $\mathcal{A}$ be an almost disjoint family of infinite subsets of ω (i.e. the intersection of every two distinct elements of $\mathcal{A}$ is finite) and let $X = \omega \cup \mathcal{A}$ be the Mrówka–Isbell space constructed from $\mathcal{A}$ [3], [2]. Then:

- (i) X is strongly star-Menger $\iff |\mathcal{A}| < \mathbf{d}$;
 - (ii) If $|\mathcal{A}| = \mathbf{c}$, then X is not star-Menger,
- where $\mathbf{d}$ is the dominating number (see [2]).

2.3. Example. There is a strongly star-Menger space X which is not strongly starcompact.

Let $X = [0, \omega_1] \times [0, \omega] \setminus \{(\omega_1, \omega)\}$ be the Tychonoff plank. It is shown in [2] that X is not strongly starcompact. We prove that X is strongly star-Menger.

Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X and let $\mathbb{N} = N_1 \cup N_2 \cup \dots$ be a partition of $\mathbb{N}$ into infinitely many finite pairwise disjoint subsets. Fix $n \in \mathbb{N}$. For each $k \in N_n$ there is an $\alpha_k < \omega_1$ such that the set $\{(\beta, k) : \alpha_k < \beta \leq \omega_1\}$ is contained in some member U of $\mathcal{U}_n$, which means that for $x_k = (\omega_1, k)$ one has $\text{St}(x_k, \mathcal{U}_n) \supset U$. Let $A_n = \{x_k : k \in N_n\}$, $\alpha_n = \sup\{\alpha_k : k \in N_n\}$ and $\alpha = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then $\alpha < \omega_1$ and $\bigcup_{n \in \mathbb{N}} \text{St}(A_n, \mathcal{U}_n) \supset (\alpha, \omega_1] \times [0, \omega]$.

Further, the subspace $T = [0, \omega_1) \times \{\omega\}$ of X is homeomorphic to $[0, \omega_1)$ and consequently T is strongly star-Menger. Thus there is a sequence $(B_n : n \in \mathbb{N})$ of finite subsets of T such that $\bigcup_{n \in \mathbb{N}} \text{St}(B_n, \mathcal{U}_n) \supset T$.

Finally, the subspace $K = [0, \alpha] \times [0, \omega]$ of X is compact and thus strongly star-Menger. There exists a sequence $(C_n : n \in \mathbb{N})$ of finite subsets of K so that $\bigcup_{n \in \mathbb{N}} \text{St}(C_n, \mathcal{U}_n) \supset K$.

For each $n \in \mathbb{N}$ put $F_n = A_n \cup B_n \cup C_n$. Then the sequence $(F_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that X is a strongly star-Menger space. $\square$

Recall that a space X is said to be *meta-compact* [*meta-Lindelöf*] if every open cover $\mathcal{U}$ of X has a point-finite [point-countable] open refinement $\mathcal{V}$ (i.e., every point of X belongs to at most finitely many [countably many] members of $\mathcal{V}$).

2.4. Theorem. Every strongly star-Menger metacompact space is a Menger space.

PROOF. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of a strongly star-Menger metacompact space X . For every $n \in \mathbb{N}$ let $\mathcal{V}_n$ be a point-finite open refinement of $\mathcal{U}_n$. As X is strongly star-Menger, there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $\bigcup_{n \in \mathbb{N}} \text{St}(F_n, \mathcal{V}_n) = X$. Elements of each F_n belong to finitely many members $V_{n,1}, \dots, V_{n,k(n)}$ of $\mathcal{V}_n$; let $\mathcal{V}'_n = \{V_{n,1}, \dots, V_{n,k(n)}\}$. Then $\text{St}(F_n, \mathcal{V}_n) = \bigcup \mathcal{V}'_n$, so that we have $\bigcup_{n \in \mathbb{N}} \bigcup \mathcal{V}'_n = X$. For every $V \in \mathcal{V}'_n$ choose a member U_V of $\mathcal{U}_n$ such that $V \subset U_V$. Then, for every n , $\mathcal{W}_n = \{U_V : V \in \mathcal{V}'_n\}$ is a finite subfamily of $\mathcal{U}_n$ and $\bigcup_{n \in \mathbb{N}} \mathcal{W}_n = X$, i.e. X is a Menger space. $\square$

2.5. Theorem. *Every strongly star-Menger meta-Lindelöf space X is Lindelöf.*

PROOF. Let $\mathcal{U}$ be an open cover of X and let $\mathcal{V}$ be a point-countable refinement of $\mathcal{U}$. Since X is strongly star-Menger there exists a sequence $\{F_n : n \in \mathbb{N}\}$ of finite subsets of X such that $\bigcup_{n \in \mathbb{N}} \text{St}(F_n, \mathcal{V}) = X$. For every $n \in \mathbb{N}$ denote by $\mathcal{W}_n$ the collection of all members of $\mathcal{V}$ which intersect F_n . Since $\mathcal{V}$ is point-countable and F_n is finite, $\mathcal{W}_n$ is countable. So the collection $\mathcal{W} = \bigcup_{n \in \mathbb{N}} \mathcal{W}_n$ is a countable subfamily of $\mathcal{V}$ and is a cover of X . For every $W \in \mathcal{W}$ pick a member $U_W \in \mathcal{U}$ such that $W \subset U_W$. Then $\{U_W : W \in \mathcal{W}\}$ is a countable subcover of $\mathcal{U}$ and X is a Lindelöf space. $\square$

It is known that in the class of Hausdorff spaces strongly starcompactness and countable compactness coincide, so that countable compact Hausdorff spaces are strongly star-Menger. From the previous theorem we obtain the next well known result [1]:

2.6. Corollary. *A countably compact meta-Lindelöf space is compact.*

2.7. Example. *There is a strongly star-Menger space which is not Menger.*

Let $X = [0, \omega_1)$ be the set of all countable ordinals with the order topology. Since X is a Hausdorff countably compact space, i.e. a strongly starcompact space, it is strongly star-Menger. On the other hand, X cannot have the Menger property because it is even not Lindelöf.

The following theorem gives an information when star-Menger spaces satisfy the Menger property.

2.8. Theorem. *For a paracompact (Hausdorff) space X the following are equivalent:*

- (a) X is a star-Menger space;
- (b) X is a star- K -Menger space;
- (c) X is a strongly star-Menger space;
- (d) X is a Menger space.

PROOF. We have to prove only that (a) implies (d). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of a paracompact star-Menger space X . By the well known Stone characterization of paracompactness [3] for every

$n \in \mathbb{N}$ let $\mathcal{V}_n$ be an open star-refinement of $\mathcal{U}_n$. Since X is star-Menger there exists a sequence $\{\mathcal{W}_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a finite subfamily of $\mathcal{V}_n$ and $\bigcup_{n \in \mathbb{N}} \text{St}(\bigcup \mathcal{W}_n, \mathcal{V}_n) = X$. For every $W \in \mathcal{W}_n$ let U_W be a member of $\mathcal{U}_n$ such that $\text{St}(W, \mathcal{V}_n) \subset U_W$. Then $\mathcal{U}'_n = \{U_W : W \in \mathcal{W}_n\}$ is a finite subfamily of $\mathcal{U}_n$ for each $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} \mathcal{U}'_n = X$ which means that X is a Menger space. $\square$

In a similar way we obtain

2.9. Theorem. *A paracompact space X is Rothberger iff it is star-Rothberger iff it is strongly star-Rothberger.*

From Theorem 2.5, Theorem 2.8 and the fact that regular Lindelöf spaces are paracompact we have

2.10. Corollary. *A regular strongly star-Menger meta-Lindelöf space is a Menger space.*

Let us observe that the following result is true without any separation axiom.

2.11. Theorem. *A paracompact space X is star- K -Menger if and only if it is a Menger space.*

PROOF. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X and let for each $n \in \mathbb{N}$, $\mathcal{V}_n$ be an open locally finite refinement of $\mathcal{U}_n$. Since X is star- K -Menger, there exists a sequence $(K_n : n \in \mathbb{N})$ of compact subspaces of X satisfying $\bigcup_{n \in \mathbb{N}} \text{St}(K_n, \mathcal{V}_n) = X$. The set $\mathcal{V}'_n$ of all members of $\mathcal{V}_n$ which meet K_n is finite because $\mathcal{V}_n$ is locally finite and $\bigcup \mathcal{V}'_n = \text{St}(K_n, \mathcal{V}_n)$. Therefore, $\bigcup_{n \in \mathbb{N}} \bigcup \mathcal{V}'_n = X$. For every $V \in \mathcal{V}'_n$ pick a $U_V \in \mathcal{U}_n$ with $V \subset U_V$ and let $\mathcal{W}_n = \{U_V : V \in \mathcal{V}'_n\}$. Then the sequence $(\mathcal{W}_n : n \in \mathbb{N})$ guarantees that X is a Menger space. $\square$

It is easy to check that the star-Menger property is preserved by countable topological sums. However, the product of two star-Menger spaces need not be star-Menger as simple examples show. The same holds for strongly star-Menger spaces.

2.12. Example. *The product of a strongly star-Menger space and a strongly star-Lindelöf space which is not strongly star-Menger.*

The ordinal space $X = [0, \omega_1)$ is strongly star-Menger. Let Y be the set $[0, \omega_1]$ with the following topology: for every $\alpha < \omega_1$ the set $\{\alpha\}$ is open; a set containing ω_1 is open iff its complement in Y is countable. Then Y is a Lindelöf space, hence a strongly star-Lindelöf space. The space $X \times Y$ is not strongly star-Menger because it is not strongly star-Lindelöf as it was shown in [2; Ex. 3.3.3].

However, we have the following result.

2.13. Theorem. *If X is a star-Menger (star-Rothberger) space and Y is a compact space, then $X \times Y$ is a star-Menger (star-Rothberger) space.*

PROOF. We shall prove the star-Menger case. Let $\{\mathcal{W}_n : n \in \mathbb{N}\}$ be a sequence of open covers of $X \times Y$; without loss of generality one can suppose that every $\mathcal{W}_n$ is a basic open cover of the form $\mathcal{U}_n \times \mathcal{V}_n$, $\mathcal{U}_n$ an open cover of X and $\mathcal{V}_n$ an open cover of Y . For a fixed $x \in X$, each $\mathcal{W}_n$ is an open cover for the compact subspace $\{x\} \times Y$ of $X \times Y$. Therefore, there exists a finite subfamily $\mathcal{U}_{n,x} \times \mathcal{V}_{n,x}$ of $\mathcal{W}_n$ such that $\cup(\mathcal{U}_{n,x} \times \mathcal{V}_{n,x}) \supset \{x\} \times Y$. Let $U_{n,x} = \cap \mathcal{U}_{n,x}$. Then $\mathcal{G}_n = \{U_{n,x} : x \in X\}$ is an open cover of X for every $n \in \mathbb{N}$. Since X has the star-Menger property there are finite $\mathcal{H}_n = \{U_{n,x_1}, \dots, U_{n,x_{k(n)}}\} \subset \mathcal{G}_n$, $n \in \mathbb{N}$, such that $\bigcup_{n \in \mathbb{N}} \text{St}(\cup \mathcal{H}_n, \mathcal{G}_n) = X$. Denote $\mathcal{W}'_n = (\mathcal{U}_{n,x_1} \times \mathcal{V}_{n,x_1}) \cup \dots \cup (\mathcal{U}_{n,x_{k(n)}} \times \mathcal{V}_{n,x_{k(n)}})$. We have that for every $n \in \mathbb{N}$, $\mathcal{W}'_n$ is a finite subfamily of $\mathcal{W}_n$ and

$$\bigcup_{n \in \mathbb{N}} \text{St}(\cup \mathcal{W}'_n, \mathcal{W}_n) \supset \bigcup_{n \in \mathbb{N}} \text{St}(\cup \mathcal{H}_n, \mathcal{G}_n) \times Y = X \times Y. \quad \square$$

Matveev observed in [8] that there is a consistent example of a strongly star-Menger space X whose product with a compact space Y is not strongly star-Menger. By Theorem 2.13 $X \times Y$ is a star-Menger space, so that we have a consistent example of a star-Menger space which is not strongly star-Menger.

We close this section by the following three questions.

2.14. Question. Characterize hereditarily (strongly) star-Menger [(strongly) star-Rothberger] spaces.

2.15. Question. Find out a space X such that all finite powers of X are (strongly) star-Menger [resp. (strongly) star-Rothberger] spaces but X^ω is not. Characterize spaces X such that X^ω (resp. every finite power of X) is (strongly) star-Menger [(strongly) star-Rothberger].

2.16. Question. Let $\mathcal{M}$ be the class of spaces X such that for every (strongly) star-Menger [(strongly) star-Rothberger] space Y the product $X \times Y$ is (strongly) star-Menger [(strongly) star-Rothberger]. Describe the class $\mathcal{M}$.

3. Other properties

In [7; Th. 1.1], it was shown that a Lindelöf space X satisfies $S_1(\Gamma, \Gamma)$ if and only if X satisfies $S_{\text{fin}}^*(\Gamma, \Gamma)$. Closely following the line of reasoning from the proof of that result we have:

3.1. Theorem. *For a Lindelöf space X we have $S_1^*(\Gamma, \Gamma) = S_{\text{fin}}^*(\Gamma, \Gamma)$.*

PROOF. Clearly, $S_1^*(\Gamma, \Gamma)$ implies $S_{\text{fin}}^*(\Gamma, \Gamma)$. Let X satisfies $S_{\text{fin}}^*(\Gamma, \Gamma)$ and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of γ -covers of X . Suppose that $\mathcal{U}_n = \{U_{n,1}, U_{n,2}, \dots\}$. We shall define a new sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of γ -covers of X as follows:

$$\mathcal{V}_n = \{V_{n,1}, V_{n,2}, \dots\}, \text{ where } V_{n,k} = U_{1,k} \cap U_{2,k} \cap \dots \cap U_{n,k}.$$

We see that $\mathcal{V}_1 = \mathcal{U}_1$, $\mathcal{V}_i$ refines $\mathcal{U}_i$ for $i \geq 2$ and $V_{n,k} \subset V_{m,k}$ whenever $n \geq m$. Let us check that every $\mathcal{V}_n$ is a γ -cover for X . Let $x \in X$. For every $i = 1, 2, \dots, n$ there is some $m_i \in \mathbb{N}$ such that $x \in U_{i,k}$ for all $k > m_i$. If $m_0 = \max\{m_1, m_2, \dots, m_n\}$, then $x \in V_{n,k}$ for all $k > m_0$.

Since X satisfies $S_{\text{fin}}^*(\Gamma, \Gamma)$ there exists a sequence $(\mathcal{W}_n : n \in \mathbb{N})$, each $\mathcal{W}_n$ a finite subset of $\mathcal{V}_n$, such that $\{\text{St}(W, \mathcal{V}_n) : W \in \mathcal{W}_n, n \in \mathbb{N}\}$ is a γ -cover of X . Now, we use the fact that every $\{\text{St}(W, \mathcal{V}_n) : W \in \mathcal{W}_n\}$ is finite while $\{\text{St}(W, \mathcal{V}_n) : W \in \mathcal{W}_n, n \in \mathbb{N}\}$ is infinite being a γ -cover.

Pick a member $V_{1,k_1} \in \mathcal{W}_1$. Then $X \setminus \text{St}(V_{1,k_1}, \mathcal{V}_1) \neq \emptyset$. Take now some $V_{2,k_2} \in \mathcal{W}_2$ such that $\text{St}(V_{2,k_2}, \mathcal{V}_2) \neq \text{St}(V_{1,k_1}, \mathcal{V}_1)$ for all $V \in \mathcal{W}_1$; we can suppose this because of the fact mentioned above. Then $X \setminus (\text{St}(V_{1,k_1}, \mathcal{V}_1) \cup \text{St}(V_{2,k_2}, \mathcal{V}_2)) \neq \emptyset$. We continue this procedure and obtain a sequence

$$(V_{n,k_n} : n \in \mathbb{N}), \quad V_{n,k_n} \in \mathcal{W}_n$$

such that, by construction, $\{\text{St}(V_{n,k_n}, \mathcal{V}_n) : n \in \mathbb{N}\}$ is a γ -cover of X . It is understood, $\{\text{St}(U_{n,k_n}, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a γ -cover of X witnessing membership of X to the class $S_1^*(\Gamma, \Gamma)$. $\square$

We need now the following simple lemma taken from [7; L. 3.2].

3.2. Lemma. *If $\mathcal{U}$ is an ω -cover of a space X , then $\{U^2 : U \in \mathcal{U}\}$ is an ω -cover of X^2 .*

3.3. Theorem. *If every finite power of a space X satisfies $S_1^*(\Omega, \mathcal{O})$, then X satisfies $S_1^*(\Omega, \Omega)$.*

PROOF. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of ω -covers of X . Let $\mathbb{N} = N_1 \cup N_2 \cup \dots \cup N_n \cup \dots$ be a partition of $\mathbb{N}$ into countably many pairwise disjoint infinite subsets. For every $i \in \mathbb{N}$ and every $j \in N_i$ let $\mathcal{V}_j = \{U^i : U \in \mathcal{U}_j\}$. According to Lemma 3.2, for every $i \in \mathbb{N}$, the sequence $(\mathcal{V}_j : j \in N_i)$ is a sequence of ω -covers of X^i . By assumption, for every $i \in \mathbb{N}$ one can choose a sequence $(U_j^i : j \in N_i)$ so that for each j , $U_j \in \mathcal{U}_j$ and $\{\text{St}(U_j^i, \mathcal{V}_j) : j \in N_i\}$ is an open cover for X^i .

We shall prove that $\{\text{St}(U_j, \mathcal{U}_j) : j \in \mathbb{N}\}$ is an ω -cover for X which witnesses that X satisfies $S_1^*(\Omega, \Omega)$. Indeed, let $A = \{a_1, a_2, \dots, a_p\}$ be a finite subset of X . Then $(a_1, a_2, \dots, a_p) \in X^p$ so that there is some $k \in N_p$ such that $(a_1, a_2, \dots, a_p) \in \text{St}(U_k^p, \mathcal{V}_k)$; it is clear that $A \subset \text{St}(U_k, \mathcal{U}_k)$. $\square$

In a similar way one may prove:

3.4. Theorem. *If every finite power of a space X is a star-Menger space, then X satisfies $S_{\text{fin}}^*(\Omega, \Omega)$.*

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