

Moufang loops of order $2m$, m odd

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Abstract. We first show that every Moufang loop L which contains an abelian associative subloop M of index two and odd order must, in fact, be a group. We then use this to address the question “For what value of $n = 2m$, m odd, must a Moufang loop of order n be associative?”

1. Introduction

This paper is motivated by a question asked by RAJAH and JAMAL in [19]: If L is a Moufang loop of order $2m$ with an abelian associative subloop M of order m , must L be a group? Generalizing a result of LEONG and TEH [13], which gives an affirmative answer in the case that $m = p^2$, p an odd prime, Rajah and Jamal prove that the answer is also affirmative if $m = p_1^2 \dots p_k^2$, or if $M \cong C_p \times C_{p^n}$. We will show that the answer is affirmative for any M of odd order.

Actually, the question raised above stems from other work done by Fook Leong and his students which investigated the question, “For what integers, n , must every Moufang loop of order n be associative?” The first result in this direction may be found in [6], where it is shown that every Moufang loop of prime order must be a group. In [3], the author extended this result to show that Moufang loops of order p^2 , p^3 , and pq , where p and q are distinct primes, must be associative. Since there are well known nonassociative Moufang loops of order 2^4 and 3^4 , it would seem that no extension of the results above is possible. However, in [7], LEONG showed that a Moufang loop of order p^4 , with $p > 3$, must be a group.

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M. PURTILL [16] extended the result to Moufang loops of orders pqr , and p^2q , (p, q and r distinct primes), although the proof of the latter result has a flaw in the case $q < p$; see [17]. Then LEONG and his students produced a spate of papers, [13], [14], [8], [9], [10], culminating in [11], in which LEONG and RAJAH show that any Moufang loop of order $p^\alpha q_1^{\alpha_1} \dots q_n^{\alpha_n}$, with $p < q_1 < \dots < q_n$ odd primes and with $\alpha \leq 3$, $\alpha_i \leq 2$, is a group, and that the same is true with $\alpha = 4$, provided that $p > 3$. Since there exist nonassociative Moufang loops of order 3^4 [1] and of order p^5 for $p > 3$ [20], and since the direct product of a nonassociative Moufang loop and a group is a nonassociative Moufang loop, this result goes a long way toward resolving the problem for odd n . The only remaining cases are $n = p_1^{\alpha_1} \dots p_k^{\alpha_k} q^\beta r_1^{\gamma_1} \dots r_m^{\gamma_m}$, where $p_1 < \dots < p_k < q < r_1 < \dots < r_m$, $k \geq 1$, $\alpha_i \leq 4$ ($\alpha_1 \leq 3$ if $p_1 = 3$), $3 \leq \beta \leq 4$, and $\gamma_i \leq 2$. RAJAH, in his doctoral dissertation [18] showed that, for p and q any odd primes, there exists a nonassociative Moufang loop of order pq^3 if and only if $q \equiv 1 \pmod{p}$, so that there exist nonassociative Moufang loops of order n , for n of the form above, provided that $q \equiv 1 \pmod{p_i}$, for at least one i , or $p_j \equiv 1 \pmod{p_i}$, for some i, j with $i < j$ and $3 \leq \alpha_j \leq 4$.

For n even, many cases are handled by a construction of the author [3] which produces a nonassociative Moufang loop, $M(G, 2)$ of order $2m$ for any nonabelian group G of order m . In particular, since the dihedral group D_r is not abelian, we get a nonassociative Moufang loop of order $4r$, for each $r \geq 3$. This leaves the case $n = 2m$, for m odd. Since there exist nonabelian groups of order p^3 and of order pq for primes $p < q$, with $q \equiv 1 \pmod{p}$, there exist nonassociative Moufang loops of orders $2p^3$ and $2pq$ for p and q as above. For $n < 64$, these account for the only nonassociative Moufang loops of order $2m$, with m odd.¹ As a result, the only the values $n = 2m$ which still need be considered, are those for which $m = p_1^{\alpha_1} \dots p_k^{\alpha_k}$, with $p_1 < \dots < p_k$ odd primes such that no p_j is congruent to 1 modulo any p_i , and with $0 \leq \alpha_i \leq 2$, for all i .

¹See [4] for a discussion of all nonassociative Moufang loops of order < 64 . Table 16 on page 81 contains all three loops of either of the forms above, $M_{42}(G_{21}, 2)$, $M_{54}(B_3, 2)$, and $M_{54}(G_{27}, 2)$, although the former is inexplicably absent from Table 28 on page 129, where it is mistakenly counted as a loop of order 40 rather than 42. Also, while I am on the subject of noting corrections to [4], I would like to thank E.G. Goodaire for observing that the loop $M_{12}(S_3, 2) \times C_3$ is missing (the error can be traced to the argument on the bottom of page 91) and that $M_{48}(5, 5, 5, 3, 6, 0) \cong M_{48}(5, 5, 5, 3, 3, 0)$ and $M_{48}(5, 5, 5, 6, 3, 6) \cong M_{48}(5, 5, 5, 3, 3, 6)$.

Leong and TEH [12] showed that any Moufang loop L of order $2pq$ with $p < q$ odd primes such that $p \nmid (q-1)$ must in fact be a group. This is not surprising since a group of order pq , for p and q as above, must be cyclic and hence, if L contains a subloop of order pq , then L would be a group, since Moufang loops are diassociative. Of course, this in itself is not a proof, since Cauchy's Theorem does not always hold for Moufang loops (for example, PAIGE's simple Moufang loop of order 120 [15] does not contain an element of order 5), and so L might not contain an element of order p or one of order q , and thus it might not contain a subloop of order pq . In a subsequent work [13], LEONG and TEH show that, in fact, a Moufang loop of order $2m$, with m odd, must contain a normal subloop of order m (and so the argument above could now be applied). This fact will be needed in order to prove Corollary 1, below.

2. The main results

Suppose that L is a Moufang loop of order $2m$, m odd, and that L contains a normal abelian subgroup M of order m .

Let u be an element of $L - M$. Then $L = \langle u, M \rangle$, and every element of L can be expressed in the form mu^α , where $m \in M$ and $0 \leq \alpha \leq 1$. Let T_u denote the inner mapping of L corresponding to conjugation by u . That is, for x in L , $uT_u x = u^{-1}xu$. Since M is a normal subloop, T_u maps M to itself. Let θ be the restriction of T_u to M . That is, for every m in M , $m\theta = u^{-1}mu$, and $mu = u(m\theta)$. By diassociativity, $m^2\theta = u^{-1}m^2u = u^{-1}muu^{-1}mu = (m\theta)^2$. Also, since u^2 must be in M , and since M is abelian, u^2 is in the center of M . Thus, $m\theta^2 = u^{-1}(u^{-1}mu)u = u^{-2}mu^2 = m$; so θ^2 is the identity mapping and $\theta^{-1} = \theta$.

By Lemma 3.2 on page 117 of [2], T_u is a semiautomorphism of L . That is, for x, y in L , $(xyx)T_u = (xT_u)(yT_u)(xT_u)$. In particular, for m_1, m_2 in M , $(m_1m_2m_1)\theta = (m_1\theta)(m_2\theta)(m_1\theta)$. But M is abelian, so $(m_1^2m_2)\theta = (m_1\theta)^2(m_2\theta) = (m_1^2\theta)(m_2\theta)$. Since M is of odd order and since the order of an element of a finite Moufang loop must divide the order of the loop, every element of M is of odd order and hence has a square root. (That is, if $|m| = 2k+1$, then $(m^{k+1})^2 = m$.) Thus, for any m, m' in M , $(mm')\theta = [(m'')^2m']\theta = [(m'')^2\theta](m'\theta) = (m\theta)(m'\theta)$, where m'' is the square root of m . Thus θ is an automorphism of M .

For m_1 and m_2 in M , let $x = (m_1u)m_2$, let $y = m_1(m_2u)$, and let $z = (m_1u)(m_2u)$. Then, by the Moufang identities and the fact that M is an abelian group, $xu = [(m_1u)m_2]u = m_1(um_2u) = m_1[u^2(m_2\theta)] = m_1[(m_2\theta)u^2] = [m_1(m_2\theta)]u^2$, so that

$$(m_1u)m_2 = x = [m_1(m_2\theta)]u.$$

Similarly,

$$\begin{aligned} uy &= u[m_1(m_2u)] = u[m_1(u(m_2\theta))] = (um_1u)(m_2\theta) \\ &= [u^2(m_1\theta)](m_2\theta) = u^2[(m_1\theta)(m_2\theta)]. \end{aligned}$$

so that

$$m_1(m_2u) = y = u[(m_1\theta)(m_2\theta)] = [(m_1\theta)(m_2\theta)]\theta u.$$

Finally, $zu = [(m_1u)(m_2u)]u = m_1(um_2u^2) = m_1[u(m_2u^2)]$, so that $uzu = u\{m_1[u(m_2u^2)]\} = (um_1u)(m_2u^2) = [u^2(m_1\theta)](m_2u^2) = [(m_1\theta)m_2]u^4$. Thus, $(z\theta)u^2 = u^2(z\theta) = uzu = [(m_1\theta)m_2]u^4$, so $z\theta = [(m_1\theta)m_2]u^2$, and

$$(m_1u)(m_2u) = z = [(m_1\theta)m_2]\theta u^2.$$

As in [4], we can summarize these results as follows: For $0 \leq \alpha, \beta \leq 1$,

$$(m_1u^\alpha)(m_2u^\beta) = [(m_1\theta^\beta)(m_2\theta^{\alpha+\beta})]\theta^\beta \cdot u^{\alpha+\beta}.$$

But θ is an endomorphism of M , and θ^2 is the identity, so

$$\begin{aligned} (m_1u^\alpha)(m_2u^\beta) &= [(m_1\theta^\beta)(m_2\theta^{\alpha+\beta})]\theta^\beta u^{\alpha+\beta} \\ &= [(m_1\theta^{2\beta})(m_2\theta^{\alpha+2\beta})]u^{\alpha+\beta} = [m_1(m_2\theta^\alpha)]u^{\alpha+\beta}. \end{aligned}$$

But then, for any $m_1u^\alpha, m_2u^\beta, m_3u^\gamma$ in L ,

$$\begin{aligned} [(m_1u^\alpha)(m_2u^\beta)](m_3u^\gamma) &= \{[m_1(m_2\theta^\alpha)]u^{\alpha+\beta}\}(m_3u^\gamma) \\ &= \{[m_1(m_2\theta^\alpha)]m_3\theta^{\alpha+\beta}\}u^{\alpha+\beta+\gamma}, \end{aligned}$$

and

$$\begin{aligned}
 (m_1 u^\alpha)[(m_2 u^\beta)(m_3 u^\gamma)] &= (m_1 u^\alpha)\{[m_2(m_3 \theta^\beta)]u^{\beta+\gamma}\} \\
 &= \{m_1[m_2(m_3 \theta^\beta)]\theta^\alpha\}u^{\alpha+\beta+\gamma} \\
 &= \{m_1[(m_2 \theta^\alpha)(m_3 \theta^{\alpha+\beta})]\}u^{\alpha+\beta+\gamma} \\
 &= \{[m_1(m_2 \theta^\alpha)](m_3 \theta^{\alpha+\beta})\}u^{\alpha+\beta+\gamma}.
 \end{aligned}$$

Thus L is associative.

We have proved the following:

Theorem. *Every Moufang loop L of order $2m$, m odd, which contains a normal abelian subgroup M of order m is a group.*

We can now settle the question of for which values of $n = 2m$ must every Moufang loop of order n be a group.

Corollary 1. *Every Moufang loop of order $2m$ is associative if and only if every group of order m is abelian.*

PROOF. If there exists a nonabelian group G of order m , then the loop $M_n(G, 2)$ is a nonassociative Moufang loop of order $n = 2m$.

As shown above, this covers all even values of m , $m \geq 6$. (There are no nonabelian groups of order less than 6, and there are no nonassociative Moufang loops of order less than 12.)

Now consider $n = 2m$, and suppose that every group of order m is abelian. If $m < 6$, then the result follows from [5], since there are no nonassociative Moufang loops of order less than 12. On the other hand, if $m \geq 6$, then m must be odd (since the dihedral group of order $2k$ is not abelian), and so, by the result of LEONG and TEH discussed above [13], any Moufang loop L of order n must contain a normal subloop M of order m . Since there exists a nonabelian group of order p^3 , for any prime p , m cannot be divisible by p^3 for any prime p . But then, M must be associative, by [11]. Furthermore, since all groups of order m are abelian, M is an abelian group. But then, by the Theorem, L is a group.

3. Some questions

We might wonder whether all of the hypotheses of the Theorem are really necessary.

Clearly it is necessary that M be abelian, since the $M(G, 2)$ construction of [3] provides examples of nonassociative Moufang loops when M is not abelian.

The proof of the Theorem clearly uses the fact that m is odd, but might there be a different proof which gives us the result for m even as well? We thank E.G. Goodaire for noting that the loop $M_{32}(D_4 \times C_2, 2)$ provides a counterexample. This nonassociative Moufang loop contains an abelian normal subgroup of index two which is isomorphic to $C_2 \times C_2 \times C_2 \times C_2$.

How about the fact that M is of index two? In the proof of the Theorem, we do not really need u^2 to be an element of M . All that is needed is that u^2 commutes with every element of M and that it associates with every pair of elements of M . That is, what is needed is that u^2 is in the center of $\langle u^2, M \rangle$. We could therefore prove the following:

Corollary 2. *If a Moufang loop L contains a normal abelian subgroup M of odd order m , such that L/M is cyclic, and if $u^2 \in Z(\langle u^2, M \rangle)$, for uM some generator of L/M , then L is a group.*

Can we dispose with the assumption that $u^2 \in Z(\langle u^2, M \rangle)$? That is,

Question 1. If a Moufang loop L contains a normal abelian subgroup M of odd order m , such that L/M is cyclic, must L be a group?

Returning to the question of whether M must be of odd order, in the counterexample above, M is of order 16 and $|L/M| = 2$. This suggests the following question:

Question 2. If a Moufang loop L contains a normal abelian subgroup M such that L/M is cyclic and such that $(|L/M|, |M|) = 1$, must L be a group?

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