

**The quasiasymptotic expansion at zero
and generalized Watson lemma
for Colombeau generalized functions**

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Abstract. Quasiasymptotic expansion at zero in the Colombeau algebra of generalized functions and its coherence with this notion for Schwartz distributions is given. A version of the Watson lemma related to the expansion of the Laplace transformation of an appropriate generalized Colombeau function is proved. In particular, the asymptotic expansion of δ^2 and the expansion of its Laplace transformation is given.

1. Introduction

Asymptotic analysis is an old subject (cf. [1]) which has a lot of applications in applied mathematics, physics and engineering. It approximates integral expressions or solutions of differential equations.

Since a generalized function g is represented by an ε -net of smooth functions g_ε with the power order growth with respect to ε , ($\varepsilon \rightarrow 0$), the order growth of ε reflects in some sense its singularity. We found that the singularity at zero is characterized through the analysis of the behaviour of $g_\varepsilon(\varepsilon x)$ as $\varepsilon \rightarrow 0$. This was already done for Schwartz distributions, using the quasiasymptotics and quasiasymptotic expansion at zero [9], but since the Colombeau space $\mathcal{G}$ contains elements which are no distributions (δ^2 , for example) we reconsider the concept of quasiasymptotic expansion

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in $\mathcal{G}$. Note that the quasiasymptotic behaviour in $\mathcal{G}$ is used in [8] for the analysis of a non-linear Cauchy problem.

Generalized asymptotic expansion in the space of Schwartz distributions is studied in [1–3], [5], [9], (see also references in [9]), and the asymptotic expansion related to geometric optics in $\mathcal{G}$ is analyzed in [6].

The results of this paper are the following.

First, by using the definition of quasiasymptotic behaviour at zero of Colombeau generalized functions [8], we derive the definition and the properties of quasiasymptotic expansions at zero for Colombeau generalized functions. We compare this notion with the corresponding one for Schwartz distributions. An $f \in \mathcal{D}'$ has a quasiasymptotic expansion if and only if the embedded Colombeau generalized function has a quasiasymptotic expansion in $\mathcal{G}$.

Second, we give an Abelian-type result for the Laplace transformation of an $f \in \mathcal{G}$ which has appropriate quasiasymptotic expansion. This is a generalized version of the classical Watson lemma [1].

Third, we find the quasiasymptotic expansion of the generalized function $\delta^2 \in \mathcal{G} \setminus \mathcal{D}'$ and the asymptotic expansion of its Laplace transformation.

2. Preliminaries

Notation

Schwartz spaces of test functions and distributions on the real line $\mathbb{R}$ are denoted by $\mathcal{D}$ and $\mathcal{D}'$, respectively; $\mathcal{S}$ is the space of rapidly decreasing functions and its dual $\mathcal{S}'$ is the space of tempered distributions. Also, we use the notions $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ where Ω is an open subset of $\mathbb{R}^n$.

Let $\alpha \in \mathbb{R}$. Denote

$$f_{\alpha+1}(x) = \begin{cases} \frac{x^\alpha H(x)}{\Gamma(\alpha+1)}, & \alpha > -1 \\ f_{\alpha+n+1}^{(n)}(x), & \alpha \leq -1, \end{cases}$$

where n is the smallest integer for which $\alpha+n > -1$, and H is the Heaviside function.

Recall that $C^\infty(\Omega)$ is a topological vector space whose topology is given by a countable set of seminorms

$$\mu_k(\varphi) = \sup \left\{ \left| \varphi^{(i)}(x) \right|; |i| \leq k, x \in \Omega_k, k \in \mathbb{N} \right\}.$$

Colombeau algebra of generalized functions

We define $\mathcal{E}_M$ as the space of locally bounded functions $R(\varepsilon) = R_\varepsilon : (0, 1) \rightarrow C^\infty(\Omega)$ such that for every $k \in \mathbb{N}$ there exists $a \in \mathbb{R}$ such that

$$\sup \left\{ \left| R_\varepsilon^{(i)}(x) \right| ; |i| \leq k, x \in \Omega_k \right\} = O(\varepsilon^a), \quad \left(\Omega_k \subset \subset \Omega_{k+1}, \bigcup_{k=1}^{\infty} \Omega_k = \Omega \right).$$

By $\mathcal{N}$ is denoted the space of all elements $H_\varepsilon \in \mathcal{E}_M$ with the property that for every $k \in \mathbb{N}$ and $a \in \mathbb{R}$

$$\sup \left\{ \left| R_\varepsilon^{(i)}(x) \right| ; |i| \leq k, x \in \Omega_k \right\} = O(\varepsilon^a).$$

The quotient space $\mathcal{G} = \mathcal{E}_M / \mathcal{N}$ is a Colombeau space.

In an appropriate way (R_ε and H_ε above do not depend on x) are defined spaces of moderate complex numbers $\mathcal{E}_0$, null spaces $\mathcal{N}_0$, and the space of generalized complex numbers $\mathcal{C} = \mathcal{E}_0 / \mathcal{N}_0$.

It is easy to verify that $\mathcal{G}(\Omega)$ is a differential algebra, where derivations are defined by $R^{(\alpha)} = [R_\varepsilon^{(\alpha)}]$ ($[\cdot]$ denotes equivalence class).

The support of a generalized function H , $\text{supp } H$, is defined as the complement of the largest open subset Ω' such that $H|_{\Omega'} = 0$.

It is said that f belongs to $\mathcal{E}_{t,M}(\mathbb{R}^n)$ if for any $k \in \mathbb{N}$ there exist $a \in \mathbb{R}$ and $m \in \mathbb{N}_0$ such that

$$(1) \quad \sup_{|\alpha| \leq k} \sup_{\mathbb{R}^n} (\langle x \rangle^{-m} |\partial^\alpha f_\varepsilon(x)|) = O(\varepsilon^a).$$

(note $\langle x \rangle \stackrel{\text{def}}{=} (1 + |x|^2)^{1/2}$).

The space of elements g of $\mathcal{E}_{t,M}(\mathbb{R}^n)$ with the property that for every k there exists $m \in \mathbb{N}$ such that (1) holds for every $a \in \mathbb{R}$, is denoted by $\mathcal{N}_t(\mathbb{R}^n)$. It is an ideal of $\mathcal{E}_{t,M}(\mathbb{R}^n)$. The quotient space $\mathcal{G}_t(\mathbb{R}^n) = \mathcal{E}_{t,M}(\mathbb{R}^n) / \mathcal{N}_t(\mathbb{R}^n)$ is called Colombeau space of tempered generalized functions. Note that $\mathcal{G}_t$ is not a subspace of G because

$$\mathcal{N}(\mathbb{R}^n) \cap \mathcal{E}_{t,M}(\mathbb{R}^n) \neq \mathcal{N}_t(\mathbb{R}^n),$$

but there is a canonical mapping: $\mathcal{G}_t(\mathbb{R}^n) \rightarrow \mathcal{G}(\mathbb{R}^n)$, and $[G_\varepsilon] = [G_\varepsilon + \mathcal{N}(\mathbb{R}^n)]$.

Let $\psi \in \mathcal{D}(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\mathcal{F}(\hat{\phi}) = \hat{\phi} \in \mathcal{D}(\mathbb{R}^n)$ and $\hat{\phi} \equiv 1$ on be neighbourhood of zero. Put $\phi_\varepsilon(x) = \frac{1}{\varepsilon^{2n}} \phi(\frac{x}{\varepsilon^2})$, $x \in \mathbb{R}^n$, $\varepsilon \in (0, 1)$. We call ϕ the “vision” function.

If $g \in \mathcal{D}'$, then by

$$\left\langle g(\xi), \epsilon^{-2n} \phi\left(\frac{x-\xi}{\epsilon^2}\right) \right\rangle, \quad x \in \mathbb{R}^n$$

is denoted the representative of the corresponding element in $\mathcal{G}$. Its class is called Colombeau regularization of g and it is denoted by $Cd \, g$.

Quasiasymptotic behaviour

Let $\mathcal{K}$ be a set of positive measurable functions c defined on $(0, 1)$ with the following property:

$$A^{-1}\epsilon^r < c(\epsilon) < A\epsilon^{-r}, \quad \epsilon \in (0, 1)$$

for some $A > 0$ and $r > 0$.

The notion of quasiasymptotic behaviour at zero in $\mathcal{G}(\Omega)$ is introduced in [8].

Let $F \in \mathcal{G}(\Omega)$. If for every $\psi \in \mathcal{D}(\Omega)$ there is $C_\psi \in \mathbb{C}$, $C_\psi \neq 0$, such that

$$\lim_{\epsilon \rightarrow 0^+} \left\langle \frac{F_\epsilon(\epsilon x)}{c(\epsilon)}, \psi(x) \right\rangle = C_\psi,$$

then it is said that F has quasiasymptotics at zero with respect to $c(\epsilon) \in \mathcal{K}$.

The consequences of this definition are contained in Proposition 2 and Proposition 3 of [8].

3. The quasiasymptotic expansion at zero of Colombeau generalized functions

We denote by Λ the set $\mathcal{N}$ or a finite set of the form $\{1, 2, \dots, N\}$, $N \in \mathbb{N}$.

Let $c_k \in K$, $k \in \Lambda$, such that

$$(1) \quad \lim_{\epsilon \rightarrow 0} \frac{c_{k+1}(\epsilon)}{c_k(\epsilon)} \rightarrow 0, \quad k = 1, \dots, N-1 \text{ (or } k \in \mathbb{N}, \text{ if } \Lambda = \mathbb{N})$$

and

$$P_k = [P_\epsilon] \in \mathcal{G}(\Omega), \quad k \in \Lambda.$$

Then $G = [G_\varepsilon] \in \mathcal{G}(\Omega)$ has quasiasymptotic expansion (strong asymptotic expansion) at zero and this equals $\sum_{k \in \Lambda} P_k$ with respect to $\{c_k(\varepsilon); k \in \Lambda\}$ if

$$\frac{(G_\varepsilon - \sum_{k=1}^m P_{k\varepsilon})(\varepsilon x)}{c_m(\varepsilon)} \rightarrow 0, \quad \varepsilon \rightarrow 0^+, \quad \text{in } \mathcal{D}'(\Omega) \text{ for every } m \in \Lambda$$

$$\left(\frac{(G_\varepsilon - \sum_{k=1}^m P_{k\varepsilon})(\varepsilon x)}{c_m(\varepsilon)} \rightarrow 0, \quad \varepsilon \rightarrow 0^+ \text{ for every } x \in \Omega, \text{ and every } m \in \Lambda \right).$$

In this case we write

$$(2) \quad G(\varepsilon x) \stackrel{\text{q.e.c.}}{\sim} \sum_{k \in \Lambda} P_k(\varepsilon x) \text{ with respect to } \{c_k(\varepsilon); k \in \Lambda\}$$

$$\left(G(\varepsilon x) \stackrel{\text{s.e.c.}}{\sim} \sum_{k \in \Lambda} P_k(\varepsilon x) \text{ with respect to } \{c_k(\varepsilon); k \in \Lambda\} \right)$$

and say that G has quasiasymptotic (strong asymptotic) expansion in the Colombeau sense.

If $G \in \mathcal{G}_t(\mathbb{R})$ and $P_k \in \mathcal{G}_t(\mathbb{R})$, $k \in \Lambda$, then with the same definitions we obtain the quasiasymptotic expansion in $\mathcal{G}_t(\mathbb{R})$.

One can simply prove that this definition does not depend on representatives.

In the sequel (if no additional condition is given), we will always assume that $\{c_k(\varepsilon); k \in \Lambda\}$ is a subset of $\mathcal{K}$ satisfying (1).

Remark 1. Let $G \in \mathcal{G}(\mathbb{R})$ ($G \in \mathcal{G}_t(\mathbb{R})$) and $P_k \in \mathcal{G}(\mathbb{R})$ ($P_k \in \mathcal{G}_t(\mathbb{R})$), $k \in \Lambda$. We define the strong asymptotic expansion at ∞ as follows:

G has strong asymptotic expansion at ∞ with respect to $\{c_k(\varepsilon); k \in \Lambda\}$ if

$$\frac{(G_\varepsilon - \sum_{k=1}^m P_{k\varepsilon})\left(\frac{x}{\varepsilon}\right)}{c_m\left(\frac{1}{\varepsilon}\right)} \rightarrow 0, \quad \varepsilon \rightarrow 0^+,$$

for every $x > 0$ and every $m \in \Lambda$.

In this case we write

$$G\left(\frac{x}{\varepsilon}\right) \stackrel{\text{s.e.c.}}{\sim} \sum_{k \in \Lambda} P_k\left(\frac{x}{\varepsilon}\right) \quad \text{at } \infty \text{ with respect to } \{c_k; k \in \Lambda\}.$$

We will use this definition in Proposition 3 below.

Remark 2. We give a definition related to a special choice of P_k , $k \in \Lambda$. Let α_k , $k \in \Lambda$ be an increasing sequence of real numbers, and $G = [G_\varepsilon] \in \mathcal{G}_t(\Omega)$.

Then G has quasiasymptotic expansion at zero as $\sum_{k \in \Lambda} A_k F_{\alpha_k+1}$ with respect to $\{c_k(\varepsilon); k \in \Lambda\}$ if there are complex numbers $A_k \neq 0$, $k \in \Lambda$, such that for any $m \in \Lambda$

$$\frac{(G_\varepsilon - \sum_{k=1}^m A_k F_{\alpha_k+1, \varepsilon})(\varepsilon x)}{c_m(\varepsilon)} \rightarrow 0, \quad \varepsilon \rightarrow 0^+ \text{ in } \mathcal{D}'(\Omega),$$

where $F_{\alpha+1, \varepsilon} = f_{\alpha+1} * \phi_{\varepsilon^2}$. We write

$$G(\varepsilon x) \stackrel{\text{q.e.c.}}{\sim} \sum_{k \in \Lambda} A_k F_{\alpha_k+1}(\varepsilon x) \quad \text{with respect to } \{c_k(\varepsilon); k \in \Lambda\}.$$

In an adequate way one defines the strong asymptotic expansion at zero as $\sum_{k \in \Lambda} A_k F_{\alpha_k+1}$.

Proposition 1. *If (2) holds, then*

- a) $G'(\varepsilon x) \stackrel{\text{q.e.c.}}{=} \sum_{k \in \Lambda} P'_k(\varepsilon x) \quad \text{with respect to } \{c_k(\varepsilon); k \in \Lambda\}.$
- b) $\varphi G(\varepsilon x) \stackrel{\text{q.e.c.}}{\sim} \sum_{k \in \Lambda} \varphi(0) P_k(\varepsilon x) \quad \text{with respect to } \{c_k(\varepsilon); k \in \Lambda\}$

where $\varphi \in C^\infty(\mathbb{R})$.

c) *The strong asymptotic expansion at zero with respect to $\{c_k(\varepsilon); k \in \Lambda\}$ implies the quasiasymptotic expansion at zero in the Colombeau sense if for every compact set $K \subset\subset \Omega$ and $m \in \Lambda$ there exists $\varepsilon_m > 0$ such that*

$$\sup \left\{ \frac{|G_\varepsilon(\varepsilon x) - \sum_{k=1}^m P_{k\varepsilon}(\varepsilon x)|}{c_m(\varepsilon)}; x \in \Omega, \varepsilon \in (0, \varepsilon_m) \right\} < \infty.$$

PROOF. Assertion a) is obvious, c) follows by Lebesgue's theorem on dominated convergence and b) follows from the equivalence of weak and strong convergence in $\mathcal{D}'(\Omega)$ since for every $\psi \in \mathcal{D}(\mathbb{R})$, $\{\psi(x)\varphi(\varepsilon x); \varepsilon \in (0, 1)\}$ is a bounded set in $\mathcal{D}(\mathbb{R})$. $\square$

Recall [5] that if $f \in \mathcal{E}'$ and

$$\frac{\left(f - \sum_{k=1}^N A_k f_{\alpha_k+1}\right)(\varepsilon x)}{c_N(\varepsilon)} \xrightarrow{\mathcal{D}'} 0, \quad \varepsilon \rightarrow 0^+, \quad \text{for every } N \in \Lambda,$$

then we say that

$$f(\varepsilon x) \stackrel{\text{q.e.}}{\sim} \sum_{k \in \Lambda} A_k f_{\alpha_k+1}(\varepsilon x) \quad \text{at zero in } \mathcal{D}'$$

with respect to the scale $\{c_k(\varepsilon); k \in \Lambda\}$.

The next assertion shows the coherence of quasiasymptotic expansions in Colombeau and Schwartz spaces.

A measurable and positive function L defined on $(0, M)$, $M > 0$, is called slowly varying at 0 if

$$\lim_{\varepsilon \rightarrow 0} \frac{L(\varepsilon t)}{L(\varepsilon)} = 1$$

uniformly for $t \in [a, b] \subset (0, M)$. A function of the form $\rho(x) = x^\alpha L(x)$, $x \in (0, M)$ is called a regularly varying function. $\square$

Proposition 2. *Let $f(x) \in \mathcal{E}'(\Omega)$. Then $f(\varepsilon x) \stackrel{\text{q.e.}}{\sim} \sum_{k \in \Lambda} A_k f_{\alpha_k+1}(\varepsilon x)$ with respect to $\{\varepsilon^{\alpha_k} L_k(\varepsilon); k \in \Lambda\}$ if and only if*

$$Cd f(\varepsilon x) \stackrel{\text{q.e.c.}}{\sim} \sum_{k \in \Lambda} A_k F_{\alpha_k+1}(\varepsilon x) \quad \text{with respect to } \{\varepsilon^{\alpha_k} L_k(\varepsilon); k \in \Lambda\}.$$

PROOF. It is proved in [7] that an $f \in \mathcal{S}'(\mathbb{R})$ has quasiasymptotic behaviour at zero in the sense of convergence in $\mathcal{S}'$ if and only if it has quasiasymptotic behaviour at zero in the sense of convergence in $\mathcal{D}'$.

The same result is true for the quasiasymptotic expansion at zero. If $f \in \mathcal{S}'(\mathbb{R})$ then $f(\varepsilon x) \stackrel{\text{q.e.}}{\sim} \sum_{k \in \Lambda} A_k f_{\alpha_k+1}(\varepsilon x)$ with respect to $\{c_k(\varepsilon); k \in \Lambda\}$ in the sense of convergence in $\mathcal{S}'$ iff $f(\varepsilon x) \stackrel{\text{q.e.}}{\sim} \sum_{k \in \Lambda} A_k f_{\alpha_k+1}(\varepsilon x)$ with respect to $\{c_k(\varepsilon); k \in \Lambda\}$ in the sense of convergence in $\mathcal{D}'$.

Let $\alpha \in \mathcal{S}$, $\eta > 0$, $\varepsilon > 0$. We have

$$\begin{aligned} & \left\langle \frac{(f * \phi_\eta)(\varepsilon x) - \sum_{k=1}^m A_m(f_{\alpha_k+1} * \phi_\eta)(\varepsilon x)}{c_m(\varepsilon)}, \alpha(x) \right\rangle \\ &= \left\langle \frac{f(x) - \sum_{k=1}^m A_k f_{\alpha_k+1}(x)}{\varepsilon c_m(\varepsilon)}, (\check{\phi}_\eta(t) * \alpha(t/\varepsilon))(x) \right\rangle \end{aligned}$$

and for $\eta = \varepsilon^2$

$$\begin{aligned} &= \left\langle \frac{(f * \phi_{\varepsilon^2})(\varepsilon x) - \sum_{k=1}^m A_k(f_{\alpha_k+1} * \phi_{\varepsilon^2})(\varepsilon x)}{c_m(\varepsilon)}, \alpha(x) \right\rangle \\ &= \left\langle \frac{f(\varepsilon x) - \sum_{k=1}^m A_k f_{\alpha_k+1}(\varepsilon x)}{c_m(\varepsilon)}, \int_{-\infty}^{\infty} \check{\phi}_{\varepsilon^2}(t) \alpha(x - t/\varepsilon) dt \right\rangle \\ &= \left\langle \frac{f(\varepsilon x) - \sum_{k=1}^m A_k f_{\alpha_k+1}(\varepsilon x)}{c_m(\varepsilon)}, \psi_\varepsilon(x) \right\rangle, \end{aligned}$$

where

$$\psi_\varepsilon(x) = \int_{-\infty}^{\infty} \check{\phi}_{\varepsilon^2}(t) \alpha(x - t/\varepsilon) dt = \int_{-\infty}^{\infty} \check{\phi}(t) \alpha(x - \varepsilon t) dt.$$

Since $\{\psi_\varepsilon; \varepsilon \in (0, 1)\}$ is a bounded set in $\mathcal{S}$ and $\psi_\varepsilon \rightarrow \alpha$ in $\mathcal{S}$, it follows

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \left\langle \frac{(f * \phi_{\varepsilon^2})(\varepsilon x) - \sum_{k=0}^m A_k(f_{\alpha_k+1} * \phi_{\varepsilon^2})(\varepsilon x)}{c_m(\varepsilon)}, \alpha(x) \right\rangle \\ &= \left\langle g(x) - \sum_{k=0}^m A_k f_{\alpha_k+1}(x), \alpha(x) \right\rangle. \end{aligned}$$

This implies the assertion in both directions.

4. Generalized Watson lemma

The Laplace transformation $\mathcal{L}_g$ for an element $G \in \mathcal{G}_t(\mathbb{R})$ supported

by $[0, \infty)$ is defined in [4] by

$$\mathcal{L}_g(G)(p) = \left[\int_{\mathbb{R}} e^{-px} G_\varepsilon(x) \eta(x) dx \right], \quad \operatorname{Re} p > 0,$$

where G_ε is a representative of G , and $\eta \in C_0^\infty(\mathbb{R})$ has the properties $|\eta| \leq 1$; $\eta = 1$ on $[-a/2, \infty)$ and $\eta = 0$ on $(-\infty, -a)$ for some $a > 0$.

We will use the notation $\mathcal{L}$ for the usual distributional or classical Laplace transformation (if it exists).

It is well known that

$$(3) \quad \begin{aligned} \mathcal{L}_g(F_{\alpha+1, \varepsilon})(\lambda) &= \mathcal{L}(f_{\alpha+1} * \phi_{\varepsilon^2})(\lambda) = \mathcal{L}(f_{\alpha+1})\mathcal{L}(\phi_{\varepsilon^2})(\lambda) \\ &= (-i\lambda)^{-\alpha-1} \mathcal{L}(\phi)(\varepsilon^2 \lambda), \quad \operatorname{Re} \lambda > 0. \end{aligned}$$

We will give an Abelian-type result for the Laplace transformation in $\mathcal{G}_t$ by using the quasiasymptotic expansion given in Remark 2.

Proposition 3. *Let $\alpha_k, k \in \mathbb{N}$, be an increasing sequence of complex numbers, $A_k, k \in \mathbb{N}$, be a sequence of real numbers and let $c_k(\varepsilon), k \in \mathbb{N}$, be a sequence in K which satisfies (1).*

Assume that $G \in \mathcal{G}_t(\mathbb{R})$, $\operatorname{supp} G \subset [0, \infty)$ and G has a representative G_ε with the property $\operatorname{supp} G_\varepsilon \subset [-b\varepsilon, \infty)$, $\varepsilon \leq \varepsilon_0$, for some $b > 0$. Let

$$G(\varepsilon x) \stackrel{\text{q.e.c.}}{\sim} \sum_{k=1}^{\infty} A_k F_{\alpha_k+1}(\varepsilon x) \quad \text{with respect to } \{c_k(\varepsilon), k \in \mathbb{N}\}.$$

Then

$$\mathcal{L}_g(G) \left(\frac{x}{\varepsilon} \right) \stackrel{\text{s.e.c.}}{\sim} \sum_{k=1}^{\infty} A_k i^{\alpha_k+1} [t^{-\alpha_k-1} \mathcal{L}(\phi)(\varepsilon^3 t)] \left(\frac{x}{\varepsilon} \right) \quad \text{at } \infty,$$

with respect to $\{c_k(\varepsilon); k \in \mathbb{N}\}$.

PROOF. We again use the fact that

$$f(\varepsilon x) \stackrel{\text{q.e.}}{\sim} \sum_{k=1}^m A_k f_{\alpha_k+1}(\varepsilon x) \quad \text{with respect to } \{c_k(\varepsilon); k \in \Lambda\}$$

in the sense of convergence in $\mathcal{S}'$ if and only if this holds in the sense of convergence in $\mathcal{D}'$.

Let $a = 2b$ and η be defined as above. We have

$$G_\varepsilon(x) = \eta\left(\frac{x}{\varepsilon}\right) G_\varepsilon(x), \quad x \in \mathbb{R}, \quad \varepsilon \leq \varepsilon_0.$$

This implies

$$\begin{aligned} \frac{1}{\varepsilon c(\varepsilon)} \mathcal{L}_g(G_\varepsilon)\left(\frac{p}{\varepsilon}\right) &= \frac{1}{\varepsilon c(\varepsilon)} \int_{\mathbb{R}} e^{-pu} G_\varepsilon(u) \eta\left(\frac{u}{\varepsilon}\right) \eta(u) du \\ &= \frac{1}{c(\varepsilon)} \int_{\mathbb{R}} e^{-pu} G_\varepsilon(u\varepsilon) \eta(u) \eta(u\varepsilon) du, \quad p > 0. \end{aligned}$$

Thus, if $\frac{G_\varepsilon(u\varepsilon)}{c(\varepsilon)}$ converges in $\mathcal{S}'$ as $\varepsilon \rightarrow 0^+$, then by using the fact that $\{e^{-pu} \eta(u\varepsilon) \eta(u); \varepsilon \in (0, 1)\}$ is bounded in $\mathcal{S}$ it follows that $\frac{\mathcal{L}_g(G_\varepsilon)\left(\frac{p}{\varepsilon}\right)}{\varepsilon c(\varepsilon)}$ converges as $\varepsilon \rightarrow 0^+$ for every $p > 0$, $\operatorname{Re} p > 0$.

By applying (3) and previous arguments to

$$\frac{1}{c_N(\varepsilon)} \left[\mathcal{L}_g(G_\varepsilon)\left(\frac{x}{\varepsilon}\right) - \sum_{k=1}^N A_k \mathcal{L}_g(F_{\alpha_k+1})\left(\frac{x}{\varepsilon}\right) \right], \quad \forall N \in \mathbb{N},$$

we obtain

$$\mathcal{L}_g(G)\left(\frac{x}{\varepsilon}\right) \stackrel{\text{s.e.c.}}{\sim} \sum_{k=1}^{\infty} A_k \left(-i \frac{x}{\varepsilon}\right)^{-\alpha_k-1} [(\mathcal{L}\phi)(\varepsilon^3 x)] \left(\frac{x}{\varepsilon}\right) \quad \text{at } \infty,$$

with respect to $\{c_k; k \in \mathbb{N}\}$. □

In the next proposition we define an element of $\mathcal{G} \setminus \mathcal{D}'$ which we call δ^2 . Note that “various squares of δ ” can be defined in this way.

Proposition 4. *Let $\delta^2 = [\frac{1}{\varepsilon^4} \phi^2(\frac{t}{\varepsilon^2})]$, where $\phi \in C_0^\infty$, $\int \phi = 1$, $\int x^m \phi(x) dx = 0$, $m = 1, \dots, N$, $N \in \mathbb{N}$. Then*

$$\text{a) } \delta^2(\varepsilon x) \stackrel{\text{q.e.c.}}{\sim} \sum_{k=0}^m (-1)^k \frac{\mu_k}{k!} \left[\frac{1}{\varepsilon^4} \phi^{(k)}\left(\frac{t}{\varepsilon^2}\right) \right] (\varepsilon x)$$

with respect to the scale $\{\varepsilon^{-3+m}, m = 0, 1, \dots, N\}$.

$$\text{b) } (\mathcal{L}_g(\delta^2))\left(\frac{x}{\varepsilon}\right) \stackrel{\text{s.e.c.}}{\sim} \sum_{k=0}^m (-1)^k \frac{\mu_k}{k!} \left[\varepsilon^{2k} t^k \mathcal{L}\left(\phi\left(\frac{u}{\varepsilon^2}\right)\right)(t) \right] \left(\frac{x}{\varepsilon}\right), \quad \text{at } \infty,$$

with respect to the scale $\{\varepsilon^{-3+m}; m = 0, 1, \dots, N\}$.

PROOF. a) Let

$$\mu_k = \int x^k \phi^2(x) dx, \quad k \leq N.$$

We have (as $\varepsilon \rightarrow 0^+$)

$$\begin{aligned} \frac{1}{\varepsilon^{-3}} \int \left(\frac{1}{\varepsilon^4} \phi^2 \left(\frac{\varepsilon x}{\varepsilon^2} \right) - \frac{\mu_0}{\varepsilon^4} \phi \left(\frac{\varepsilon x}{\varepsilon^2} \right) \right) \psi(x) dx &\rightarrow 0, \\ \frac{1}{\varepsilon^{-2}} \int \left(\frac{1}{\varepsilon^4} \phi^2 \left(\frac{\varepsilon x}{\varepsilon^2} \right) - \frac{\mu_0}{\varepsilon^4} \phi \left(\frac{\varepsilon x}{\varepsilon^2} \right) + \frac{\mu_1}{\varepsilon^4} \phi' \left(\frac{\varepsilon x}{\varepsilon^2} \right) \right) \psi(x) dx &\rightarrow 0, \\ \dots\dots\dots \\ \frac{1}{\varepsilon^s} \int \left(\frac{1}{\varepsilon^4} \phi^2 \left(\frac{\varepsilon x}{\varepsilon^2} \right) - \frac{\mu_0}{\varepsilon^4} \phi \left(\frac{\varepsilon x}{\varepsilon^2} \right) + \dots + \frac{(-1)^{s+2}}{\varepsilon^4 (s+3)!} \phi^{(s+3)} \left(\frac{\varepsilon x}{\varepsilon^2} \right) \right) \\ &\times \psi(x) dx \rightarrow 0, \quad s \in \mathbb{N}. \end{aligned}$$

Thus,

$$\left[\frac{1}{\varepsilon^4} \phi^2 \left(\frac{t}{\varepsilon^2} \right) \right] (\varepsilon x) \stackrel{\text{q.e.c.}}{\sim} \sum_{k=0}^m (-1)^k \frac{\mu_k}{k!} \left[\frac{1}{\varepsilon^4} \phi^{(k)} \left(\frac{t}{\varepsilon^2} \right) \right] (\varepsilon x)$$

with respect to the scale $\{\varepsilon^{-3+m}; m = 0, 1, \dots, N\}$.

b) Let $m \leq N$. We have

$$\begin{aligned} (\mathcal{L}_g(\delta^2)) \left(\frac{x}{\varepsilon} \right) &= \left[\mathcal{L} \left(\frac{1}{\varepsilon^4} \phi^2 \left(\frac{t}{\varepsilon^2} \right) \right) \right] \left(\frac{x}{\varepsilon} \right) \\ &= \left[\frac{1}{\varepsilon^2} (\mathcal{L}(\phi^2)) (\varepsilon x) \right], \quad x > 0, \\ \mathcal{L} \left(\phi^{(k)} \left(\frac{t}{\varepsilon^2} \right) \right) (x) &= \int \exp(-xt) \phi^{(k)} \left(\frac{t}{\varepsilon^2} \right) dt \\ &= \varepsilon^{2k} x^k \left(\mathcal{L} \phi \left(\frac{t}{\varepsilon^2} \right) \right) (x), \quad x > 0, \end{aligned}$$

and

$$\frac{1}{\varepsilon^{-3+m}} \mathcal{L} \left(\frac{1}{\varepsilon^4} \phi^2 \left(\frac{\varepsilon t}{\varepsilon^2} \right) - \frac{1}{\varepsilon^4} \sum_{k=0}^m (-1)^k \frac{\mu_k}{k!} \phi^{(k)} \left(\frac{\varepsilon t}{\varepsilon^2} \right) \right) (x) \rightarrow 0,$$

$$x > 0, \text{ as } \varepsilon \rightarrow 0^+.$$

This implies

$$\frac{1}{\varepsilon^{-2+m}} \mathcal{L} \left(\frac{1}{\varepsilon^4} \phi^2 \left(\frac{t}{\varepsilon^2} \right) \right) \left(\frac{x}{\varepsilon} \right) - \frac{1}{\varepsilon^{-2+m}} \sum_{k=0}^m \frac{(-1)^k \mu_k}{k!} \mathcal{L} \left(\phi^{(k)} \left(\frac{t}{\varepsilon^2} \right) \right) \left(\frac{x}{\varepsilon} \right) \rightarrow 0,$$

$x > 0$, $\varepsilon \rightarrow 0^+$, and

$$\mathcal{L}_g(\delta^2) \left(\frac{x}{\varepsilon} \right) \stackrel{\text{s.e.c.}}{\sim} \sum_{k=0}^m (-1)^k \frac{\mu_k}{k!} \left[\varepsilon^{2k} t^k \mathcal{L} \left(\phi \left(\frac{t}{\varepsilon^2} \right) \right) (t) \right] \left(\frac{x}{\varepsilon} \right) \quad \text{at } \infty,$$

with respect to the scale $\{\varepsilon^{-3+m}; m = 0, 1, \dots, N\}$.

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