

## Submanifolds of a quasi constant curvature manifold

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**Abstract.** In this paper, we study submanifolds in a quasi constant curvature space and generalize a result due to Chern–do Carmo–Kobayashi and Li–Li.

### 0. Introduction

Let  $M$  be an  $n$ -dimensional compact minimal submanifold immersed into the  $(n + p)$ -dimensional unit sphere  $S^{n+p}$ . Denote by  $S$  the square of the length of the second fundamental form of  $M$ . The following result is well known ([1, 2]):

**Theorem A.** *Let  $M$  be an  $n$ -dimensional compact minimal submanifold in  $S^{n+p}$ . If  $S$  satisfies*

$$S \leq \frac{n}{1 + \frac{1}{2} \operatorname{sgn}(p-1)},$$

*then  $M$  is a totally geodesic submanifold, and the Clifford torus or the Veronese surface is in  $S^4$ , where  $\operatorname{sgn}(x) = 1$  for  $x > 0$  and  $\operatorname{sgn}(x) = 0$  for  $x \leq 0$ .*

The following definition was introduced by CHEN and YANO [3].

*Definition.* A Riemannian manifold is said to be a quasi constant curvature manifold if its curvature tensor satisfies

$$\begin{aligned} (*) \quad K_{ABCD} = & a(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}) \\ & + b(\delta_{AC}v_Bv_D - \delta_{AD}v_Bv_D + \delta_{BD}v_Av_C - \delta_{BC}v_Av_D), \end{aligned}$$

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where  $a, b$  are some scalar functions and  $v_A$  the component of a unit vector field which is called the generator of the manifold.

By the definition we see that when  $b \equiv 0$ , the quasi constant curvature manifold becomes a constant curvature manifold.

From now on, we use of the following convention on the ranges of the indices:

$$1 \leq A, B, \dots \leq n + p; \quad 1 \leq i, j, \dots \leq n; \quad n + 1 \leq \alpha, \beta, \dots \leq n + p.$$

The purpose of this paper is to study the case that the ambient space is a quasi constant curvature manifold  $N^{n+p}$  and to generalize Theorem A. We obtain

**Theorem 1.** *Let  $M$  be an  $n$ -dimensional compact minimal submanifold in an  $(n + p)$ -dimensional quasi constant curvature manifold  $N^{n+p}$  and let  $a, b$  be constants. Suppose that*

(a)  $a > 0$ , the generator is orthogonal to  $M$  and

$$S < \frac{na}{1 + \frac{1}{2} \operatorname{sgn}(p - 1)},$$

or else that

(b)  $na + b - n|b| > 0$ , the generator is parallel to  $M$ , and

$$S < \frac{na + b - n|b|}{1 + \frac{1}{2} \operatorname{sgn}(p - 1)}.$$

Then each of these two sets of conditions implies that  $M$  is a totally geodesic submanifold.

**Theorem 2.** *Let  $M$  be an  $n$ -dimensional compact minimal submanifold in an  $(n + p)$ -dimensional quasi constant curvature manifold  $N^{n+p}$ . Suppose  $a$  is a positive number. If the generator is orthogonal to  $M$  and*

$$S = \frac{na}{1 + \frac{1}{2} \operatorname{sgn}(p - 1)},$$

then either  $M = S^m \left( \sqrt{\frac{m}{(n-m)a+m}} \right) \times S^{n-m} \left( \sqrt{\frac{n-m}{ma+n-m}} \right)$  or  $n = p = 2$  and with respect to an adapted dual orthonormal frame  $\omega_1, \omega_2, \omega_3, \omega_4$ ,

the connection form  $(\omega_{AB})$  of  $N^4$ , restricted to  $M$ , is given by

$$\begin{pmatrix} 0 & \omega_{12} & g\omega_1 & g\omega_2 \\ \omega_{21} & 0 & -g\omega_2 & g\omega_1 \\ -g\omega_1 & g\omega_2 & 0 & 2\omega_{12} \\ -g\omega_2 & -g\omega_1 & -2\omega_{12} & 0 \end{pmatrix}, \quad g = \sqrt{\frac{1}{3}a}.$$

*Remark.* When  $a = 1$  and  $b = 0$ , from Theorems 1, 2 we can get Theorem A immediately.

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### 1. Fundamental formulas

Let  $M$  be an  $n$ -dimensional compact minimal submanifold immersed in the  $(n+p)$ -dimensional quasi constant curvature manifold  $N^{n+p}$  with the curvature tensor as (\*). Suppose that  $\{e_A\}$  is an orthonormal frame field on  $N^{n+p}$  such that  $e_1, \dots, e_n$  are tangent to  $M$ , and let  $\{\omega_A\}$  be the dual frame field. Then the structure equations of  $N^{n+p}$  are given by

$$\begin{aligned} d\omega_A &= - \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} &= - \sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D, \end{aligned}$$

where  $K_{ABCD}$  satisfies (\*).

Restricting these forms to  $M$ , we have

$$\begin{aligned} \omega_\alpha &= 0, \quad \omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha, \\ d\omega_i &= - \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} &= - \sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \end{aligned}$$

where

$$(1.1) \quad \begin{aligned} R_{ijkl} &= K_{ijkl} + \sum_{\alpha} (h_{ik}^{\alpha} h_{jl}^{\alpha} - h_{il}^{\alpha} h_{jk}^{\alpha}), \\ d\omega_{\alpha\beta} &= - \sum_{\gamma} \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \frac{1}{2} \sum_{kl} R_{\alpha\beta kl} \omega_k \wedge \omega_l, \\ R_{\alpha\beta kl} &= K_{\alpha\beta kl} + \sum_i (h_{ki}^{\alpha} h_{il}^{\beta} - h_{ki}^{\beta} h_{il}^{\alpha}). \end{aligned}$$

We call  $h = \sum_{ij\alpha} h_{ij}^{\alpha} \omega_i \omega_j e_{\alpha}$  and  $\xi = \frac{1}{n} \sum_{i\alpha} h_{ii}^{\alpha} e_{\alpha}$  the second fundamental form and the mean curvature vector of the immersion, respectively.  $M$  is said to be minimal, if  $\xi \equiv 0$ . Denote by  $S = \sum_{ij\alpha} (h_{ij}^{\alpha})^2$  the square of the length of  $h$ . Define  $h_{ijk}^{\alpha}$  and  $h_{ijkl}^{\alpha}$  by

$$\sum_k h_{ijk}^{\alpha} \omega_k = dh_{ij}^{\alpha} - \sum_k h_{kj}^{\alpha} \omega_{ki} - \sum_k h_{ik}^{\alpha} \omega_{kj} - \sum_{\beta} h_{ij}^{\beta} \omega_{\beta\alpha}$$

and by

$$\sum_l h_{ijkl}^{\alpha} \omega_l = dh_{ijk}^{\alpha} - \sum_l h_{ljk}^{\alpha} \omega_{li} - \sum_l h_{ilk}^{\alpha} \omega_{lj} - \sum_l h_{ijl}^{\alpha} \omega_{lk} - \sum_{\beta} h_{ijk}^{\beta} \omega_{\beta\alpha},$$

respectively. We know that

$$(1.2) \quad h_{ijk}^{\alpha} - h_{ikj}^{\alpha} = K_{\alpha ikj} = -K_{\alpha ijk}$$

and

$$(1.3) \quad h_{ijk}^{\alpha} - h_{ijlk}^{\alpha} = \sum_m h_{im} R_{mjkl} + \sum_m h_{mj} R_{mikl} + \sum_{\beta} h_{ij}^{\beta} R_{\beta\alpha kl}.$$

## 2. Proofs of the theorems

In order to prove our theorems, we need the following

**Lemma 1** ([1, 2]). *Let  $H_{\alpha}$ ,  $\alpha \geq 2$  be symmetric  $(n \times n)$ -matrices,  $S_{\alpha} = \text{tr } H_{\alpha}^2$ ,  $S = \sum_{\alpha} S_{\alpha}$ . Then*

$$\sum_{\alpha, \beta} \text{tr}(H_{\alpha} H_{\beta} - H_{\beta} H_{\alpha})^2 - \sum_{\alpha, \beta} (\text{tr } H_{\alpha} H_{\beta})^2 \geq - \left[1 + \frac{1}{2} \text{sgn}(p-1)\right] S^2,$$

and equality holds if and only if all  $H_\alpha = 0$  or there exist two  $H_\alpha$ 's different from zero. Moreover, if  $H_{n+1} \neq 0$ ,  $H_{n+2} \neq 0$ ,  $H_\alpha = 0$ ,  $\alpha \neq n+1, n+2$ , then  $S_{n+1} = S_{n+2}$  and there exists an orthogonal  $(n \times n)$  matrix  $T$  such that

$$TH_{n+1} {}^tT = \begin{pmatrix} f & 0 & 0 \\ 0 & -f & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad TH_{n+2} {}^tT = \begin{pmatrix} 0 & f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where  $f = \sqrt{\frac{S_1}{2}}$ .

If  $a$  and  $b$  are constants and  $M$  is minimal, then using (\*), (1.1)–(1.3) we get

$$\begin{aligned} \frac{1}{2}\Delta S &= \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 \\ &+ \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{mk}^\alpha R_{mijk} + \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{im}^\alpha R_{mkjk} \\ &+ \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ki}^\beta R_{\beta\alpha jk} + \sum_{i,j,k,\alpha} h_{ij}^\alpha \nabla_k K_{\alpha ikj} \\ &+ \sum_{i,j,k,\alpha} h_{ij}^\alpha \nabla_j K_{\alpha kki} = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + naS + bS \sum_k v_k^2 \\ &+ nb \sum_{i,j,m,\alpha} h_{ij}^\alpha h_{im}^\alpha v_m v_j - n \sum_{i,j,\alpha} h_{ij}^\alpha \nabla_j (bv_\alpha v_j) \\ &+ \sum_{\alpha,\beta} \text{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha,\beta} (\text{tr} H_\alpha H_\beta)^2. \end{aligned} \tag{2.1}$$

Now, if we assume that the generator  $v = \sum_A v_A e_A$  is orthogonal to  $M$ , then we see that  $v_i = 0$  and (2.1) becomes

$$\begin{aligned} \frac{1}{2}\Delta S &= \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + naS + \sum_{\alpha,\beta} \text{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 \\ &- \sum_{\alpha,\beta} (\text{tr} H_\alpha H_\beta)^2. \end{aligned} \tag{2.2}$$

Applying Lemma 1 to (2.2) we get

$$(2.3) \quad \begin{aligned} \frac{1}{2}\Delta S &\geq \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + naS - \left[1 + \frac{1}{2} \operatorname{sgn}(p-1)\right] S^2 \\ &\geq naS - \left[1 + \frac{1}{2} \operatorname{sgn}(p-1)\right] S^2. \end{aligned}$$

When  $a$  is a positive number, it follows from the compactness of  $M$  and (2.3) that if

$$(2.4) \quad S \leq \frac{na}{1 + \frac{1}{2} \operatorname{sgn}(p-1)},$$

then  $S$  is a constant and (2.4) leads to

$$(2.5) \quad \{na - [1 + \frac{1}{2} \operatorname{sgn}(p-1)] S\} S = 0.$$

Thus, if

$$S < \frac{na}{1 + \frac{1}{2} \operatorname{sgn}(p-1)},$$

then from (2.5) we see that  $S = 0$ , hence  $M$  is totally geodesic.

If the generator  $v = \sum_A v_A e_A$  is parallel to  $M$ , then we see that  $v_\alpha = 0$  and  $\sum_i v_i^2 = 1$ . Applying Lemma 1 to (2.1) we get

$$(2.6) \quad \begin{aligned} \frac{1}{2}\Delta S &\geq \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + naS + bS + nb \sum_{i,j,m,\alpha} h_{ij}^\alpha h_{im}^\alpha v_m v_j \\ &\quad - \left[1 + \frac{1}{2} \operatorname{sgn}(p-1)\right] S^2. \end{aligned}$$

We claim that, for any  $\alpha$

$$\sum_{i,j,m} h_{ij}^\alpha h_{im}^\alpha v_m v_j \leq \sum_{i,j} (h_{ij}^\alpha)^2.$$

In fact, since both sides of the formula above are independent of  $e_i$ , we can choose  $e_1, \dots, e_n$  such that  $h_{ij}^\alpha = h_{ii}^\alpha \delta_{ij}$  for fixed  $\alpha$ , and hence

$$\sum_{ijm} h_{ij}^\alpha h_{im}^\alpha v_m v_j = \sum_i (h_{ii}^\alpha)^2 v_i^2 \leq \sum_i (h_{ii}^\alpha)^2 \sum_i v_i^2 = \sum_{ij} (h_{ij}^\alpha)^2.$$

Thus

$$(2.7) \quad \sum_{i,j,m,\alpha} h_{ij}^\alpha h_{im}^\alpha v_m v_j \leq \sum_{i,j,\alpha} (h_{ij}^\alpha)^2 = S.$$

Combining (2.7) with (2.6) we get

$$(2.8) \quad \begin{aligned} \frac{1}{2} \Delta S &\geq \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + naS + nS - nbS - [1 + \frac{1}{2} \operatorname{sgn}(p-1)] S^2 \\ &\geq \{na + b - n|b| - [1 + \frac{1}{2} \operatorname{sgn}(p-1)] S\} S. \end{aligned}$$

Using the same arguments as above we see that if

$$S < \frac{na + b - n|b|}{1 + \frac{1}{2} \operatorname{sgn}(p-1)},$$

then  $M$  is totally geodesic. This completes the proof of Theorem 1.

By (1.1), the scalar curvature of  $M$  satisfies

$$R = (n-1) \left( na + 2b \sum_i v_i^2 \right) - S,$$

which shows that if the generator is orthogonal to  $M$ , then

$$(2.9) \quad R = (n-1)na - S$$

and if the generator is parallel to  $M$ , then we have

$$(2.10) \quad R = (n-1)(na + 2b) - S.$$

Therefore, combining (2.9) with (2.10), Theorem 1 implies the

**Corollary.** *Let  $M$  be an  $n$ -dimensional compact minimal submanifold in an  $(n+p)$ -dimensional quasi constant curvature manifold  $N^{n+p}$  and let  $a$  and  $b$  be constants. Suppose that*

(a)  $a > 0$ , the generator is orthogonal to  $M$ , and

$$R > na \left[ n - 1 - \frac{1}{1 + \frac{1}{2} \operatorname{sgn}(p-1)} \right]$$

or else that

(b)  $na + b - n|b| > 0$ , the generator is parallel to  $M$  and

$$R > (n-1)(na+2b) - \frac{na+b-n|b|}{1+\frac{1}{2}\operatorname{sgn}(p-1)}.$$

Then each of these two sets of conditions implies that  $M$  is a totally geodesic submanifold.

PROOF of Theorem 2. If

$$(2.11) \quad S = \frac{na}{1+\frac{1}{2}\operatorname{sgn}(p-1)},$$

and  $p = 1$ , (2.3) and (2.11) show that  $\sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 = 0$ . We can choose  $e_1, \dots, e_n$  so that  $h_{ij}^{n+1} = h_{ii}^{n+1} \delta_{ij}$ . Since  $v_i = 0$  and using the same method as in [1] we get

$$(2.12) \quad h_{ii}^{n+1} h_{jj}^{n+1} = -a$$

for  $h_{ii}^{n+1} \neq h_{jj}^{n+1}$ . (2.12) implies that  $M$  has two distinct principal curvatures  $\lambda, \mu$  and these satisfy

$$m\lambda + (n-m)\mu = 0, \quad m\lambda^2 + (n-m)\mu^2 = na, \quad \lambda\mu = -a,$$

hence

$$\lambda^2 + 1 = \frac{(n-m)a+m}{m}, \quad \mu^2 + 1 = \frac{ma+n-m}{n-m}.$$

So by the same arguments as in [1] we see that  $M$  is locally congruent to

$$S^m \left( \sqrt{\frac{m}{(n-m)a+m}} \right) \times S^{n-m} \left( \sqrt{\frac{n-m}{ma+n-m}} \right).$$

On the other hand, for  $p \geq 2$ , (2.2), (2.3) and (2.6) imply that  $\sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 = 0$  and the equality

$$\sum_{\alpha,\beta} \operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha,\beta} (\operatorname{tr} H_\alpha H_\beta)^2 = -\frac{3}{2} S^2$$

holds. Thus applying Lemma 1 we may assume that

$$H_{n+1} = \begin{pmatrix} g & 0 & 0 \\ 0 & -g & \\ 0 & & 0 \end{pmatrix}, \quad H_{n+2} = \begin{pmatrix} 0 & g & 0 \\ g & 0 & \\ 0 & & 0 \end{pmatrix},$$

where we have  $g = \sqrt{\frac{S_1}{2}} = \sqrt{\frac{1}{6}na}$ . Noting that  $v_i = 0$ , and using the method of [1], we get  $n = p = 2$ . Thus we have

$$\omega_{13} = \omega_{24} = \sqrt{\frac{1}{3}}a\omega_1, \quad \omega_{14} = \sqrt{\frac{1}{3}}a\omega_2, \quad \omega_{23} = -\sqrt{\frac{1}{3}}a\omega_2, \quad \omega_{34} = 2\omega_{12},$$

and so with respect to an adapted dual orthonormal frame field  $\omega_1, \omega_2, \omega_3, \omega_4$ , the connection forms  $\{\omega_{AB}\}$  of  $N^4$ , restricted to  $M$ , are given by

$$\begin{pmatrix} 0 & \omega_{12} & g\omega_1 & g\omega_2 \\ \omega_{12} & 0 & -g\omega_2 & g\omega_1 \\ -g\omega_1 & g\omega_2 & 0 & 2\omega_{12} \\ -g\omega_2 & -g\omega_1 & -2\omega_{12} & 0 \end{pmatrix}, \quad g = \sqrt{\frac{1}{3}a}.$$

This completes the proof of Theorem 2.  $\square$

When  $a = 1, b = 0$ , it follows from Theorems 1 and 2 that we can immediately prove Theorem A of the Introduction.

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