

A note on powers of Pisot numbers

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Abstract. Let b be an algebraic number greater than one such that the fractional part of its powers tends to zero or one. We show that in the first case b is an integer and in the second case b is a certain type of Pisot number.

For $x \in \mathbb{R}$ let $\{x\}$ denote the fractional part of x , and let $\|x\|$ denote the distance from x to the nearest integer. A theorem of Koksma states that the sequence $\{b^n\}_{n \geq 1}$ is uniformly distributed in the interval $[0; 1]$ for almost all real numbers $b > 1$. However Pisot (or Pisot–Vijayaraghavan) numbers represent a nontrivial exceptional set in this theorem. A real algebraic integer greater than 1 is called a Pisot (or PV) number if all its remaining conjugates (if any) lie strictly inside the unit circle. A theorem of Pisot and Vijayaraghavan (see, e.g., [1], [4]) implies that for an algebraic number b greater than one $\|b^n\| \rightarrow 0$ ($n \rightarrow \infty$) iff b is a PV number. In estimating the number of lattice points below a logarithmic curve [5] the question arises whether there exists any $b > 1$ with $\{b^n\} \rightarrow 0$ as $n \rightarrow \infty$. Leaving transcendental numbers out of consideration G. KUBA [5] asked the following:

Question. Are there any PV numbers $b \notin \mathbb{Z}$ with $\{b^n\} \rightarrow 0$ as $n \rightarrow \infty$?

A related question is the following one: find all real numbers $b > 1$ for which $\{b^n\} \rightarrow 1$ as $n \rightarrow \infty$. If b is algebraic then this condition implies that b is a PV number. We say that b is a *strong PV number* if it is a

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PV number such that one of its remaining conjugates $b_2 > 0$ is strictly greater than any of the absolute values of the remaining ones (if any) $b_2 > \max_{3 \leq j \leq d} |b_j|$. Clearly, the strong PV numbers of degree $d = 2$ are given by $b = \frac{1}{2}(p + \sqrt{p^2 - 4q})$, where p, q and $p^2 - 4q$ are positive integers and $p^2 - 4q$ is not a perfect square. The referee noted that the set of strong PV numbers was also considered by D. BOYD [3].

The following theorem describes the algebraic numbers with the limit of fractional part of powers zero or one. In particular, it shows that the answer to the question in [5] is negative.

Theorem. *If $b > 1$ is an algebraic number such that $\{b^n\} \rightarrow 0$ as $n \rightarrow \infty$ then $b \in \mathbb{Z}$. If with the same hypotheses $\{b^n\} \rightarrow 1$ as $n \rightarrow \infty$ then b is a strong PV number.*

PROOF. Suppose first that $\{b^n\} \rightarrow 0$ as $n \rightarrow \infty$. The theorem of Pisot and Vijayaraghavan implies that either $b \in \mathbb{Z}$ or b is a PV number of degree $d \geq 2$. We will show that the second case is impossible. Indeed, let $b_2, b_3, \dots, b_d$ be the conjugates of b . Clearly, the sum

$$b^n + b_2^n + b_3^n + \dots + b_d^n$$

is an integer if n is a positive integer. Since

$$S(n) = b_2^n + b_3^n + \dots + b_d^n$$

is a real number and $S(n) \rightarrow 0$ as $n \rightarrow \infty$, it suffices to show that $S(n)$ is positive for an infinite number of n 's.

Let $M = \max_{2 \leq j \leq d} |b_j|$. A result of C. SMYTH [7] shows that if b is a PV number then it has no two conjugates of equal modulus, except for pairs of complex conjugates (see [2], [6] for related results). Without loss of generality we assume that $|b_2| = M$. There are three possibilities: $b_2 = M$, $b_2 = -M$ and $b_2 = Me^{i\theta}$ with $\theta \in (0; \pi)$. In the first two cases we have $S(2n) > 0$ for all sufficiently large n 's and the first part of the theorem follows. In the third case let $b_3 = Me^{-i\theta}$. Then

$$b_2^n + b_3^n = 2M^n \cos(n\theta) \geq M^n$$

if $\|n\theta/2\pi\| \leq 1/6$. By Dirichlet's theorem, for each $x \in \mathbb{R}$ and $\delta > 0$ the inequality $\|nx\| < \delta$ has infinite number of solutions in positive integers n (here $x = \theta/2\pi$, $\delta = 1/6$). Hence, if $m = \max_{4 \leq j \leq d} |b_j|$ for these n we have

$$S(n) \geq M^n - (d-3)m^n$$

which is positive for sufficiently large n . This completes the proof of the first part of the theorem.

Suppose now that $\{b^n\} \rightarrow 1$ as $n \rightarrow \infty$. Then again $S(n) \rightarrow 0$ as $n \rightarrow \infty$. In the first case ($b_2 = M$) b is a strong PV number. In the second case ($b_2 = -M$) $S(n)$ is negative for all sufficiently large odd n which contradicts the condition $\{b^n\} \rightarrow 1$ ($n \rightarrow \infty$). In the third case

$$b_2^n + b_3^n = 2M^n \cos(n\theta) \leq -M^n$$

if $1/3 \leq \|n\theta/2\pi\| \leq 2/3$. Clearly, for $x \in \mathbb{R}$, $x \notin \mathbb{Z}$ the inequality $1/3 \leq \|nx\| \leq 2/3$ has infinite number of solutions in positive integers n (for $x \notin \mathbb{Q}$ this follows from the uniform distribution of $\{nx\}$ and for $x \in \mathbb{Q}$ this is an easy exercise). So that for these n we have

$$S(n) \leq -M^n + (d-3)m^n$$

which is negative for sufficiently large n . This completes the proof of the second part of the theorem. $\square$

References

- [1] M. J. BERTIN, A. DECOMPS-GUILLOUX, M. GRANDET-HUGOT, M. PATHIAUX-DELEFOSSE and J. P. SCHREIBER, Pisot and Salem numbers, *Birkhäuser*, 1992.
- [2] D. BOYD, Irreducible polynomials with many roots of maximal modulus, *Acta Arith.* **68** (1994), 85–88.
- [3] D. BOYD, Linear recurrence relations for some generalized Pisot sequences, *Advances in Number Theory* (F.Q. Gouvea and N. Yui, eds.), Proc. of the 1991 CNTA conference, *Oxford University Press*, 1993, 333–340.
- [4] J. W. S. CASSELS, An introduction to Diophantine approximation, *Cambridge*, 1957.
- [5] G. KUBA, The number of lattice points below a logarithmic curve, *Arch. Math.* **69** (1997), 156–163.
- [6] M. MIGNOTTE, Sur les conjuguées des nombres de Pisot, *C.R. Acad. Sci. Paris Sér. I Math.* **298** (1984), 21.

- [7] C. SMYTH, The conjugates of algebraic integers, *Amer. Math. Monthly* **82** (1975), 86.

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