

On cosymplectic quasi-Sasakian manifolds with quasi-Reeb vector field

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Abstract. A cosymplectic quasi-Sasakian manifold M (see [O]) with quasi-Reeb vector field is considered. We study some distinguished vector fields on M : skew symmetric Killing vector fields [MRV] and vector fields which define strong automorphisms of the symplectic structure. Some foliations on M are obtained.

Let $M(\phi, \Omega, \eta, \xi, g)$ be a $(2m + 1)$ -dimensional cosymplectic quasi-Sasakian manifold (abbr. CQS) in the sense of [O], i.e. the structure tensors satisfy:

$$(0.1) \quad \phi^2 = -I + \eta \otimes \xi, \quad d\Omega = 0, \quad d\eta = 0, \quad \xi(\eta) = 1.$$

If J means the anti-invariant operator of square $+1$ [R3], then [BR] have initiated the case when the covariant differential of the structure vector ξ satisfies:

$$(0.2) \quad \nabla \xi = c(J \circ \phi)dp,$$

where c is a non vanishing constant (called the essential constant) and dp the soldering form of M . Such a manifold is called a CQS manifold with quasi-Reeb vector field ξ (abbr. CQSQR).

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Clearly the distribution $\{Z \in \Gamma TM; \eta(X) = 0\}$ is a horizontal involutive distribution.

In the present paper, we study some properties of skew symmetric Killing vector fields [R1] (abbr. SSK) and of vector fields which define strong automorphisms of the $(1 \times Sp(2m, \mathbb{R}))$ -structure considered, i.e. $\mathcal{L}_Z \Omega = 0$, $\mathcal{L}_Z \eta = 0$, where $\mathcal{L}_Z$ is the Lie derivative with respect to Z .

In Section 2 it is shown that the existence of an SSK vector field X is assured by an exterior differential system in involution (in the sense of [C]) and the following properties are proved:

- (i) M is foliated by surfaces M_X of constant Ricci curvature, tangent to X and its generative $\mathcal{T}$.
- (ii) $\|X\|^2$ is an isoparametric function [W], where $\|X\|^2 = g(X, X)$.
- (iii) the conditions:
 - a) $\|X\|^2$ is an eigenfunction of Δ ;
 - b) X is an affine vector field,
 are mutually equivalent.

In Section 3 we obtain a necessary and sufficient condition for a strong automorphism of the $(1 \times Sp(2m, \mathbb{R}))$ -structure to be a Killing vector field.

In Section 4 one considers on the horizontal hypersurface M_ξ defined by $\eta = 0$ two associated principal vector fields W and $\bar{W}$ in the sense of [Ph]. Then if W and $\bar{W}$ are SSK vector fields having ξ as generative, this implies that both define strong automorphisms of the $(1 \times Sp(2m, \mathbb{R}))$ -structure under consideration.

1. Preliminaries

Let (M, g) be a $(2m+1)$ -dimensional oriented C^∞ -manifold with Riemannian metric g . Let ΓTM be the set of sections of the tangent bundle and ∇ be the covariant derivative operator defined by g . Assume that M carries the quadruple of structure tensors $(\phi, \Omega, \eta, \xi)$, where ϕ is a $(1, 1)$ tensor field, Ω is a closed 2-form of rank $2m$, η a closed Pfaffian and $\xi = \eta^\sharp$ the structure vector field (one may also call ξ the quasi-Reeb vector field (abbr. QR)). Then, if these tensor fields satisfy:

$$(1.1) \quad \begin{cases} \phi^2 = -I + \eta \otimes \xi, & \eta(\xi) = 1, & \phi\xi = 0, \\ g(\phi Z, \phi Z') = g(Z, Z') - \eta(Z)\eta(Z'), & \eta(Z) = g(\xi, Z), \\ d\Omega = 0, & \Omega(Z, Z') = g(Z, \phi Z'), & \Omega^m \wedge \eta \neq 0, \end{cases}$$

and

$$(1.2) \quad d\eta = 0,$$

one says [O] that M is a quasi-Sasakian manifold endowed with a cosymplectic structure $(1 \times Sp(2m, \mathbb{R}))$, and the distribution $D_\eta = \{Z \in \Gamma TM; \eta(Z) = 0\}$, which is called the horizontal distribution, is always involutive.

We also recall that $\flat : TM \rightarrow T^*M$, $\sharp : T^*M \rightarrow TM$ mean the musical isomorphisms defined by g , and

$$(1.3) \quad \Omega^\flat : TM \rightarrow T^*M, \quad Z \mapsto -i_Z \Omega = {}^\flat Z, \quad Z \in \Gamma TM$$

denotes the symplectic isomorphism, where i_Z is the interior product operator with respect to Z .

Further, if we set $A^q(M, TM) = \text{Hom}(\Lambda^q TM, TM)$ (elements of $A^q(M, TM)$ are vector valued q -forms), then following [P], $d^\nabla : A^q(M, TM) \rightarrow A^{q+1}(M, TM)$ denotes the exterior covariant operator with respect to ∇ .

It should be noticed that generally $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$. If $p \in M$, then the vector valued 1-form $dp \in A^1(M, TM)$ is the canonical vector valued 1-form of M and is called the soldering form [Di]. A (non-parallel) vector field X on a Riemannian (or pseudo-Riemannian) manifold is, following [R2], is said to be exterior concurrent (abbr. EC) if

$$(1.4) \quad d^\nabla(\nabla X) = \nabla^2 X = r \wedge dp$$

for some 1-form r , called the concurrence form associated with X . The above formula is equivalent to

$$(1.5) \quad \nabla^2 X = -\frac{1}{n-1} \text{Ric}(X) X^\flat \wedge dp,$$

where $\text{Ric}(X)$ denotes the Ricci curvature of M with respect to X and $n = \dim M$.

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is isoparametric [W] if $\|\nabla f\|^2$ and $\text{div}(\nabla f)$ are functions of f ($\nabla f = \text{grad } f$).

Let $\mathcal{O} = \text{vect}\{e_A, A = 1, \dots, n\}$ be a local field of adapted vectorial frames over M , and let $\mathcal{O}^* = \text{covect}\{\omega^A\}$ be its associated coframe. Then the soldering form dp is expressed by

$$(1.6) \quad dp = \omega^A \otimes e_A,$$

and E. Cartan's structure equations written in the indexless manner are:

$$(1.7) \quad \nabla e = \theta \otimes e,$$

$$(1.8) \quad d\omega = -\theta \wedge \omega,$$

$$(1.9) \quad d\theta = -\theta \wedge \theta + \Theta.$$

In the above equations, θ (resp. Θ) are the local connection forms in the tangent bundle TM (resp. the curvature 2-forms of M).

On a $(2m+1)$ -dimensional manifold carrying the structure tensors ϕ and Ω one sets generally

$$(1.10) \quad \Omega = \omega^i \wedge \omega^{i^*}, \quad i \in \{1, \dots, m\}, \quad i^* = i + m,$$

and the $(1,1)$ tensor field ϕ induces the Kaehlerian relations for the horizontal connection forms

$$(1.11) \quad \theta_j^i = \theta_{j^*}^{i^*}, \quad \theta_j^{i^*} = \theta_i^{j^*}.$$

Further, following [R3] (see also [VR]) the anti-invariant operator with respect to ϕ is defined by

$$(1.12) \quad J e_i = e_{i^*}, \quad J e_{i^*} = e_i, \quad J^2 = I,$$

and one has

$$(1.13) \quad J \circ \phi + \phi \circ J = 0, \quad J\xi = 0.$$

In order to simplify, we set $\mathcal{A} = J \circ \phi$ and agree to call $\mathcal{A}$ the mixed anti-invariant operator (abbr. MA). By (1.7) we write:

$$(1.14) \quad \nabla \xi = c(J \circ \phi)dp, \quad c = \text{const.},$$

and it is easily seen that equations (1.1) and (1.2) are satisfied.

Such a quasi-Sasakian manifold is defined as a cosymplectic quasi-Sasakian manifold with $J\phi$ -structure vector field ξ . We agree to call it a quasi-Reeb vector field. One may write (1.14) as

$$(1.15) \quad \nabla \xi = c(\omega^i \otimes e_i - \omega^{i^*} \otimes e_{i^*}), \quad c \neq 0, \quad c = \text{const.},$$

and the constant c will be called the essential constant. By (1.15), we notice that a short calculation gives

$$(1.16) \quad \text{div } \xi = 0.$$

2. Skew symmetric Killing vector fields on a CQSQR-manifold

In this section we study some properties of skew symmetric Killing vector fields X on a CQSQR manifold $M(\phi, \Omega, \eta, \xi, J, g)$ defined by (0.1) and (0.2). Following [R1] such a vector field is defined by

$$(2.1) \quad \nabla X = X \wedge \mathcal{T} = \tau \otimes X - X^\flat \otimes \mathcal{T},$$

where $\tau = \mathcal{T}^\flat$ and the vector field $\mathcal{T}$ is called the generative of X (see also [MRV]), and as in [R1] we assume that $\mathcal{T}$ is a closed torse forming (abbr. TF) [Y].

If $Z \in \Gamma TM$ is any vector field on M , then by reference to (1.7) and (1.15) its covariant differential is expressed by

$$(2.2) \quad \begin{aligned} \nabla Z = & (dZ^i + Z^a \theta_a^i + cZ^0 \omega^i) \otimes e_i + (dZ^{i*} + Z^a \theta_a^{i*} - cZ^0 \omega^{i*}) \otimes e_{i*} \\ & + (dZ^0 - c(Z^i \omega^i - Z^{i*} \omega^{i*})) \otimes \xi, \end{aligned}$$

where $a \in \{1, \dots, 2m\}$.

If X coincides with the SSK vector field, then one derives by (2.1)

$$(2.3) \quad dX^\flat = 2\tau \wedge X^\flat,$$

and so one refinds ROSCA's lemma for SSK vector fields [R1], i.e. $X^\flat$ is an exterior recurrent [D] form, having τ as recurrence form. In addition, if $\mathcal{T}$ is a closed TF, then one has

$$(2.4) \quad \nabla \mathcal{T} = fdp - \tau \otimes \mathcal{T}, \quad f \in C^\infty M,$$

and it is easily seen that

$$(2.5) \quad d\tau = 0.$$

Setting $s = g(X, \mathcal{T})$, one quickly derives from (2.1) that

$$(2.6) \quad ds \wedge X^\flat = 0,$$

and so, we may set

$$(2.7) \quad s = s_0 = \text{const.}$$

Further, from (2.1) and (2.3) a short calculation gives

$$(2.8) \quad ds = (f - \|\mathcal{T}\|^2)X^\flat \Rightarrow f = \|\mathcal{T}\|^2,$$

and under these conditions one has

$$(2.9) \quad [X, \mathcal{T}] = 0,$$

which shows that X and $\mathcal{T}$ commute. Moreover, considering $\langle \mathcal{T}, \mathcal{T} \rangle$ and taking account of (2.5), it follows from (2.8) that $d\|\mathcal{T}\|^2 = 0$, and so by (2.8) one may write

$$(2.10) \quad f = \|\mathcal{T}\|^2 = \text{const.}$$

Operating now on (2.1) and (2.4) by d^∇ , one quickly derives by (2.10)

$$(2.11) \quad \begin{cases} \nabla^2 X = fX^\flat \wedge dp \\ \nabla^2 \mathcal{T} = f\mathcal{T}^\flat \wedge dp. \end{cases}$$

This proves the significant fact that both X and $\mathcal{T}$ are exterior concurrent vector fields with the constant conformal factor f . Hence, following [MRV], one may write:

$$f = -\frac{1}{2m} \text{Ric}(X) = -\frac{1}{2m} \text{Ric}(\mathcal{T}).$$

Clearly, by (2.3) the distribution $D_X = \{X, \mathcal{T}\}$ is involutive, and since the property of exterior concurrency is preserved by linearity, one may say that D_X is an autoparallel exterior concurrent distribution whose leaves are surfaces of constant Ricci curvature.

On the other hand, one derives from (2.1):

$$(2.12) \quad \nabla \|X\|^2 = c \|X\|^2 \mathcal{T} - 2s_0 X, \quad s_0 = \text{const.},$$

and one may write

$$(2.13) \quad \|\nabla \|X\|^2\|^2 = 8\|X\|^4 f + 2s_0^2 \|X\|^2,$$

and one also infers from (2.12):

$$(2.14) \quad \text{div}(\nabla \|X\|^2) = 2(2m+1)f \|X\|^2 - 2s_0.$$

Hence, since $\|\nabla\|X\|^2\|^2$ and $\operatorname{div}(\nabla\|X\|^2)$ are functions of $\|X\|^2$, we conclude that $\|X\|^2 : \mathbb{R}^{2m+1} \rightarrow \mathbb{R}$ is an isoparametric function (see 1).

Further, by the well known formula $\Delta\mu = -\operatorname{div}\nabla\mu, \mu \in C^\infty M$, it follows from (2.14) that

$$(2.15) \quad \Delta\|X\|^2 = -2(2m+1)f\|X\|^2 + 2s_0.$$

This equation affirms that the necessary and sufficient condition in order that $\|X\|^2$ be an eigenfunction of Δ is that the constant s_0 vanishes. In this case, since the constant $f = \|\mathcal{T}\|^2$ is positive definite, it follows by a known Proposition that the manifold M under consideration cannot be compact (see also [BR]).

In another order of ideas, remember that a vector field Z is affine if $\mathcal{L}_Z\nabla Z = 0$.

Then, coming back to the case under discussion, one finds by (2.9) and (2.10):

$$(2.16) \quad \mathcal{L}_X\nabla X = s_0 X^\flat \otimes \mathcal{T},$$

and so by (2.15) and (2.16) we may assert that the conditions

- (i) $\|X\|^2$ is an eigenfunction of Δ ;
- (ii) X is an affine vector field

are equivalent.

Finally, denote by Σ the exterior differential system which determines the vector field X . Then by (2.3) and (2.5) it is seen that the characteristic numbers (or E. Cartan's numbers) of Σ are $r = 2, s_0 = 0, s_1 = 2$. Since $r = s_0 + s_1$, it follows that Σ is in involution and by E. CARTAN's test [C], we conclude that the existence of X is determined by an arbitrary function of one argument.

Summing up, we state the

Theorem 2.1. *Let $M(\phi, \Omega, \eta, \xi, J)$ be the CQSQR manifold of dimension $2m+1$ under consideration. The existence of an SSK vector field X having a TF vector field $\mathcal{T}$ as generative is assured by an exterior differential system in involution and the following properties hold:*

- (i) M is foliated by surfaces M_X of constant Ricci curvature, tangent to X and $\mathcal{T}$;
- (ii) $\|X\|^2$ is an isoparametric function;
- (iii) the conditions $\|X\|^2$ is an eigenfunction of Δ and X is an affine vector field are equivalent.

3. Strong automorphisms

Let Y be any vector field on a cosymplectic quasi-Sasakian manifold M and let Ω (resp. η) be the structure 2-form (resp. the structure 1-form) which defines the cosymplectic structure $(1 \times Sp(2m, \mathbb{R}))$ of M .

Following a known definition, if Y defines an infinitesimal automorphism of both Ω and η , i.e.

$$(3.1) \quad \mathcal{L}_Y \Omega = 0, \quad \mathcal{L}_Y \eta = 0,$$

one says that Y is a strong automorphism of $(1 \times Sp(2m, \mathbb{R}))$.

Assume that M is a CQSQR manifold and set

$$(3.2) \quad Y = Y^a e_a + Y^0 \xi, \quad a \in \{1, \dots, 2m\}.$$

Since $d\Omega = 0$ and $\mathcal{L}_Y = di_Y + i_Y d$, one may write

$$(3.3) \quad \mathcal{L}_Y \Omega = 0 \iff d^b Y = 0 \iff d(\phi Y)^b = 0,$$

where ${}^b Y$ is the symplectic isomorphism.

In addition, since $d\eta = 0$, it is seen that $X\eta(Y) = 0$ (i.e. $Y^0 = \text{const.}$).

One finds after some calculations

$$(3.4) \quad (\phi Y)^b = \Sigma(Y^i \omega^{i*} - Y^{i*} \omega^i),$$

then from (1.8), (1.11) and (3.4), the equation (3.3) is expressed by

$$(3.5) \quad \begin{cases} dY^i + Y^a \theta_a^i - cY^i \eta = \lambda \omega^i, \\ dY^{i*} + Y^a \theta_a^{i*} + cY^{i*} \eta = -\lambda \omega^{i*}, \end{cases}$$

where λ is a certain scalar field.

Now, using (2.2) and carrying out the calculations one derives:

$$(3.6) \quad \nabla Y = \mathcal{A}((\lambda + cY^0)dp + c(Y \wedge \xi)) - c(Y^i \omega^i - Y^{i*} \omega^{i*}) \otimes \xi,$$

where $\mathcal{A} = \phi \circ J$ is the mixed anti-invariant operator.

From (3.6) we quickly find

$$g(\nabla_Z Y, Z') + g(\nabla_{Z'} Y, Z) = 2(\lambda + cY^0)g(Z, \mathcal{A}Z'), \quad Z, Z' \in \Gamma TM,$$

which says that in order that Y be a Killing vector, the necessary and sufficient condition is that the conformal scalar associated with Y satisfies

$$\lambda + cY^0 = 0.$$

Theorem 3.1. *Let Y be a strong automorphism in the CQSQR manifold defined in Section 2, $Y^0 = \eta(Y)$ the constant vertical component of Y and λ the associated conformal scalar of Y . Then the necessary and sufficient condition in order that Y be a Killing vector is that*

$$\lambda + cY^0 = 0$$

holds good.

4. Principal vector fields

Let M_ξ be a hypersurface defined by $\eta = 0$, which foliates the manifold $M(\phi, \Omega, \eta, \xi, \mathcal{A})$ under consideration and let

$$L : TM_\xi \rightarrow TM_\xi, \quad LV = \nabla_V \xi$$

be the Weingarten map.

One finds from (1.15)

$$(4.1) \quad \begin{cases} L(JV + \phi V) = -c(JV + \phi V), \\ L(JV - \phi V) = c(JV - \phi V), \end{cases}$$

where J is the anti-invariant operator on M_ξ and V denotes any horizontal vector field.

The vector fields

$$W = JV + \phi V, \quad \bar{W} = JV - \phi V, \quad \eta(V) = 0,$$

have been defined in [BR] as the principal vector fields of M_ξ (see also [Ph]).

Taking into account (1.7) and the operators J and ϕ , one finds

$$(4.3) \quad \nabla W = dW^i \otimes e_{i^*} + W^i (\theta_{i^*}^a \otimes e_a + c\omega^{i^*} \otimes \xi),$$

and expressing that W is an SSK vector field having ξ as generative, one refinds Rosca's lemma

$$(4.4) \quad dW^b = 2\eta \wedge W^b,$$

and in addition

$$(4.5) \quad c = -1,$$

$$(4.6) \quad \begin{cases} dW^i + W^j \theta_j^{i*} = W^i \eta, \\ W^i \theta_j^{i*} = 0. \end{cases}$$

In these conditions one finds

$$(4.7) \quad (\phi W)^b = -W^i \omega^i = -i_W \Omega,$$

and making use of (1.1) and $\mathcal{L}_W = di_W + i_W d$, one infers

$$(4.8) \quad d(\phi W)^b = 0 \Leftrightarrow \mathcal{L}_W \Omega = 0.$$

Also, we find that W is a horizontal vector field, i.e. $\eta(W) = 0$, if and only if $\mathcal{L}_W \eta = 0$. Thus W defines a strong automorphism of the cosymplectic structure $(1 \times Sp(2m, \mathbb{R}))$ of M .

Proceeding in a similar manner for the associated principal vector field $\bar{W}$ of W , one finds that the essential scalar c is equated by $+1$ and like W , the vector field $\bar{W}$ defines a strong automorphism of the $(1 \times Sp(2m, \mathbb{R}))$ -structure considered.

On the other hand, it is easily seen that one has $d\|W\|^2 = 2\|W\|^2 \eta$ and $d\|\bar{W}\|^2 = 2\|\bar{W}\|^2 \eta$ and similarly as for $\|X\|^2$, we may prove that $\|W\|^2$ and $\|\bar{W}\|^2$ are isoparametric functions.

Theorem 4.1. *Let M_ξ be the hypersurface defined by $\eta = 0$ and let W and $\bar{W}$ be the principal vector fields defined by the Weingarten map L . If W and $\bar{W}$ are SSK vector fields having $\xi = \eta^\sharp$ as generative, then both W and $\bar{W}$ define a strong automorphism of the $(1 \times Sp(2m, \mathbb{R}))$ -structure carried by the manifold M (CQSQR) under consideration.*

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