

On the norm form inequality $|F(\mathbf{x})| \leq h$

By JAN-HENDRIK EVERTSE (Leiden)

To Professor Kálmán Győry on his 60-th birthday

Abstract. Let $F \in \mathbb{Z}[X_1, \dots, X_n]$ be a non-degenerate norm form of degree r . In his paper [17] from 1990, SCHMIDT conjectured that for the number $Z_F(h)$ of solutions of the inequality $|F(\mathbf{x})| \leq h$ in $\mathbf{x} \in \mathbb{Z}^n$ one has $Z_F(h) \leq c(n, r)h^{n/r}$, with $c(n, r)$ depending on n and r only. In this paper, we show that

$$Z_F(h) \leq (16r)^{\frac{1}{3}(n+11)^3} h^{\left(n + \sum_{m=2}^{n-1} \frac{1}{m}\right)/r} (1 + \log h)^{\frac{1}{2}n(n-1)}.$$

1. Introduction

We start with recalling some results about inequalities as in the title in two variables, i.e., Thue inequalities

$$(1.1) \quad |F(x, y)| \leq h \quad \text{in } x, y \in \mathbb{Z}$$

where $F(X, Y) = a_r X^r + a_{r-1} X^{r-1} Y + \dots + a_0 Y^r \in \mathbb{Z}[X, Y]$ is a binary form which is irreducible over $\mathbb{Q}$. Assume that F has degree $r \geq 3$. In 1933, MAHLER [10] showed that for the number $Z_F(h)$ of solutions of (1.1) one has

$$Z_F(h) = C_F \cdot h^{2/r} + O\left(h^{1/(r-1)}\right) \quad \text{as } h \rightarrow \infty \quad \text{with } C_F = \iint_{|F(x,y)| \leq 1} dx dy$$

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where the constant implied by the O -symbol depends on F . Note that the main term $C_F \cdot h^{2/r}$ is just the area of the region $\{(x, y) \in \mathbb{R}^2 : |F(x, y)| \leq h\}$.

In 1987, BOMBIERI and SCHMIDT [3] showed that the Thue equation $|F(x, y)| = 1$ has only $\ll r$ solutions in $x, y \in \mathbb{Z}$ (where here and below constants implied by $\ll$ are absolute) and that the dependence on r is best possible. Also in 1987, SCHMIDT [15] proved more generally that $Z_F(h) \ll rh^{2/r}(1 + \frac{1}{r} \log h)$ for every $h \geq 1$ and he conjectured that the $\log h$ -factor is unnecessary. THUNDER [18], [19] showed that $Z_F(h) \ll rh^{2/r}$ if $\log \log m > r^9$ and $Z_F(h) \ll (r^{10}/\log r)h^{2/r}$ otherwise.

In 1993, BEAN [1] showed that for every binary form $F \in \mathbb{Z}[X, Y]$ of degree $r \geq 3$ one has $C_F < 16$. In certain special cases, MUELLER and SCHMIDT [11] and THUNDER [18], [19] obtained explicit estimates $|Z_F(h) - C_F h^{2/r}| \leq c(r)h^{d(r)}$ with $c(r)$ and $d(r)$ depending only on r and $d(r) < 2/r$. Recently, THUNDER [20] showed that for a binary cubic form $F \in \mathbb{Z}[X, Y]$ of discriminant $D(F)$ which is irreducible over $\mathbb{Q}$, one has $|Z_F(h) - C_F h^{2/3}| < 9 + 2008h^{1/2}|D(F)|^{-1/12} + 3156h^{1/3}$.

Now let F be a norm form of degree r in $n \geq 3$ variables, that is,

$$(1.2) \quad F = cN_{K/\mathbb{Q}}(\alpha_1 X_1 + \cdots + \alpha_n X_n) = c \prod_{i=1}^r \left(\alpha_1^{(i)} X_1 + \cdots + \alpha_n^{(i)} X_n \right),$$

where $K = \mathbb{Q}(\alpha_1, \dots, \alpha_n)$ is a number field of degree r , $\alpha \mapsto \alpha^{(i)}$ ($i = 1, \dots, r$) denote the isomorphic embeddings of K into $\mathbb{C}$, and c is a non-zero rational number such that F has its coefficients in $\mathbb{Z}$. To F we associate the $\mathbb{Q}$ -vector space

$$(1.3) \quad V := \{\alpha_1 x_1 + \cdots + \alpha_n x_n : x_1, \dots, x_n \in \mathbb{Q}\}.$$

For each subfield J of K we define the linear subspace of V ,

$$(1.4) \quad V^J := \{\xi \in V : \xi \lambda \in V \text{ for every } \lambda \in J\}.$$

It is easy to see that $\xi \lambda \in V^J$ for $\xi \in V^J$, $\lambda \in J$, so V^J is the largest subspace of V closed under multiplication by elements from J . The norm form F is said to be *non-degenerate* if $\alpha_1, \dots, \alpha_n$ are linearly independent over $\mathbb{Q}$ and if $V^J = (0)$ for each subfield J of K which is not equal to $\mathbb{Q}$ or to an imaginary quadratic field. It is easy to show that this notion of non-degeneracy does not depend on the choice of $c, \alpha_1, \dots, \alpha_n$ in (1.2).

Denote by $Z_F(h)$ the number of solutions of the norm form inequality

$$(1.5) \quad |F(\mathbf{x})| \leq h \quad \text{in } \mathbf{x} \in \mathbb{Z}^n,$$

where $h > 0$. SCHMIDT's famous result on norm form equations from 1971 ([14], Satz 2, p. 5) can be rephrased as follows:

$Z_F(h)$ is finite for every $h > 0 \iff F$ is non-degenerate.

In view of Mahler's result one expects that for arbitrary non-degenerate norm forms F there is an asymptotic formula

$$(1.6) \quad Z_F(h) = C_F \cdot h^{n/r} + O(h^{d(n,r)}) \quad \text{as } h \rightarrow \infty$$

where C_F is the volume of the region $\{\mathbf{x} \in \mathbb{R}^n : |F(\mathbf{x})| \leq 1\}$ and where $d(n, r) < n/r$. By a result of BEAN and THUNDER [2] we have $C_F \leq n^{cn}$ for some absolute constant c . Note that the main term is precisely the volume of the region $\{\mathbf{x} \in \mathbb{R}^n : |F(\mathbf{x})| \leq h\}$.

As yet, only for norm forms from a restricted class such an asymptotic formula has been derived. In 1969, RAMACHANDRA [12] proved that for norm forms F of the special shape $F = cN_{K/\mathbb{Q}}(X_1 + \alpha X_2 + \alpha^2 X_3 + \cdots + \alpha^{n-1} X_n)$, where $K = \mathbb{Q}(\alpha)$ is a number field of degree r and $r \geq 8n^6$, one has an asymptotic formula (1.6) with $(n-1)/(r-n+2) < d(n, r) < n/r$. This was generalised recently by DE JONG [9], who showed that there is an asymptotic formula (1.6) for norm forms F as in (1.2) satisfying the following three conditions: a) each n -tuple among the linear factors of F is linearly independent; b) the Galois group of the normal closure of K over $\mathbb{Q}$ acts $n-1$ times transitively on the set of conjugates $\{\alpha^{(1)}, \dots, \alpha^{(r)}\}$ of $\alpha \in K$; c) $r \geq 2n^{5/3}$. In the results of Ramachandra and de Jong, the constant in the error term depends on F and is ineffective. THUNDER [21] obtained a formula (1.6) for norm forms F in $n = 3$ variables satisfying de Jong's conditions a) and b) and no further restriction on r with an effective error term depending on F . For arbitrary norm forms F in $n \geq 4$ variables, THUNDER [21] could show only that the set of solutions of (1.5) can be divided into two sets, S_1 and S_2 , say, where for the cardinality of S_1 we have an effective asymptotic formula like the right-hand side of (1.6) and where the set S_2 lies in the union of not more than $c(F)(1 + \log h)^{n-1}$ proper linear subspaces of $\mathbb{Q}^n$.

In this paper we do not consider the problem to derive an asymptotic formula such as (1.6) but instead to derive an explicit upper bound for $Z_F(h)$. In 1989, SCHMIDT [17] showed that for arbitrary non-degenerate norm forms F of degree r in n variables, the number $Z_F(1)$ of solutions of $|F(\mathbf{x})| = 1$ in $\mathbf{x} \in \mathbb{Z}^n$ is at most $\min \left(r^{2^{30n}r^2}, r^{(2n)^{n^4+4}} \right)$.

This was improved by the author [6] to $(2^{33}r^2)^{n^3}$. Also in his paper [17], SCHMIDT conjectured that in general one has $Z_F(h) \leq c(n, r)h^{n/r}$ where $c(n, r)$ depends only on n and r .¹

What we can prove is much less. Our main result is as follows:

Theorem 1. *Let $F \in \mathbb{Z}[X_1, \dots, X_n]$ be a non-degenerate norm form of degree r in $n \geq 2$ variables and let $h \geq 1$. Then for the number of solutions $Z_F(h)$ of $|F(\mathbf{x})| \leq h$ in $\mathbf{x} \in \mathbb{Z}^n$ one has*

$$(1.7) \quad Z_F(h) \leq (16r)^{\frac{1}{3}(n+11)^3} \cdot h^{(n+\sum_{m=2}^{n-1} \frac{1}{m})/r} \cdot (1 + \log h)^{\frac{1}{2}n(n-1)}.$$

Except for Ramachandra's, all results on norm form equations mentioned above use Schmidt's Subspace theorem in a qualitative or quantitative form; in particular, the results giving explicit upper bounds for $Z_F(1)$ use SCHMIDT's quantitative Subspace Theorem from 1989 [16] or improvements of the latter. In our proof of Theorem 1 we use a recent quantitative version of the Subspace Theorem due to SCHLICKWEI and the author [8]. In fact, using this we first compute an upper bound for the number of proper linear subspaces of $\mathbb{Q}^n$ containing the set of solutions of (1.5) (cf. Theorem 2 below) and then obtain Theorem 1 by induction on n .

We introduce some notation used in the statement of Theorem 2. For a linear form $L = \alpha_1 X_1 + \dots + \alpha_n X_n$ with complex coefficients, we write $\bar{L} := \bar{\alpha}_1 X_1 + \dots + \bar{\alpha}_n X_n$ where $\bar{\alpha}$ denotes the complex conjugate of $\alpha \in \mathbb{C}$. Let F be the norm form given by (1.2). We assume henceforth that the isomorphic embeddings of K into $\mathbb{C}$ are so ordered that $\alpha \mapsto \alpha^{(i)}$ ($i = 1, \dots, r_1$) map K into $\mathbb{R}$ and that $\alpha^{(i+r_2)} = \overline{\alpha^{(i)}}$ for $i = r_1 + 1, \dots, r_1 + r_2$, where $r_1 + 2r_2 = r$. There are linear forms $L_1, \dots, L_r$ in n variables such that

$$(1.8) \quad \begin{cases} F = \pm L_1 \dots L_r, \\ L_1, \dots, L_{r_1} \text{ have real coefficients,} \\ L_{i+r_2} = \bar{L}_i \text{ for } i = r_1 + 1, \dots, r_1 + r_2 \end{cases}$$

¹Schmidt's conjecture has been proved very recently by Thunder.

(for instance, one may take $L_i = \sqrt[r]{|c|} \cdot (\alpha_1^{(i)} X_1 + \cdots + \alpha_n^{(i)} X_n)$ for $i=1, \dots, r$). Linear forms $L_1, \dots, L_r$ are not uniquely determined by (1.8). For any set of linear forms $L_1, \dots, L_r$ with (1.8) we define the quantity

$$(1.9) \quad \Delta(L_1, \dots, L_r) := \max_{\{i_1, \dots, i_n\} \subset \{1, \dots, r\}} |\det(L_{i_1}, \dots, L_{i_n})|,$$

where $\det(L_{i_1}, \dots, L_{i_n})$ is the coefficient determinant of $L_{i_1}, \dots, L_{i_n}$, and where the maximum is taken over all subsets $\{i_1, \dots, i_n\}$ of $\{1, \dots, r\}$ of cardinality n . We define the *invariant height* of F by

$$(1.10) \quad H^*(F) := \inf \Delta(L_1, \dots, L_r),$$

where the infimum is taken over all r -tuples of linear forms $L_1, \dots, L_r$ with (1.8). By Lemma 1 in Section 2 of the present paper, we have $H^*(F) \geq 1$.

As usual, we write e for $2.7182\dots$. Let again F be the norm form given by (1.2) and V the vector space given by (1.3). Theorem 2 below holds for norm forms F satisfying instead of non-degeneracy the weaker condition

$$(1.11) \quad \begin{cases} \alpha_1, \dots, \alpha_n & \text{are linearly independent over } \mathbb{Q} \\ V^J \subsetneq V & \text{for each subfield } J \text{ of } K \text{ not equal to } \mathbb{Q} \\ & \text{or an imaginary quadratic number field.} \end{cases}$$

Theorem 2. *Let $F \in \mathbb{Z}[X_1, \dots, X_n]$ be a norm form of degree r in $n \geq 2$ variables with (1.11) and let P be any real ≥ 1 . Then the set of solutions of (1.5) is contained in the union of not more than*

$$(16r)^{(n+10)^2} \cdot \max \left(1, \left(\frac{h^{n/r} \cdot P}{H^*(F)} \right)^{\frac{1}{n-1}} \cdot \left(1 + \frac{\log(eh \cdot H^*(F))}{\log eP} \right)^{n-1} \right)$$

proper linear subspaces of $\mathbb{Q}^n$.

In the proof of Theorem 2 we make as usual a distinction between “small” and “large” solutions. We estimate the number of subspaces containing the small solutions by means of a gap principle which is derived in Section 5. In the proof of this gap principle we partly use arguments from SCHMIDT [16]; the main new idea is probably Lemma 5 in Section 5. We deal with the large solutions by reducing eq. (1.5) to a number of inequalities of the type occurring in the Subspace Theorem (where we more or less follow [6]) and then applying the quantitative result from [8].

We state another consequence of Theorem 2. For a homogeneous polynomial $Q \in \mathbb{C}[X_1, \dots, X_n]$ and a non-singular complex $n \times n$ -matrix B we define the homogeneous polynomial

$$Q^B(\mathbf{X}) := Q(\mathbf{X}B),$$

where $\mathbf{X} = (X_1, \dots, X_n)$ is the row vector consisting of the n variables. Now let $F \in \mathbb{Z}[X_1, \dots, X_n]$ be a norm form of degree r and $L_1, \dots, L_r$ linear forms with (1.8). Further, let B be a non-singular $n \times n$ -matrix with entries in $\mathbb{Z}$. From definition (1.9) it follows at once that

$$\Delta(L_1^B, \dots, L_r^B) = |\det B| \cdot \Delta(L_1, \dots, L_r).$$

Now clearly, if $L_1, \dots, L_r$ run through all factorisations of F with (1.8), then $L_1^B, \dots, L_r^B$ run through all factorisations of F^B with (1.8). Hence the invariant height defined by (1.10) satisfies

$$(1.12) \quad H^*(F^B) = |\det B| \cdot H^*(F).$$

Two norm forms $F, G \in \mathbb{Z}[X_1, \dots, X_n]$ are said to be *equivalent* if $G = F^B$ for some matrix $B \in GL_n(\mathbb{Z})$, i.e., with $\det B = \pm 1$. Thus, a special case of (1.12) is that

$$(1.13) \quad H^*(G) = H^*(F) \quad \text{for equivalent norm forms } F, G.$$

For a norm form $F \in \mathbb{Z}[X_1, \dots, X_n]$, let $\|F\|$ denote the maximum of the absolute values of its coefficients. In [17], SCHMIDT developed a reduction theory for norm forms, which implies that for every norm form $F \in \mathbb{Z}[X_1, \dots, X_n]$ of degree r there is a matrix $B \in GL_n(\mathbb{Z})$ such that $\|F^B\|$ is bounded from above in terms of r, n and $H^*(F)$. By combining this with Theorem 2, we show in an explicit form the following: there is a finite union of equivalence classes depending on h , such that for all norm forms F outside this union, the set of solutions of (1.5) is contained in the union of at most a quantity independent of h proper linear subspaces of $\mathbb{Q}^n$.

Theorem 3. *Let $h \geq 1$ and let $F \in \mathbb{Z}[X_1, \dots, X_n]$ be a norm form of degree r in $n \geq 2$ variables with (1.11) and with*

$$(1.14) \quad \min_{B \in GL_n(\mathbb{Z})} \|F^B\| \geq (32n)^{nr/2} h^{2n}.$$

Then the set of solutions of (1.5) is contained in the union of not more than

$$(16r)^{(n+11)^2}$$

proper linear subspaces of $\mathbb{Q}^n$.

The case $n = 2$ of Theorem 3 was considered earlier by GYÖRY and the author in [7]. Note that each one-dimensional subspace of $\mathbb{Q}^2$ contains precisely two primitive points, i.e., points with coordinates in $\mathbb{Z}$ whose gcd is equal to 1. Combining the method of BOMBIERI and SCHMIDT [3] with linear forms in logarithms estimates, Györy and the author showed that the Thue inequality $|F(x, y)| \leq h$ has at most $12r$ primitive solutions $(x, y) \in \mathbb{Z}^2$ provided that $F \in \mathbb{Z}[X, Y]$ is an irreducible binary form of degree $r \geq 400$ with $\min_{B \in GL_2(\mathbb{Z})} \|F^B\| \geq \exp(c_1(r)h^{10(r-1)^2})$ for some effectively computable function $c_1(r)$ of r . Theorem 3 gives the much worse upper bound $2 \times (16r)^{169}$ for the number of primitive solutions of $|F(x, y)| \leq h$ but subject to the much weaker constraints that F have degree $r \geq 3$ and $\min_{B \in GL_2(\mathbb{Z})} \|F^B\| \geq 2^{6r}h^4$. The polynomial dependence on h of this last condition is because in the proof of Theorem 3 no linear forms in logarithms estimates were used. Under a similar condition on F and for all $r \geq 3$, Györy obtained the upper bound $28r$ (personal communication).

One may wonder whether there is a sharpening of Theorem 3 which gives for all norm forms F in $n \geq 3$ variables lying outside some union of finitely many equivalence classes, an upper bound independent of h for the number of primitive solutions of equation (1.5) instead of just for the number of subspaces. It was already explained in [5] that such a sharpening does not exist. To construct a counterexample, one takes a number field K of degree r and fixes $\mathbb{Q}$ -linearly independent $\alpha_1, \dots, \alpha_{n-1} \in K$ and $c \in \mathbb{Q}^*$ such that the norm form $cN_{K/\mathbb{Q}}(\alpha_1 X_1 + \dots + \alpha_{n-1} X_{n-1})$ has its coefficients in $\mathbb{Z}$. Now if α_n runs through all algebraic integers of K , then $F := cN_{K/\mathbb{Q}}(\alpha_1 X_1 + \dots + \alpha_{n-1} X_{n-1} + \alpha_n X_n)$ runs through infinitely many pairwise inequivalent norm forms in $\mathbb{Z}[X_1, \dots, X_n]$. Clearly, one has $|F(\mathbf{x})| \leq h$ for every algebraic integer $\alpha_n \in K$ and each primitive vector $\mathbf{x} = (x_1, \dots, x_{n-1}, 0)$ with $x_i \in \mathbb{Z}$, $|x_i| \ll h^{1/r}$ for $i = 1, \dots, n-1$, the constant implied by $\ll$ depending only on $K, c, \alpha_1, \dots, \alpha_{n-1}$. Thus, there are infinitely many pairwise inequivalent norm forms $F \in \mathbb{Z}[X_1, \dots, X_n]$ of degree r such that for every such F and for every $h \gg 1$, the inequality $|F(\mathbf{x})| \leq h$ has $\gg h^{(n-1)/r}$ primitive solutions $\mathbf{x} \in \mathbb{Z}^n$ lying in the subspace $x_n = 0$.

2. Proof of Theorem 1

In this section, we deduce Theorem 1 from Theorem 2. Let $F \in \mathbb{Z}[X_1, \dots, X_n]$ be the norm form of degree r given by (1.2) and let K , $\alpha_1, \dots, \alpha_n$ and c be as in (1.2). Assume that F is non-degenerate. The cardinality of a set $\mathcal{I}$ is denoted by $|\mathcal{I}|$. We need the following lemma:

Lemma 1. $H^*(F) \geq 1$.

PROOF. Choose linear forms $L_1, \dots, L_r$ with (1.8). Let $\mathcal{I}$ denote the collection of ordered n -tuples $(i_1, \dots, i_n)$ from $\{1, \dots, r\}$ for which $\det(L_{i_1}, \dots, L_{i_n}) \neq 0$. According to SCHMIDT ([17], p. 203), the semi-discriminant

$$D(F) := \prod_{(i_1, \dots, i_n) \in \mathcal{I}} |\det(L_{i_1}, \dots, L_{i_n})|$$

is a positive integer. This implies that

$$\Delta(L_1, \dots, L_r) = \max_{(i_1, \dots, i_n) \in \mathcal{I}} |\det(L_{i_1}, \dots, L_{i_n})| \geq D(F)^{1/|\mathcal{I}|} \geq 1.$$

By taking the infimum over all $L_1, \dots, L_r$ with (1.8) we obtain Lemma 1. $\square$

Assume that F is non-degenerate. Denote by $Z_F^*(h)$ the number of primitive solutions of (1.5), i.e., with $\gcd(x_1, \dots, x_n) = 1$.

Lemma 2. $Z_F^*(h) \leq \frac{1}{3} \cdot (16r)^{\frac{1}{3}(n+11)^3} \cdot h^{\left(n + \sum_{m=2}^{n-1} \frac{1}{m}\right)/r} \cdot (1 + \log h)^{\frac{1}{2}n(n-1)}.$

PROOF. Denote by $A(n, r, h)$ the right-hand side of the inequality in Lemma 2. We proceed by induction on n . First, let $n = 2$. Since F is non-degenerate, condition (1.11) is satisfied. On applying Theorem 2 with $P = H^*(F)$ (which is allowed by Lemma 1), we infer that the set of solutions of (1.5) is contained in the union of not more than

$$(16r)^{64} h^{2/r} \left(1 + \frac{\log(eh \cdot H^*(F))}{\log eH^*(F)}\right) \leq (16r)^{64} h^{2/r} (2 + \log h) \leq \frac{1}{2} A(2, r, h)$$

proper one-dimensional linear subspaces of $\mathbb{Q}^2$. Using that each one-dimensional subspace contains at most two primitive solutions, we get $Z_F^*(h) \leq A(2, r, h)$.

Now let $n \geq 3$. Again, (1.11) holds since F is non-degenerate, and again from Theorem 2 with $P = H^*(F)$ we infer that the set of solutions of (1.5) is contained in the union of not more than

$$\begin{aligned} & (16r)^{(n+10)^2} h^{\frac{n}{(n-1)r}} \left(1 + \frac{\log(eh \cdot H^*(F))}{\log eH^*(F)} \right)^{n-1} \\ & \leq (16r)^{(n+10)^2} 2^{n-1} \cdot h^{\frac{n}{(n-1)r}} (1 + \log h)^{n-1} =: B(n, r, h) \end{aligned}$$

proper linear subspaces of $\mathbb{Q}^n$.

There is no loss of generality to assume that these subspaces have dimension $n-1$. Let T be one of these subspaces and consider the solutions of (1.5) lying in T . Fix a basis $\{\mathbf{a}_i = (a_{i1}, \dots, a_{in}) : i = 1, \dots, n-1\}$ of the $\mathbb{Z}$ -module $T \cap \mathbb{Z}^n$ and define the norm form $G \in \mathbb{Z}[Y_1, \dots, Y_{n-1}]$ in $n-1$ variables by

$$G := F(Y_1 \mathbf{a}_1 + \dots + Y_{n-1} \mathbf{a}_{n-1}).$$

Clearly, there is a one-to-one correspondence between the primitive solutions of (1.5) lying in T and the primitive solutions of

$$(2.1) \quad |G(\mathbf{y})| \leq h \quad \text{in } \mathbf{y} \in \mathbb{Z}^{n-1}.$$

Note that since $F = cN_{K/\mathbb{Q}}(\alpha_1 X_1 + \dots + \alpha_n X_n)$ we have

$$G = cN_{K/\mathbb{Q}}(\beta_1 Y_1 + \dots + \beta_{n-1} Y_{n-1})$$

$$\text{with } \beta_i = \sum_{j=1}^n a_{ij} \alpha_j \text{ for } i = 1, \dots, n-1.$$

The vector space associated to G is $W := \{\beta_1 y_1 + \dots + \beta_{n-1} y_{n-1} : y_1, \dots, y_{n-1} \in \mathbb{Q}\}$. As usual, for each subfield J of K we define $W^J := \{\xi \in W : \lambda \xi \in W \text{ for every } \lambda \in J\}$. We verify that G is non-degenerate. First, the numbers $\beta_1, \dots, \beta_{n-1}$ are linearly independent over $\mathbb{Q}$ since $\alpha_1, \dots, \alpha_n$ are $\mathbb{Q}$ -linearly independent and since the vectors $\mathbf{a}_1, \dots, \mathbf{a}_{n-1}$ are $\mathbb{Q}$ -linearly independent. Second, since $W \subset V$ and F is non-degenerate we have that $W^J \subset V^J = (0)$ if J is not equal to $\mathbb{Q}$ or to an imaginary quadratic field.

We infer from the induction hypothesis that the number of primitive solutions of (2.1), and hence the number of primitive solutions of (1.5) lying

in T , is at most $A(n-1, r, h)$. Since we have at most $B(n, r, h)$ possibilities for T , we conclude that the total number of primitive solutions of (1.5) is at most

$$\begin{aligned}
& A(n-1, r, h) \cdot B(n, r, h) \\
&= \frac{1}{3} \cdot 2^{n-1} (16r)^{\frac{1}{3}(n+10)^3 + (n+10)^2} \cdot h^{(n-1 + (\sum_{m=2}^{n-2} \frac{1}{m}) + \frac{n}{n-1})/r} \\
&\quad \cdot (1 + \log h)^{\frac{1}{2}(n-1)(n-2) + n-1} \\
&\leq \frac{1}{3} \cdot (16r)^{\frac{1}{3}(n+11)^3} \cdot h^{(n + \sum_{m=2}^{n-1} \frac{1}{m})/r} \cdot (1 + \log h)^{\frac{1}{2}n(n-1)} \\
&= A(n, r, h). \quad \square
\end{aligned}$$

PROOF of Theorem 1. We have to prove that $Z_F(h) \leq \psi(h)h^{n/r}$, where

$$\psi(h) = (16r)^{\frac{1}{3}(n+11)^3} h^{(\sum_{m=2}^{n-1} \frac{1}{m})/r} (1 + \log h)^{\frac{1}{2}n(n-1)}.$$

For $c = 0, \dots, h$, denote by $a(c)$ the number of primitive solutions $\mathbf{x} \in \mathbb{Z}^n$ of $|F(\mathbf{x})| = c$ and by $b(c)$ the number of all solutions $\mathbf{x} \in \mathbb{Z}^n$ of $|F(\mathbf{x})| = c$. Thus,

$$a(0) = 0, \quad b(0) = 1, \quad b(c) = \sum_{d: d^r | c, d > 0} a(c/d^r) \quad \text{for } c > 0,$$

$$Z_F(h) = \sum_{c=0}^h b(c), \quad Z_F^*(h) = \sum_{c=0}^h a(c), \quad \text{for } h \geq 0.$$

This implies, on interchanging the summation and then using $a(c) = Z_F^*(c) - Z_F^*(c-1)$ for $c \geq 1$,

$$\begin{aligned}
Z_F(h) &= 1 + \sum_{c=1}^h \sum_{d: d^r | c, d > 0} a(c/d^r) \leq 1 + \sum_{c=1}^h a(c) \sqrt[r]{h/c} \\
&= 1 + Z_F^*(h) + \sum_{c=1}^{h-1} \left(\sqrt[r]{h/c} - \sqrt[r]{h/(c+1)} \right) \cdot Z_F^*(c).
\end{aligned}$$

By Lemma 2 we have $Z_F^*(c) \leq \frac{1}{3}\psi(h)c^{n/r}$ for $c \leq h$. Hence

$$\begin{aligned} Z_F(h) &\leq 1 + \frac{1}{3}\psi(h) \left(h^{n/r} + \sum_{c=1}^{h-1} \left(\sqrt[r]{h/c} - \sqrt[r]{h/(c+1)} \right) \cdot c^{n/r} \right) \\ &= 1 + \frac{1}{3}\psi(h) \sum_{c=1}^h \sqrt[r]{h/c} \cdot (c^{n/r} - (c-1)^{n/r}) \\ &\leq 1 + \frac{1}{3}\psi(h) \int_0^h \sqrt[r]{h/x} \cdot \frac{n}{r} x^{\frac{n}{r}-1} dx = 1 + \frac{1}{3}\psi(h) \frac{n}{n-1} h^{n/r} \\ &\leq \psi(h) h^{n/r}. \end{aligned}$$

This proves Theorem 1. $\square$

3. Proof of Theorem 3

Let $F \in \mathbb{Z}[X_1, \dots, X_n]$ be a norm form of degree r satisfying (1.2), (1.11). Define the quantity

$$H_2^*(F) := \inf \left(\sum_{\{i_1, \dots, i_n\} \subset \{1, \dots, r\}} |\det(L_{i_1}, \dots, L_{i_n})|^2 \right)^{1/2},$$

where the sum is taken over all subsets of $\{1, \dots, r\}$ of cardinality n and the infimum over all r -tuples of linear forms $L_1, \dots, L_r$ with (1.8). By SCHMIDT's reduction theory for norm forms (cf. [17], Lemma 4) we have

$$\min_{B \in GL_n(\mathbb{Z})} \|F^B\| \leq (2^n n^{3/2} V(n)^{-1})^r \cdot H_2^*(F)^r,$$

where $V(n)$ is the volume of the n -dimensional Euclidean ball with radius 1. Together with $V(n) \geq (n!)^{1/2}$ and $H_2^*(F) \leq \binom{r}{n}^{1/2} H^*(F)$, this implies

$$\min_{B \in GL_n(\mathbb{Z})} \|F^B\| \leq (32n)^{nr/2} \cdot H^*(F)^r.$$

Now assume that F satisfies (1.14). Then it follows that

$$H^*(F) \geq h^{2n/r}.$$

By applying Theorem 2 with $P = H^*(F)^{1/2}$, we infer that the set of solutions of (1.5) is contained in the union of at most

$$\begin{aligned} & (16r)^{(n+10)^2} \cdot \left(1 + \frac{\log(eh \cdot H^*(F))}{\log eH^*(F)^{1/2}}\right)^n \\ & \leq (16r)^{(n+10)^2} \cdot \left(1 + \frac{\log eH^*(F)^{1+(r/2n)}}{\log eH^*(F)^{1/2}}\right)^{n-1} \leq (16r)^{(n+11)^2} \end{aligned}$$

proper linear subspaces of $\mathbb{Q}^n$. This proves Theorem 3. $\square$

4. Choice of the linear factors

By $\overline{\mathbb{Q}}$ we denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We agree that algebraic number fields occurring in this paper are contained in $\overline{\mathbb{Q}}$. Vectors from $\mathbb{Q}^n$, $\mathbb{R}^n$, etc., will always be row vectors.

Let $F \in \mathbb{Z}[X_1, \dots, X_n]$ be a norm form of degree r satisfying (1.2) for some number field K , some $\alpha_1, \dots, \alpha_n \in K$, and some non-zero $c \in \mathbb{Q}$. As before, we order the isomorphic embeddings of K into $\mathbb{C}$ in such a way that $\alpha \mapsto \alpha^{(i)}$ map K into $\mathbb{R}$ for $i = 1, \dots, r_1$ and $\alpha^{(i+r_2)} = \overline{\alpha^{(i)}}$ for $i = r_1 + 1, \dots, r_1 + r_2$, where $r_1 + 2r_2 = r = [K : \mathbb{Q}]$. In this section we choose appropriate linear factors $L_1, \dots, L_r$ of F satisfying (1.8). The quantities $\Delta(L_1, \dots, L_r)$ and $H^*(F)$ are defined by (1.9), (1.10), respectively.

Lemma 3. *There are linear forms $L_1, \dots, L_r$ which satisfy (1.8) and which have the following additional properties:*

$$(4.1) \quad L_i = \sqrt[k]{\beta^{(i)}} \cdot (\alpha_1^{(i)} X_1 + \dots + \alpha_n^{(i)} X_n) \quad (i = 1, \dots, r)$$

for some $\beta \in K$, $k \in \mathbb{Z}_{>0}$ and choices for the roots $\sqrt[k]{\beta^{(1)}}, \dots, \sqrt[k]{\beta^{(r)}}$;

$$(4.2) \quad L_1, \dots, L_r \text{ have algebraic integer coefficients}$$

$$(4.3) \quad H^*(F) \leq \Delta(L_1, \dots, L_r) \leq 2H^*(F).$$

PROOF. We will frequently use that if $L_1, \dots, L_r$ satisfy (1.8) then $c_1 L_1, \dots, c_r L_r$ satisfy (1.8) if and only if $c_i \in \mathbb{R}$ for $i = 1, \dots, r_1$, $c_{i+r_2} = \overline{c_i}$ for $i = r_1 + 1, \dots, r_1 + r_2$ and $c_1 \dots c_r = \pm 1$.

Suppose that K has class number h . Let $\mathfrak{a}$ denote the fractional ideal in K generated by $\alpha_1, \dots, \alpha_r$. Then $\mathfrak{a}^h$ is a principal ideal with generator $\gamma \in K$, say. Choose roots $\delta_i := \sqrt[2h]{(\gamma^{(i)})^2}$ such that $\delta_i \in \mathbb{R}_{>0}$ for $i = 1, \dots, r_1$ and such that $\delta_{i+r_2} = \overline{\delta_i}$ for $i = r_1 + 1, \dots, r_1 + r_2$ (note that for $i = 1, \dots, r_1$, $(\gamma^{(i)})^2$ is positive so that it has a positive real $2h$ -th root).

Define the linear forms

$$(4.4) \quad \tilde{L}_i := \sqrt[r]{|c| \cdot \delta_1 \dots \delta_r} \cdot \delta_i^{-1} (\alpha_1^{(i)} X_1 + \dots + \alpha_n^{(i)} X_n) \quad \text{for } i = 1, \dots, r,$$

where the r -th root is a positive real. From (1.2) it follows easily that $\tilde{L}_1, \dots, \tilde{L}_r$ satisfy (1.8). We claim that $\tilde{L}_i$ has algebraic integer coefficients for $i = 1, \dots, r$. Let M be a finite extension of K containing the numbers $\alpha_i^{(j)}$, δ_j ($i = 1, \dots, n$, $j = 1, \dots, r$) and $\sqrt[r]{|c| \cdot \delta_1 \dots \delta_r}$. For $\beta_1, \dots, \beta_m \in M$, denote by $[\beta_1, \dots, \beta_m]$ the fractional ideal in M generated by $\beta_1, \dots, \beta_m$. For a polynomial $Q \in M[X_1, \dots, X_n]$, denote by $[Q]$ the fractional ideal in M generated by the coefficients in Q . By the choice of the δ_i we have $[\delta_i]^{2h} = [\gamma^{(i)}]^2 = [\alpha_1^{(i)}, \dots, \alpha_n^{(i)}]^{2h}$, hence $[\delta_i] = [\alpha_1^{(i)}, \dots, \alpha_n^{(i)}]$. Therefore, $[\tilde{L}_i] = [\sqrt[r]{|c| \cdot \delta_1 \dots \delta_r}]$. But according to Gauss' lemma for Dedekind domains we have $[F] = [\tilde{L}_1] \dots [\tilde{L}_r] = [|c| \cdot \delta_1 \dots \delta_r]$. By assumption, F has its coefficients in $\mathbb{Z}$, so $|c| \cdot \delta_1 \dots \delta_r$ is an algebraic integer. This proves our claim.

Let $\theta > 0$. From the definition of $H^*(F)$ it follows at once that there are complex numbers $c_1, \dots, c_r$ with $c_1, \dots, c_{r_1} \in \mathbb{R}$, $c_{i+r_2} = \overline{c_i}$ for $i = r_1 + 1, \dots, r_1 + r_2$ and $c_1 \dots c_r = \pm 1$ such that

$$(4.5) \quad \Delta(c_1 \tilde{L}_1, \dots, c_r \tilde{L}_r) \leq (1 + \theta) H^*(F).$$

We approximate $c_1, \dots, c_r$ by algebraic units. Let U_K denote the unit group of the ring of integers of K . According to Dirichlet's unit theorem, the set $\{(\log |\varepsilon^{(1)}|, \dots, \log |\varepsilon^{(r)}|) : \varepsilon \in U_K\}$ is a lattice which spans the linear subspace $H \subset \mathbb{R}^r$ given by the equations $x_1 + \dots + x_r = 0$, $x_{i+r_2} = x_i$ for $i = r_1 + 1, \dots, r_1 + r_2$. This implies that there is a constant $C_K > 0$ such that for every positive integer m there is an $\varepsilon \in U_K$ with

$$|\log |\varepsilon^{(i)}| - m \log |c_i|| \leq C_K \quad \text{for } i = 1, \dots, r.$$

Choose m so large that $C_K < m \log(1 + \theta)$. Choose roots $\eta_i := \sqrt[2m]{(\varepsilon^{(i)})^2}$ such that $\eta_i \in \mathbb{R}$ for $i = 1, \dots, r_1$ and $\eta_{i+r_2} = \overline{\eta_i}$ for $i = r_1 + 1, \dots, r_1 + r_2$. Thus, $(\log |\eta_1|, \dots, \log |\eta_r|) \in H$ and

$$(4.6) \quad |\log |\eta_i| - \log |c_i|| \leq \log(1 + \theta) \quad \text{for } i = 1, \dots, r.$$

Define the linear forms

$$L_i := \eta_i \cdot \tilde{L}_i = \eta_i \sqrt[r]{|c| \delta_1 \dots \delta_r} \cdot \delta_i^{-1} (\alpha_1^{(i)} X_1 + \dots + \alpha_n^{(i)} X_n) \quad \text{for } i = 1, \dots, r.$$

Note that with $k = 2hmr$ we have

$$\begin{aligned} (\eta_i \sqrt[r]{|c| \delta_1 \dots \delta_r} \cdot \delta_i^{-1})^k &= (\varepsilon^{(i)})^{2hr} |c|^{2hm} (\gamma^{(1)} \dots \gamma^{(r)})^{2m} \\ (\gamma^{(i)})^{-2mr} &= \beta^{(i)} \quad \text{with } \beta := \varepsilon^{2hr} |c|^{2hm} N_{K/\mathbb{Q}}(\gamma)^{2m} \gamma^{-2mr} \in K. \end{aligned}$$

Hence $L_1, \dots, L_r$ satisfy (4.1) for some β, k . It is easy to check that $L_1, \dots, L_r$ satisfy (1.8) and (4.2). To verify (4.3), we observe that by (4.6), (4.5) we have, on choosing $i_1, \dots, i_n \in \{1, \dots, r\}$ with $\Delta(L_1, \dots, L_r) = |\det(L_{i_1}, \dots, L_{i_n})|$,

$$\begin{aligned} \Delta(L_1, \dots, L_r) &= \frac{|\eta_{i_1} \dots \eta_{i_n}|}{|c_{i_1} \dots c_{i_n}|} \cdot |\det(c_{i_1} \tilde{L}_{i_1}, \dots, c_{i_n} \tilde{L}_{i_n})| \\ &\leq (1 + \theta)^n \cdot |\det(c_{i_1} \tilde{L}_{i_1}, \dots, c_{i_n} \tilde{L}_{i_n})| \\ &\leq (1 + \theta)^{n+1} H^*(F) \leq 2H^*(F) \end{aligned}$$

for sufficiently small θ . Lastly, since $L_1, \dots, L_r$ satisfy (1.8) we have $\Delta(L_1, \dots, L_r) \geq H^*(F)$. This proves Lemma 3. $\square$

We recall that for a homogeneous polynomial $Q \in \mathbb{C}[X_1, \dots, X_n]$ and a non-singular complex $n \times n$ -matrix B we define $Q^B(\mathbf{X}) := Q(\mathbf{X}B)$. Further, we denote by $\|Q\|$ the maximum of the absolute values of the coefficients of Q .

Lemma 4. *Let $L_1, \dots, L_r$ be linear forms with (1.8), (4.1), (4.2), (4.3). Then there is a matrix $B \in GL_n(\mathbb{Z})$ such that*

$$(4.7) \quad \|L_i^B\| \leq (2n)^{n+1} H^*(F) \quad \text{for } i = 1, \dots, r.$$

PROOF. We modify an argument of SCHMIDT from [17]. We will apply Minkowski's theorem on successive minima to the symmetric convex body

$$\mathcal{C} := \{\mathbf{x} \in \mathbb{R}^n : |L_i(\mathbf{x})| \leq 1 \text{ for } i = 1, \dots, r\}.$$

We need a lower bound for the volume of $\mathcal{C}$. Recall (1.8). Let $M_1, \dots, M_r$ be the linear forms with real coefficients given by

$$(4.8) \quad \begin{cases} M_i &:= L_i & (i = 1, \dots, r_1), \\ M_i &:= \frac{1}{2}(L_i + \overline{L_i}) = \frac{1}{2}(L_i + L_{i+r_2}) & (i = r_1 + 1, \dots, r_1 + r_2), \\ M_{i+r_2} &:= \frac{1}{2\sqrt{-1}}(L_i - \overline{L_i}) \\ &= \frac{1}{2\sqrt{-1}}(L_i - L_{i+r_2}) & (i = r_1 + 1, \dots, r_1 + r_2). \end{cases}$$

Let $\{j_1, \dots, j_n\}$ be a subset of $\{1, \dots, r\}$ for which $|\det(M_{j_1}, \dots, M_{j_n})|$ is maximal. Suppose that $1 \leq j_1 < \dots < j_s \leq r_1 < j_{s+1} < \dots < j_n$. By (4.8) we have

$$\det(M_{j_1}, \dots, M_{j_n}) = \sum_I \varepsilon_I \det(L_{j_1}, \dots, L_{j_s}, L_{i_{s+1}}, \dots, L_{i_n})$$

where the sum is taken over tuples $I = (i_{s+1}, \dots, i_n)$ with precisely two possibilities for each index i_j and where $|\varepsilon_I| = 2^{s-n}$ for each tuple I . Together with (4.3) this implies

$$(4.9) \quad |\det(M_{j_1}, \dots, M_{j_n})| \leq \Delta(L_1, \dots, L_r) \leq 2H^*(F).$$

From (4.8) it follows that $\text{rank}\{M_1, \dots, M_r\} = n$. Hence $|\det(M_{j_1}, \dots, M_{j_n})| \neq 0$. So there are $c_{ik} \in \mathbb{C}$ with

$$(4.10) \quad L_i = \sum_{k=1}^n c_{ik} M_{j_k} \quad \text{for } i = 1, \dots, r.$$

We estimate $|c_{ik}|$ from above. First suppose $i \leq r_1$. Then $L_i = M_i$ by (4.8), so

$$|c_{ik}| = \frac{|\det(M_{j_1}, \dots, M_i, \dots, M_{j_n})|}{|\det(M_{j_1}, \dots, M_{j_k}, \dots, M_{j_n})|} \leq 1.$$

Now suppose $r_1 + 1 \leq i \leq r_1 + r_2$. Then by (4.8) we have $L_i = M_i + \sqrt{-1}M_{i+r_2}$, so

$$\begin{aligned} |c_{ik}| &= \frac{|\det(M_{j_1}, \dots, L_i, \dots, M_{j_n})|}{|\det(M_{j_1}, \dots, M_{j_k}, \dots, M_{j_n})|} \\ &\leq \frac{|\det(M_{j_1}, \dots, M_i, \dots, M_{j_n})|}{|\det(M_{j_1}, \dots, M_{j_k}, \dots, M_{j_n})|} + \frac{|\det(M_{j_1}, \dots, M_{i+r_2}, \dots, M_{j_n})|}{|\det(M_{j_1}, \dots, M_{j_k}, \dots, M_{j_n})|} \leq 2. \end{aligned}$$

We have a similar estimate for $|c_{ik}|$ for $r_1 + r_2 + 1 \leq i \leq r$. Hence $|c_{ik}| \leq 2$ for $i = 1, \dots, r$, $k = 1, \dots, n$. Together with (4.10) this implies

$$\mathcal{C} \supseteq \mathcal{D} := \{\mathbf{x} \in \mathbb{R}^n : |M_{j_k}(\mathbf{x})| \leq (2n)^{-1} \text{ for } k = 1, \dots, n\}.$$

So by (4.9),

$$\begin{aligned} \text{vol}(\mathcal{C}) &\geq \text{vol}(\mathcal{D}) = 2^n (2n)^{-n} |\det(M_{j_1}, \dots, M_{j_n})|^{-1} \\ (4.11) \quad &\geq \frac{1}{2} n^{-n} H^*(F)^{-1}. \end{aligned}$$

Denote by $\lambda_1, \dots, \lambda_n$ the successive minima of $\mathcal{C}$ with respect to $\mathbb{Z}^n$. Thus, there are linearly independent vectors $\mathbf{b}_1, \dots, \mathbf{b}_n \in \mathbb{Z}^n$ with $\mathbf{b}_j \in \lambda_j \mathcal{C}$, i.e., with $|L_i(\mathbf{b}_j)| \leq \lambda_j$ for $i = 1, \dots, r$, $j = 1, \dots, n$. By Minkowski's theorem and (4.11) we have

$$\lambda_1 \dots \lambda_n \leq 2^n \text{vol}(\mathcal{C})^{-1} \leq 2^{n+1} n^n H^*(F).$$

Further, by (1.8) we have $1 \leq |F(\mathbf{b}_1)| = \prod_{i=1}^n |L_i(\mathbf{b}_1)| \leq \lambda_1^n$. Hence

$$(4.12) \quad \lambda_n \leq 2^{n+1} n^n H^*(F).$$

By a result of Mahler (cf. CASSELS [4], Lemma 8, p. 135), $\mathbb{Z}^n$ has a basis $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ with $\mathbf{b}_j \in j\lambda_j \mathcal{C}$ for $j = 1, \dots, n$. Together with (4.12) this implies

$$|L_i(\mathbf{b}_j)| \leq n\lambda_n \leq (2n)^{n+1} H^*(F) \quad \text{for } i = 1, \dots, r, \quad j = 1, \dots, n.$$

Now Lemma 4 holds for the matrix B with rows $\mathbf{b}_j$ ($j = 1, \dots, n$). □

Let $L_1, \dots, L_r$ be linear forms with (1.8), (4.1)–(4.3) and let B be the matrix from Lemma 4. We now write F for F^B , L_i for L_i^B and replace

everywhere the old forms F, L_i by the new ones just chosen. This affects neither the minimal number of subspaces of $\mathbb{Q}^n$ containing the set of solutions of (1.5) nor the invariant height $H^*(F)$. Further, the conditions (1.8) and (4.1)–(4.3) remain valid (but with different $\alpha_1, \dots, \alpha_n$ in (4.1)). Lastly, condition (4.7) holds but with B being replaced by the identity matrix.

So it suffices to prove Theorem 2 for these newly chosen forms $F, L_1, \dots, L_r$ and we will proceed further with these forms. This means that in the remainder of this paper, F is a norm form in $\mathbb{Z}[X_1, \dots, X_n]$ of degree r of the shape (1.2) satisfying (1.11), K is the number field and $\alpha_1, \dots, \alpha_n$ are the elements of K from (1.2), and $L_1, \dots, L_r$ are linear forms with the following properties:

$$(4.13) \quad F = \pm L_1 \dots L_r;$$

$$(4.14) \quad L_1, \dots, L_{r_1} \text{ have real coefficients};$$

$$(4.15) \quad L_{i+r_2} = \bar{L}_i \quad \text{for } i = r_1 + 1, \dots, r_1 + r_2;$$

$$(4.16) \quad L_i = \sqrt[k]{\beta^{(i)}} \cdot (\alpha_1^{(i)} X_1 + \dots + \alpha_n^{(i)} X_n) \quad (i = 1, \dots, r)$$

for some $\beta \in K$, $k \in \mathbb{Z}_{>0}$ and choices for the roots $\sqrt[k]{\beta^{(1)}}, \dots, \sqrt[k]{\beta^{(r)}}$;

$$(4.17) \quad H^*(F) \leq \Delta(L_1, \dots, L_r) \leq 2H^*(F);$$

$$(4.18) \quad \|L_i\| \leq (2n)^{n+1} H^*(F) \quad \text{for } i = 1, \dots, r;$$

$$(4.19) \quad L_1, \dots, L_r \text{ have algebraic integer coefficients.}$$

We fix once and for all a finite, normal extension $N \subset \mathbb{C}$ of $\mathbb{Q}$ such that N contains K , the images of the isomorphic embeddings $\alpha \mapsto \alpha^{(i)}$ ($i = 1, \dots, r$) of K into $\mathbb{C}$, the coefficients of $L_1, \dots, L_r$ and the k -th roots of unity, where k is the integer from (4.16). Let

$$(4.20) \quad d := [N : \mathbb{Q}]$$

and denote by $\text{Gal}(N/\mathbb{Q})$ the Galois group of $N/\mathbb{Q}$. Clearly, for each $\sigma \in \text{Gal}(N/\mathbb{Q})$ there is a permutation $\sigma^*(1), \dots, \sigma^*(r)$ of $1, \dots, r$ such that

$$(4.21) \quad \sigma(\alpha^{(i)}) = \alpha^{(\sigma^*(i))} \quad \text{for } \alpha \in K, i = 1, \dots, r.$$

For each pair $i, j \in \{1, \dots, r\}$, we have

$$(4.22) \quad \sigma^*(i) = j \text{ for precisely } d/r \text{ elements } \sigma \in \text{Gal}(N/\mathbb{Q}),$$

since the $\mathbb{Q}$ -isomorphism $\alpha^{(i)} \mapsto \alpha^{(j)}$ ($\alpha \in K$) can be extended in exactly d/r ways to an automorphism of N . For a linear form $L = \alpha_1 X_1 + \dots + \alpha_n X_n$ with coefficients in N and for $\sigma \in \text{Gal}(N/\mathbb{Q})$ define $\sigma(L) := \sigma(\alpha_1)X_1 + \dots + \sigma(\alpha_n)X_n$. From (4.21) and (4.16) it follows that there are k -th roots of unity $\rho_{\sigma,i}$ such that

$$(4.23) \quad \sigma(L_i) = \rho_{\sigma,i} L_{\sigma^*(i)} \quad \text{for } i = 1, \dots, r, \sigma \in \text{Gal}(N/\mathbb{Q}).$$

Denote by ι the restriction to N of the complex conjugation. Note that $\iota \in \text{Gal}(N/\mathbb{Q})$. Recall that the conjugates of $\alpha \in K$ were so ordered that $\alpha^{(i)} \in \mathbb{R}$ for $i = 1, \dots, r_1$ and $\alpha^{(i+r_2)} = \overline{\alpha^{(i)}}$ for $i = r_1 + 1, \dots, r_1 + r_2$. By (4.21) we have that $\iota^*(i) = i$ for $i = 1, \dots, r_1$ and that ι^* interchanges i and $i + r_2$ for $i = r_1 + 1, \dots, r_1 + r_2$. Together with (4.14) (i.e., $\overline{L_i} = L_i$ for $i = 1, \dots, r_1$) and (4.15) this implies

$$(4.24) \quad L_{\iota^*(i)} = \overline{L_i} \quad \text{for } i = 1, \dots, r.$$

5. The small solutions

In this section, we develop a gap principle to deal with the small solutions of (1.5). We need a preparatory lemma.

Lemma 5. *Let $D \geq 1$ and let $\mathcal{S}$ be a subset of $\mathbb{Z}^n$ with the property that*

$$(5.1) \quad |\det(\mathbf{x}_1, \dots, \mathbf{x}_n)| \leq D \quad \text{for each } n\text{-tuple } \mathbf{x}_1, \dots, \mathbf{x}_n \in \mathcal{S}.$$

Then $\mathcal{S}$ is contained in the union of not more than

$$100^n \cdot D^{\frac{1}{n-1}}$$

proper linear subspaces of $\mathbb{Q}^n$.

PROOF. We assume without loss of generality that $\mathcal{S}$ is not contained in a single proper linear subspace of $\mathbb{Q}^n$, i.e., that $\mathcal{S}$ contains n linearly

independent vectors, $\mathbf{x}_1, \dots, \mathbf{x}_n$, say. Then every $\mathbf{x} \in \mathcal{S}$ can be expressed as $\mathbf{x} = \sum_{i=1}^n z_i \mathbf{x}_i$ for certain $z_i \in \mathbb{Q}$. By (5.1) we have for such an $\mathbf{x}$,

$$|z_i| = \frac{|\det(\mathbf{x}_1, \dots, \mathbf{x}, \dots, \mathbf{x}_n)|}{|\det(\mathbf{x}_1, \dots, \mathbf{x}_i, \dots, \mathbf{x}_n)|} \leq D \quad \text{for } i = 1, \dots, n.$$

This implies that $\mathcal{S}$ is finite.

Let $\mathcal{S} = \{\mathbf{x}_1, \dots, \mathbf{x}_m\}$. Denote by $\mathcal{C}$ the smallest convex body which contains $\mathcal{S}$ and which is symmetric about $\mathbf{0}$, i.e.,

$$\mathcal{C} = \left\{ \sum_{i=1}^m z_i \mathbf{x}_i : z_i \in \mathbb{R}, \sum_{i=1}^m |z_i| \leq 1 \right\}.$$

Let $\lambda_1, \dots, \lambda_n$ denote the successive minima of $\mathcal{C}$ with respect to $\mathbb{Z}^n$. The body $\mathcal{C}$ contains n linearly independent points from $\mathbb{Z}^n$ since $\mathcal{S}$ does. Therefore,

$$(5.2) \quad 0 < \lambda_1 \leq \dots \leq \lambda_n \leq 1.$$

Further, we have

$$(5.3) \quad \lambda_1 \dots \lambda_n \geq D^{-1}.$$

To show this, take linearly independent vectors $\mathbf{y}_1, \dots, \mathbf{y}_n \in \mathbb{Z}^n$ with $\mathbf{y}_i \in \lambda_i \mathcal{C}$ for $i = 1, \dots, n$. Then $\lambda_i^{-1} \mathbf{y}_i \in \mathcal{C}$, i.e., $\lambda_i^{-1} \mathbf{y}_i = \sum_{j=1}^m z_{ij} \mathbf{x}_j$ for certain $z_{ij} \in \mathbb{R}$ with $\sum_{j=1}^m |z_{ij}| \leq 1$. In view of (5.1) this implies

$$\begin{aligned} (\lambda_1 \dots \lambda_n)^{-1} &\leq |\det(\lambda_1^{-1} \mathbf{y}_1, \dots, \lambda_n^{-1} \mathbf{y}_n)| \\ &\leq \sum_{j_1=1}^m \dots \sum_{j_n=1}^m |z_{1,j_1}| \dots |z_{n,j_n}| \cdot |\det(\mathbf{x}_{j_1}, \dots, \mathbf{x}_{j_n})| \leq D, \end{aligned}$$

which is (5.3).

To $\mathcal{C}$ we associate the vector norm on $\mathbb{R}^n$ given by

$$\|\mathbf{x}\| := \min\{\lambda : \mathbf{x} \in \lambda \mathcal{C}\}.$$

According to a result of SCHLICKWEI [13] (p. 176, Proposition 4.2), the lattice $\mathbb{Z}^n$ has a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ such that

$$(5.4) \quad \|\mathbf{x}\| \geq 4^{-n} \max(|z_1| \cdot \|\mathbf{e}_1\|, \dots, |z_n| \cdot \|\mathbf{e}_n\|) \quad \text{for } \mathbf{x} \in \mathbb{Z}^n,$$

where $z_1, \dots, z_n$ are the integers determined by $\mathbf{x} = \sum_{i=1}^n z_i \mathbf{e}_i$. Assuming $\|\mathbf{e}_1\| \leq \dots \leq \|\mathbf{e}_n\|$ as we may, we have

$$(5.5) \quad \|\mathbf{e}_i\| \geq \lambda_i \quad \text{for } i = 1, \dots, n.$$

Since $\mathcal{S} \subseteq \mathcal{C}$ we have $\|\mathbf{x}\| \leq 1$ for $\mathbf{x} \in \mathcal{S}$. So by (5.4), (5.5) we have for all $\mathbf{x} \in \mathcal{S}$,

$$(5.6) \quad |z_i| \leq 4^n \lambda_i^{-1} \quad \text{for } i = 1, \dots, n.$$

Now since the mapping $\mathbf{x} \mapsto (z_1, \dots, z_n)$ is a linear isomorphism from $\mathbb{Z}^n$ to itself, it suffices to prove that the set of vectors $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{Z}^n$ with (5.6) is contained in the union of not more than $100^n \cdot D^{\frac{1}{n-1}}$ proper linear subspaces of $\mathbb{Q}^n$.

We construct a collection of $(n-1)$ -dimensional linear subspaces of $\mathbb{Q}^n$ whose union contains the set of vectors with (5.6), or rather a set of linear forms with integer coefficients such that each vector $\mathbf{z} \in \mathbb{Z}^n$ with (5.6) is a zero of at least one of these forms. We first determine an index s such that λ_t is not too small for $t > s$. Let $s \in \{0, \dots, n-2\}$ be the index for which $(\lambda_{s+1} \dots \lambda_n)^{\frac{1}{n-s-1}}$ is maximal. From (5.3) it follows that

$$(5.7) \quad (\lambda_{s+1} \dots \lambda_n)^{\frac{1}{n-s-1}} \geq (\lambda_1 \dots \lambda_n)^{\frac{1}{n-1}} \geq D^{-\frac{1}{n-1}}.$$

Moreover, we have

$$(5.8) \quad \lambda_t \geq (\lambda_{s+1} \dots \lambda_n)^{\frac{1}{n-s-1}} \quad \text{for } t = s+1, \dots, n.$$

Indeed, for $s = n-2$ this follows at once from (5.2). Suppose $s < n-2$. Then from the definition of s it follows that

$$(\lambda_{s+1} \dots \lambda_n)^{\frac{1}{n-s-1}} \geq (\lambda_{s+2} \dots \lambda_n)^{\frac{1}{n-s-2}}.$$

This implies $\lambda_{s+1}^{\frac{1}{n-s-2}} \geq (\lambda_{s+1} \dots \lambda_n)^{\frac{1}{n-s-2} - \frac{1}{n-s-1}}$, whence

$\lambda_{s+1} \geq (\lambda_{s+1} \dots \lambda_n)^{\frac{1}{n-s-1}}$, and this certainly implies (5.8).

Using an argument similar to the proof of Siegel's lemma, we show that for each vector $\mathbf{z}$ with (5.6), there is a non-zero vector $\mathbf{c} = (c_{s+1}, \dots, c_n) \in \mathbb{Z}^{n-s}$ with

$$(5.9) \quad c_{s+1} z_{s+1} + \dots + c_n z_n = 0,$$

$$(5.10) \quad |c_i| \leq \lambda_i \cdot \left(\frac{n \cdot 4^n + 1}{\lambda_{s+1} \dots \lambda_n} \right)^{\frac{1}{n-s-1}} \quad \text{for } i = s+1, \dots, n.$$

Put

$$B := \left(\frac{n \cdot 4^n + 1}{\lambda_{s+1} \dots \lambda_n} \right)^{\frac{1}{n-s-1}}.$$

Consider all vectors $\mathbf{c} = (c_{s+1}, \dots, c_n) \in \mathbb{Z}^{n-s}$ with

$$(5.11) \quad 0 \leq c_i \leq \lambda_i B \quad \text{for } i = s+1, \dots, n.$$

Let $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{Z}^n$ be a vector with (5.6), and suppose that $z_i > 0$ for exactly r indices $i \in \{s+1, \dots, n\}$, where $r \geq 0$. Then for vectors $\mathbf{c} \in \mathbb{Z}^{n-s}$ with (5.11) we have

$$-(n-s-r)4^n B \leq c_{s+1}z_{s+1} + \dots + c_n z_n \leq r \cdot 4^n B.$$

So the number of possible values for $c_{s+1}z_{s+1} + \dots + c_n z_n$ is at most

$$[r \cdot 4^n B] + [(n-s-r) \cdot 4^n B] + 1 \leq [(n-s) \cdot 4^n B] + 1.$$

Further, the number of vectors $\mathbf{c} \in \mathbb{Z}^{n-s}$ with (5.11) is equal to

$$\prod_{i=s+1}^n ([\lambda_i B] + 1).$$

By the choice of B , this number is larger than

$$\lambda_{s+1} \dots \lambda_n B^{n-s} = \lambda_{s+1} \dots \lambda_n B^{n-s-1} \cdot B = (n \cdot 4^n + 1)B \geq [(n-s)4^n B] + 1,$$

noting that by (5.2) we have $B \geq 1$. Therefore, there are two different vectors $\mathbf{c}' = (c'_{s+1}, \dots, c'_n)$, $\mathbf{c}'' = (c''_{s+1}, \dots, c''_n) \in \mathbb{Z}^{n-s}$ with $0 \leq c'_i, c''_i \leq \lambda_i B$ for $i = s+1, \dots, n$ and $c'_{s+1}z_{s+1} + \dots + c'_n z_n = c''_{s+1}z_{s+1} + \dots + c''_n z_n$. Now clearly, the vector $\mathbf{c} := \mathbf{c}' - \mathbf{c}''$ is non-zero and satisfies (5.9), (5.10).

For each non-zero $\mathbf{c} \in \mathbb{Z}^{n-s}$, (5.9) defines a proper linear subspace of $\mathbb{Q}^n$. By estimating from above the number of vectors $\mathbf{c} \in \mathbb{Z}^{n-s}$ with (5.10), we conclude that the set of vectors $\mathbf{z} \in \mathbb{Z}^n$ with (5.6) is contained

in the union of at most

$$\begin{aligned}
& \prod_{i=s+1}^n \left\{ 2\lambda_i \left(\frac{n \cdot 4^n + 1}{\lambda_{s+1} \dots \lambda_n} \right)^{\frac{1}{n-s-1}} + 1 \right\} \\
& \leq \prod_{i=s+1}^n \left\{ 3\lambda_i \left(\frac{n \cdot 4^n + 1}{\lambda_{s+1} \dots \lambda_n} \right)^{\frac{1}{n-s-1}} \right\} \quad (\text{by (5.8)}) \\
& = 3^{n-s} (n \cdot 4^n + 1)^{\frac{n-s}{n-s-1}} (\lambda_{s+1} \dots \lambda_n)^{1 - \frac{n-s}{n-s-1}} \\
& \leq 100^n \cdot (\lambda_{s+1} \dots \lambda_n)^{-\frac{1}{n-s-1}} \leq 100^n \cdot D^{\frac{1}{n-1}} \quad (\text{by (5.7)})
\end{aligned}$$

proper linear subspaces of $\mathbb{Q}^n$. This proves Lemma 5. $\square$

The next gap principle is a generalisation of Lemma 3.1 of SCHMIDT [16]. For $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{C}^n$ we put $\|\mathbf{x}\| := \max(|x_1|, \dots, |x_n|)$.

Lemma 6. *Let P, Q, B be reals with*

$$(5.12) \quad P \geq 1, \quad Q \geq 1, \quad B \geq 1$$

and let $M_1, \dots, M_n$ be linearly independent linear forms in $X_1, \dots, X_n$ with complex coefficients. Then the set of $\mathbf{x} \in \mathbb{Z}^n$ with

$$(5.13) \quad |M_1(\mathbf{x}) \dots M_n(\mathbf{x})| \leq |\det(M_1, \dots, M_n)| \cdot \frac{Q}{P},$$

$$(5.14) \quad \|\mathbf{x}\| \leq B$$

is contained in the union of not more than

$$(100n^2)^n \cdot Q^{\frac{1}{n-1}} \cdot \left(1 + \frac{\log eB}{\log eP} \right)^{n-1}$$

proper linear subspaces of $\mathbb{Q}^n$.

PROOF. Put

$$(5.15) \quad T := (n-1) \left(1 + \left\lceil \frac{\log eB}{\log eP} \right\rceil \right).$$

We assume that

$$(5.16) \quad \|M_i\| = 1 \quad \text{for } i = 1, \dots, n$$

(recall that $\|M_i\|$ is the maximum of the absolute values of the coefficients of M_i). This is no loss of generality, since (5.13) does not change if $M_1, \dots, M_n$ are replaced by constant multiples. As a consequence, the solutions $\mathbf{x}$ of (5.13), (5.14) satisfy

$$|M_i(\mathbf{x})| \leq nB \quad \text{for } i = 1, \dots, n.$$

This implies that for every solution $\mathbf{x} \in \mathbb{Z}^n$ of (5.13), (5.14) either there is an index $j \in \{1, \dots, n-1\}$ such that

$$(5.17) \quad |M_j(\mathbf{x})| < (nB)^{1-n},$$

or there are integers $c_1, \dots, c_{n-1}$ with

$$(5.18) \quad (nB)^{c_i/T} \leq |M_i(\mathbf{x})| \leq (nB)^{(c_i+1)/T} \quad \text{for } i = 1, \dots, n-1,$$

$$(5.19) \quad -(n-1)T \leq c_i \leq T-1 \quad \text{for } i = 1, \dots, n-1.$$

(The linear form $M_n(\mathbf{x})$ does not have to be taken into consideration.)

We consider first the solutions $\mathbf{x} \in \mathbb{Z}^n$ of (5.13), (5.14) which satisfy (5.17) for some fixed $j \in \{1, \dots, n-1\}$. Let $\mathbf{x}_1 = (x_{11}, \dots, x_{1n}), \dots, \mathbf{x}_n$ be any such solutions. Let $M_i = \alpha_1 X_1 + \dots + \alpha_n X_n$ with $\|M_i\| = |\alpha_t|$, say. Then $|\alpha_t| = 1$ by (5.16) and so the absolute value of the determinant $\det(\mathbf{x}_1, \dots, \mathbf{x}_n)$ does not change if we replace its t -th column by $M_i(\mathbf{x}_1), \dots, M_i(\mathbf{x}_n)$. Hence

$$\begin{aligned} |\det(\mathbf{x}_1, \dots, \mathbf{x}_n)| &= \left| \det \begin{pmatrix} x_{11} & \dots & M_i(\mathbf{x}_1) & \dots & x_{1n} \\ \vdots & & \vdots & & \vdots \\ x_{n1} & \dots & M_i(\mathbf{x}_n) & \dots & x_{nn} \end{pmatrix} \right| \\ &\leq n! \cdot \prod_{\substack{k=1 \\ k \neq t}}^n \max_{j=1, \dots, n} |x_{jk}| \cdot \max_{j=1, \dots, n} |M_i(\mathbf{x}_j)| \\ &< n! \cdot B^{n-1} (nB)^{1-n} \leq 1. \end{aligned}$$

Now since $\det(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbb{Z}$, this implies $\det(\mathbf{x}_1, \dots, \mathbf{x}_n) = 0$. Hence $\mathbf{x}_1, \dots, \mathbf{x}_n$ lie in a single subspace of $\mathbb{Q}^n$. We infer that for each $i \in \{1, \dots, n-1\}$, the set of solutions of (5.13), (5.14) satisfying (5.17) is contained in a single proper linear subspace of $\mathbb{Q}^n$.

We now deal with the solutions $\mathbf{x}$ of (5.13), (5.14) which satisfy (5.18) for some fixed tuple $c_1, \dots, c_{n-1}$ with (5.19). Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be any such solutions. Then

$$(5.20) \quad |\det(\mathbf{x}_1, \dots, \mathbf{x}_n)| = |\det(M_1, \dots, M_n)|^{-1} \cdot |\det(M_i(\mathbf{x}_j))_{1 \leq i, j \leq n}|.$$

The last determinant is a sum of $n!$ terms

$$\pm M_1(\mathbf{x}_{\sigma(1)}) \dots M_n(\mathbf{x}_{\sigma(n)})$$

where σ is a permutation of $1, \dots, n$. Consider such a term with $\sigma(n) = j$. Using that by (5.18) we have

$$|M_k(\mathbf{x}_l)| \leq (nB)^{1/T} |M_k(\mathbf{x}_j)| \quad \text{for } k = 1, \dots, n-1, l \neq j$$

we get

$$\begin{aligned} |M_1(\mathbf{x}_{\sigma(1)}) \dots M_n(\mathbf{x}_{\sigma(n)})| &\leq (nB)^{(n-1)/T} |M_1(\mathbf{x}_j) \dots M_n(\mathbf{x}_j)| \\ &\leq (nB)^{(n-1)/T} |\det(M_1, \dots, M_n)| \cdot \frac{Q}{P} && \text{using (5.13)} \\ &\leq e \cdot (n/e)^{(n-1)/T} |\det(M_1, \dots, M_n)| \cdot Q && \text{using (5.15)} \end{aligned}$$

By inserting this into (5.20) and using $T \geq n-1$ we obtain

$$|\det(\mathbf{x}_1, \dots, \mathbf{x}_n)| \leq n \cdot n! \cdot Q.$$

Now Lemma 5 implies that the set of solutions of (5.13), (5.14) satisfying (5.18) for some fixed $c_1, \dots, c_{n-1}$ is contained in the union of at most

$$100^n \cdot (n \cdot n! \cdot Q)^{\frac{1}{n-1}}$$

proper linear subspaces of $\mathbb{Q}^n$.

We have $n-1$ inequalities (5.17), each giving rise to a single subspace of $\mathbb{Q}^n$. Further, in view of (5.19) we have $(nT)^{n-1}$ systems of inequalities (5.18). Together with (5.15), this implies that the set of solutions $\mathbf{x} \in \mathbb{Z}^n$ of (5.13), (5.14) is contained in the union of at most

$$\begin{aligned} n-1 + 100^n (n \cdot n!)^{\frac{1}{n-1}} Q^{\frac{1}{n-1}} \cdot (n(n-1))^{n-1} \left(1 + \frac{\log eB}{\log eP}\right)^{n-1} \\ \leq (100n^2)^n \cdot Q^{\frac{1}{n-1}} \cdot \left(1 + \frac{\log eB}{\log eP}\right)^{n-1} \end{aligned}$$

proper linear subspaces of $\mathbb{Q}^n$. This proves Lemma 6. $\square$

Now let $F \in \mathbb{Z}[X_1, \dots, X_n]$ be the norm form of degree r with (1.2), (1.11), and $L_1, \dots, L_r$ the linear forms with (4.13)–(4.19) which we have fixed in Section 4.

Lemma 7. *For every solution $\mathbf{x}$ of (1.5) there are $i_1, \dots, i_n \in \{1, \dots, r\}$ such that $L_{i_1}, \dots, L_{i_n}$ are linearly independent and such that*

$$(5.21) \quad |L_{i_1}(\mathbf{x}) \dots L_{i_n}(\mathbf{x})| \leq |\det(L_{i_1}, \dots, L_{i_n})| \cdot \frac{h^{n/r}}{H^*(F)}.$$

PROOF. We closely follow the proof of Lemma 3 of SCHMIDT [17]. For $\mathbf{x} = \mathbf{0}$ we have $L_i(\mathbf{x}) = 0$ for $i = 1, \dots, r$ and (5.21) is trivial. Let $\mathbf{x}$ be a non-zero solution of (1.5). Define linear forms

$$L'_i = \frac{|F(\mathbf{x})|^{1/r}}{|L_i(\mathbf{x})|} \cdot L_i \quad (i = 1, \dots, r).$$

From (4.13)–(4.15) it follows that $L'_1, \dots, L'_r$ satisfy (1.8). Pick $i_1, \dots, i_n$ such that $|\det(L'_{i_1}, \dots, L'_{i_n})|$ is maximal. Then from (1.5) and the definition of $H^*(F)$ it follows that

$$\begin{aligned} |L_{i_1}(\mathbf{x}) \dots L_{i_n}(\mathbf{x})| &= \frac{|\det(L_{i_1}, \dots, L_{i_n})|}{|\det(L'_{i_1}, \dots, L'_{i_n})|} \cdot |F(\mathbf{x})|^{n/r} \\ &= \frac{|\det(L_{i_1}, \dots, L_{i_n})|}{\Delta(L'_1, \dots, L'_r)} \cdot |F(\mathbf{x})|^{n/r} \leq |\det(L_{i_1}, \dots, L_{i_n})| \cdot \frac{h^{n/r}}{H^*(F)}. \quad \square \end{aligned}$$

By combining Lemmata 7 and 6 we arrive at the following result for the small solutions of norm form inequality (1.5):

Proposition 1. *Let $P \geq 1$, $B \geq 1$. Then the set of solutions $\mathbf{x}$ of (1.5) with*

$$(5.22) \quad \|\mathbf{x}\| \leq B$$

is contained in the union of at most

$$(300rn)^n \cdot \max \left(1, \left(\frac{h^{n/r} P}{H^*(F)} \right)^{\frac{1}{n-1}} \right) \cdot \left(1 + \frac{\log eB}{\log eP} \right)^{n-1}$$

proper linear subspaces of $\mathbb{Q}^n$.

PROOF. From Lemma 6 with

$$Q = \max\left(1, \frac{h^{n/r} P}{H^*(F)}\right)$$

and from Lemma 7 we infer that for each tuple $\{i_1, \dots, i_n\}$, the set of solutions of (1.5) satisfying (5.21) and (5.22) is contained in the union of not more than

$$(100n^2)^n \cdot \max\left(1, \left(\frac{h^{n/r} P}{H^*(F)}\right)^{\frac{1}{n-1}}\right) \cdot \left(1 + \frac{\log eB}{\log eP}\right)^{n-1}$$

proper linear subspaces of $\mathbb{Q}^n$. Now Proposition 1 follows, on noting that we have at most $\binom{r}{n}$ possibilities for $\{i_1, \dots, i_n\}$ and that $(100n^2)^n \cdot \binom{r}{n} \leq (300rn)^n$. $\square$

6. The quantitative Subspace Theorem

We recall a special case of the quantitative Subspace Theorem from [8] and then specialise it to a situation relevant for eq. (1.5). We must first introduce the Euclidean height which has been used in the statement of the quantitative Subspace theorem of [8].

Let $\mathbf{x} = (x_1, \dots, x_n) \in \overline{\mathbb{Q}}^n$ with $\mathbf{x} \neq \mathbf{0}$. To define the height of $\mathbf{x}$, we choose a number field K containing $x_1, \dots, x_n$. Let $d = [K : \mathbb{Q}]$ and let $\sigma_1, \dots, \sigma_d$ denote the isomorphic embeddings of K into $\mathbb{C}$. Further, denote by $N_{K/\mathbb{Q}}(x_1, \dots, x_n)$ the absolute norm of the fractional ideal in K generated by $x_1, \dots, x_n$. Then the Euclidean height of $\mathbf{x}$ is defined by

$$H_2(\mathbf{x}) := \left(\frac{\prod_{i=1}^d \left(\sum_{j=1}^n |\sigma_i(x_j)|^2 \right)^{1/2}}{N_{K/\mathbb{Q}}(x_1, \dots, x_n)} \right)^{1/d}.$$

It is easy to see that this is independent of the choice of K . Moreover, we have $H_2(\lambda \mathbf{x}) = H_2(\mathbf{x})$ for every non-zero $\lambda \in \overline{\mathbb{Q}}$. This implies $H_2(\mathbf{x}) \geq 1$. Note that

$$H_2(\mathbf{x}) = \frac{(|x_1|^2 + \dots + |x_n|^2)^{1/2}}{\gcd(x_1, \dots, x_n)} \quad \text{for } \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n \setminus \{\mathbf{0}\}.$$

For a non-zero linear form $L = \alpha_1 X_1 + \cdots + \alpha_n X_n$ with $\mathbf{a} = (\alpha_1, \dots, \alpha_n) \in \overline{\mathbb{Q}}^n$ we put $H_2(L) := H_2(\mathbf{a})$.

The following result is a special case of Theorem 3.1 of [8]. Except for the better quantitative bound, this result is of the same nature as the first quantitative version of the Subspace Theorem, obtained by SCHMIDT [16].

Quantitative Subspace Theorem. *Let $0 < \delta \leq 1$ and let $M_1, \dots, M_n$ be linearly independent linear forms in $X_1, \dots, X_n$ such that*

$$(6.1) \quad \text{the coefficients of } M_1, \dots, M_n \text{ generate an algebraic number field of degree } D.$$

Then the set of $\mathbf{x} \in \mathbb{Z}^n$ with

$$(6.2) \quad |M_1(\mathbf{x}) \cdots M_n(\mathbf{x})| \leq |\det(M_1, \dots, M_n)| \cdot H_2(\mathbf{x})^{-\delta},$$

$$(6.3) \quad H_2(\mathbf{x}) \geq \max \left(n^{4n/\delta}, H_2(M_1), \dots, H_2(M_n) \right)$$

is contained in the union of at most

$$(6.4) \quad 16^{(n+9)^2} \cdot \delta^{-2n-4} \log(4D) \cdot \log \log(4D)$$

proper linear subspaces of $\mathbb{Q}^n$.

Let F be the norm form with (1.2), (1.11) and $L_1, \dots, L_r$ the linear forms with (4.13)–(4.19) which we have fixed throughout the paper. Let K be the number field associated to F as in (1.2), N the finite, normal extension of $\mathbb{Q}$ introduced at the end of Section 4 containing the coefficients of $L_1, \dots, L_r$ and $d = [N : \mathbb{Q}]$. We want to apply the quantitative Subspace Theorem to any set of n linearly independent forms from $L_1, \dots, L_r$. We need the following estimates:

Lemma 8. (i) $H_2(L_i) \leq \sqrt{n}(2n)^{n+1} H^*(F)$ for $i = 1, \dots, r$.
(ii) For each linearly independent subset $\{L_{i_1}, \dots, L_{i_n}\}$ of $\{L_1, \dots, L_r\}$ we have

$$|\det(L_{i_1}, \dots, L_{i_n})| \geq (2H^*(F))^{1-\binom{r}{n}}.$$

PROOF. (i) From (4.19), (4.23), (4.18) it follows that

$$\begin{aligned} H_2(L_i) &\leq \left(\prod_{\sigma \in \text{Gal}(N/\mathbb{Q})} \|\sigma(L_i)\|_2 \right)^{1/d} = \left(\prod_{\sigma \in \text{Gal}(N/\mathbb{Q})} \|L_{\sigma^*(i)}\|_2 \right)^{1/d} \\ &\leq \sqrt{n}(2n)^{n+1} H^*(F). \end{aligned}$$

(ii) By Schmidt's result ([17], p. 203) the semi-discriminant

$$D(F) := \prod_{(j_1, \dots, j_n)} |\det(L_{j_1}, \dots, L_{j_n})|$$

is a positive integer, where the product is taken over all ordered n -tuples $(j_1, \dots, j_n)$ for which $\det(L_{j_1}, \dots, L_{j_n}) \neq 0$. Further, by (4.17) we have for each such n -tuple that $|\det(L_{j_1}, \dots, L_{j_n})| \leq 2H^*(F)$. Denote by $\mathcal{J}$ the collection of all unordered subsets $\{j_1, \dots, j_n\}$ of $\{1, \dots, r\}$ for which $\det(L_{j_1}, \dots, L_{j_n}) \neq 0$. Then $\mathcal{J}$ has cardinality $\leq \binom{r}{n}$. Hence

$$\begin{aligned} |\det(L_{i_1}, \dots, L_{i_n})| &\geq \prod_{\{j_1, \dots, j_n\} \in \mathcal{J}} |\det(L_{j_1}, \dots, L_{j_n})| \cdot (2H^*(F))^{1 - \binom{r}{n}} \\ &= |D(F)|^{1/n!} (2H^*(F))^{1 - \binom{r}{n}} \geq (2H^*(F))^{1 - \binom{r}{n}}. \end{aligned} \quad \square$$

Below we have stated our basic tool for dealing with the large solutions of (1.5). As before, $\|\mathbf{x}\|$ denotes the maximum norm of $\mathbf{x}$. After the proof of Proposition 2, we will not use anymore Euclidean heights.

Proposition 2. *Let $L_{i_1}, \dots, L_{i_n}$ be linearly independent linear forms among $L_1, \dots, L_r$ and let $0 < \delta \leq 1$. Then the set of primitive $\mathbf{x} \in \mathbb{Z}^n$ with*

$$(6.5) \quad |L_{i_1}(\mathbf{x}) \dots L_{i_n}(\mathbf{x})| \leq \|\mathbf{x}\|^{-\delta},$$

$$(6.6) \quad \|\mathbf{x}\| \geq (eH^*(F))^{(4r)^{n+1}/\delta}$$

is contained in the union of not more than

$$(6.7) \quad 16^{(n+10)^2} \cdot \delta^{-2n-4} \log(4r) \cdot \log \log(4r)$$

proper linear subspaces of $\mathbb{Q}^n$.

PROOF. Inequality (6.2) does not change if the linear forms $M_1, \dots, M_n$ are replaced by constant multiples. Therefore, we may replace (6.1) by the weaker condition that $M_1, \dots, M_n$ are constant multiples of linear forms $M'_1, \dots, M'_n$ such that the coefficients of $M'_1, \dots, M'_n$ generate an algebraic number field of degree D .

Let $\mathbf{x} \in \mathbb{Z}^n$ be a primitive solution of (6.5), (6.6). Then

$$|L_{i_1}(\mathbf{x}) \dots L_{i_n}(\mathbf{x})| \leq (2H^*(F))^{1-\binom{r}{n}} \cdot (\sqrt{n} \cdot \|\mathbf{x}\|)^{-3\delta/4} \quad \text{by (6.5), (6.6)}$$

$$\leq |\det(L_{i_1}, \dots, L_{i_n})| \cdot H_2(\mathbf{x})^{-3\delta/4} \quad \text{by } H_2(\mathbf{x}) \leq \sqrt{n} \cdot \|\mathbf{x}\| \text{ and Lemma 8 (ii).}$$

Therefore, (6.2) holds with $L_{i_1}, \dots, L_{i_n}$ replacing $M_1, \dots, M_n$ and $3\delta/4$ replacing δ . Further, from Lemma 8 (i) and (6.6) it follows that $\mathbf{x}$ satisfies (6.3) with $L_{i_1}, \dots, L_{i_n}$ and $3\delta/4$ replacing $M_1, \dots, M_n$ and δ . Finally, from the construction of $L_1, \dots, L_r$ in Section 4 it follows that for $i = 1, \dots, r$, L_i is a constant multiple of a linear form with coefficients in $K^{(i)}$, where $K^{(i)}$ is a conjugate of K , whence has degree r . Therefore, there are constant multiples of $L_{i_1}, \dots, L_{i_n}$ whose coefficients generate a number field of degree at most r^n . Now by applying the quantitative Subspace Theorem with $L_{i_1}, \dots, L_{i_n}$, $3\delta/4$ and r^n replacing $M_1, \dots, M_n$, δ and D , we obtain that the set of primitive $\mathbf{x} \in \mathbb{Z}^n$ with (6.5) and (6.6) is contained in the union of at most

$$16^{(n+9)^2} (4/3\delta)^{2n+4} \log(4r^n) \log \log(4r^n)$$

proper linear subspaces of $\mathbb{Q}^n$. This is smaller than the quantity in (6.7). $\square$

7. Reduction to the Subspace Theorem

We will use some results from [5] and follow the arguments of Sections 7, 8 of [6]. Like before, the norm form F satisfies (1.2) and (1.11) and the linear forms $L_1, \dots, L_r$ satisfy (4.13)–(4.19). Further, $N \subset \mathbb{C}$ is the normal extension of $\mathbb{Q}$ chosen in Section 4 and for each $\sigma \in \text{Gal}(N/\mathbb{Q})$, $(\sigma^*(1), \dots, \sigma^*(r))$ is the permutation of $(1, \dots, r)$ defined by (4.21). For $\sigma \in \text{Gal}(N/\mathbb{Q})$, $I \subseteq \{1, \dots, r\}$ we write $\sigma^*(I) := \{\sigma^*(i) : i \in I\}$. We denote by ι the restriction to N of the complex conjugation on $\mathbb{C}$.

We define a hypergraph $\mathcal{H}$ as follows. The vertices of $\mathcal{H}$ are the indices $1, \dots, r$. Further, the edges of $\mathcal{H}$ are the sets I of cardinality ≥ 2 such that $\{L_i : i \in I\}$ is a linearly dependent set of linear forms (over N), whereas for each proper subset I' of I , the set $\{L_i : i \in I'\}$ is linearly independent. As usual, two vertices i, j of $\mathcal{H}$ are said to be connected if there is a sequence of edges $I_1, \dots, I_m$ of $\mathcal{H}$ such that $i \in I_1$, $I_j \cap I_{j+1} \neq \emptyset$ for $j = 1, \dots, m-1$

and $j \in I_m$. Denote by $C_1, \dots, C_t$ the connected components of $\mathcal{H}$. It follows at once from (4.23) that if $\{L_i : i \in I\}$ is linearly (in)dependent, then so is $\{L_i : i \in \sigma^*(I)\}$ for each $\sigma \in \text{Gal}(N/\mathbb{Q})$. Hence if I is an edge of $\mathcal{H}$, then so is $\sigma^*(I)$ for each $\sigma \in \text{Gal}(N/\mathbb{Q})$. This implies that for each connected component C_i of $\mathcal{H}$ and for each $\sigma \in \text{Gal}(N/\mathbb{Q})$, $\sigma^*(C_i)$ is also a connected component of $\mathcal{H}$.

Lemma 9. *Either $\mathcal{H}$ is connected, or $\mathcal{H}$ has precisely two connected components, C_1 and C_2 , say, and $\iota^*(C_1) = C_2$.*

PROOF. We assume that the embedding $\alpha \mapsto \alpha^{(1)}$ is the identity on K and that the index $1 \in C_1$. Define the subfield J of N by

$$\text{Gal}(N/J) = \{\sigma \in \text{Gal}(N/\mathbb{Q}) : \sigma^*(C_1) = C_1\}.$$

By (4.21) we have

$$\text{Gal}(N/J) \supseteq \{\sigma \in \text{Gal}(N/\mathbb{Q}) : \sigma^*(1) = 1\} = \text{Gal}(N/K);$$

so $J \subseteq K$. It is easy to see that for $\sigma \in \text{Gal}(N/\mathbb{Q})$, the left coset $\sigma \text{Gal}(N/J)$ is equal to $\{\tau \in \text{Gal}(N/\mathbb{Q}) : \tau^*(C_1) = C_i\}$ where $\sigma^*(C_1) = C_i$. Moreover, (4.21) implies that for each index $j \in \{1, \dots, r\}$, there is a $\sigma \in \text{Gal}(N/\mathbb{Q})$ with $\sigma^*(1) = j$ and so for each $i \in \{1, \dots, t\}$ there is a $\sigma \in \text{Gal}(N/\mathbb{Q})$ with $\sigma^*(C_1) = C_i$. This implies that there are exactly t left cosets of $\text{Gal}(N/J)$ in $\text{Gal}(N/\mathbb{Q})$, and so

$$(7.1) \quad [J : \mathbb{Q}] = t.$$

Let V as before be the vector space defined by (1.3). We have

$$(7.2) \quad V^J = V.$$

This follows from some theory from Section 4, pp. 191–193 of [5]. By (4.16) of the present paper, the linear form L_i is proportional to $L'_i := \alpha_1^{(i)}X_1 + \dots + \alpha_n^{(i)}X_n$ for $i = 1, \dots, r$, so the hypergraph $\mathcal{H}$ does not change if in its definition, L_i is replaced by L'_i . For the space V and the field L on p. 192 of [5] we take $\mathbb{Q}^n$ and the field N of the present paper. In our situation, the quantity u defined by (4.5) of [5] is equal to 1, and the field K_1 defined by (4.3) on p. 192 of [5] is equal to J . Consider the injective map from K to N^r ,

$$\psi : \xi \mapsto (\xi^{(1)}, \dots, \xi^{(r)}).$$

Note that ψ maps V onto the space $\{(L'_1(\mathbf{x}), \dots, L'_r(\mathbf{x})) : \mathbf{x} \in \mathbb{Q}^n\}$. Further, from Lemma 6, (ii) on pp. 192–193 of [5] it follows that ψ maps J onto

$$\Lambda(F) := \{\mathbf{c} = (c_1, \dots, c_r) \in N^r : \text{for every } \mathbf{x} \in \mathbb{Q}^n \text{ there is an } \mathbf{y} \in \mathbb{Q}^n \\ \text{with } L'_i(\mathbf{y}) = c_i L'_i(\mathbf{x}) \text{ for } i = 1, \dots, r\}.$$

Denoting the images of $(c_1, \dots, c_r)$, $(L'_1(\mathbf{x}), \dots, L'_r(\mathbf{x}))$, $(L'_1(\mathbf{y}), \dots, L'_r(\mathbf{y}))$ under ψ^{-1} by λ , ξ , η , respectively, we obtain that J is the set of $\lambda \in K$ such that for every $\xi \in V$ there is an $\eta \in V$ with $\eta = \lambda\xi$. This implies (7.2).

By (7.2) and (1.11) we have that either $J = \mathbb{Q}$ in which case it follows from (7.1) that $\mathcal{H}$ is connected; or that J is an imaginary quadratic field, in which case (7.1) implies that $\mathcal{H}$ has precisely two connected components C_1 and C_2 . Moreover, in this case we have that ι is not the identity on J , so $\iota^*(C_1) \neq C_1$, which implies $\iota^*(C_1) = C_2$. This completes the proof of Lemma 9. $\square$

Let $\mathbf{x} \in \mathbb{Z}^n$. We write

$$u_i := L_i(\mathbf{x}) \quad (i = 1, \dots, r), \quad \mathbf{u} = (u_1, \dots, u_r).$$

From (4.22) it follows that for each $i \in \{1, \dots, r\}$ we have

$$(7.3) \quad \prod_{\sigma \in \text{Gal}(N/\mathbb{Q})} |u_{\sigma^*(i)}| = \prod_{\sigma \in \text{Gal}(N/\mathbb{Q})} |L_{\sigma^*(i)}(\mathbf{x})| = |F(\mathbf{x})|^{d/r},$$

where as before $d = [N : \mathbb{Q}]$. From (1.11) it follows that if $\mathbf{x} \neq \mathbf{0}$, then $F(\mathbf{x}) \neq 0$ and so $u_i \neq 0$ for $i = 1, \dots, r$. Further, (7.3) implies

$$(7.4) \quad \prod_{\sigma \in \text{Gal}(N/\mathbb{Q})} |u_{\sigma^*(i)}| \geq 1 \quad \text{if } \mathbf{x} \neq \mathbf{0}.$$

For each subset I of $\{1, \dots, r\}$ we define a suitable height,

$$H_I(\mathbf{u}) := \left(\prod_{\sigma \in \text{Gal}(N/\mathbb{Q})} \max_{i \in I} |u_{\sigma^*(i)}| \right)^{1/d}.$$

From (7.4) it follows at once that

$$(7.5) \quad H_I(\mathbf{u}) \geq 1 \quad \text{for each non-empty subset } I \text{ of } \{1, \dots, r\}.$$

Further, we have

$$(7.6) \quad H_{I_1 \cup I_2}(\mathbf{u}) \leq H_{I_1}(\mathbf{u}) \cdot H_{I_2}(\mathbf{u}) \quad \text{if } I_1 \cap I_2 \neq \emptyset.$$

For if $i \in I_1 \cap I_2$, then for each $\sigma \in \text{Gal}(N/\mathbb{Q})$ we have

$$\begin{aligned} |u_{\sigma^*(i)}| \cdot \max_{j \in I_1 \cup I_2} |u_{\sigma^*(j)}| &= \max_{j \in I_1 \cup I_2} |u_{\sigma^*(i)} \cdot u_{\sigma^*(j)}| \\ &\leq \max_{j \in I_1} |u_{\sigma^*(j)}| \cdot \max_{j \in I_2} |u_{\sigma^*(j)}|, \end{aligned}$$

whence

$$\left(\prod_{\sigma \in \text{Gal}(N/\mathbb{Q})} |u_{\sigma^*(i)}| \right)^{1/d} \cdot H_{I_1 \cup I_2}(\mathbf{u}) \leq H_{I_1}(\mathbf{u}) \cdot H_{I_2}(\mathbf{u}),$$

which together with (7.4) implies (7.6).

In what follows, we assume that the collection of edges of $\mathcal{H}$ is not empty. We deal with the cases that $\mathcal{H}$ is connected and that $\mathcal{H}$ has two connected components, simultaneously. Let C_1 be a connected component of $\mathcal{H}$ (so either the whole vertex set $\{1, \dots, r\}$ or one of the two components).

Lemma 10. *Let S be a maximal subset of C_1 such that $\{L_j : j \in S\}$ is linearly independent. Then for each non-zero $\mathbf{x} \in \mathbb{Z}^n$ there is an edge I of $\mathcal{H}$ contained in C_1 such that*

$$(7.7) \quad H_S(\mathbf{u}) \leq H_I(\mathbf{u})^{n-1}.$$

PROOF. Fix $\mathbf{x} \in \mathbb{Z}^n$, $\mathbf{x} \neq \mathbf{0}$. We use the argument on p. 208 of [5]. We have

$$L_i = \sum_{j \in D_i} c_{ij} L_j \quad \text{for } i \in C_1,$$

where $D_i \subseteq S$ and where $c_{ij} \neq 0$ for $j \in D_i$. As has been explained on p. 208 of [5], for each subset D of S with $D \neq \emptyset$, $D \subsetneq S$, there is an i such that $D_i \cap D \neq \emptyset$, $D_i \not\subseteq D$. This implies in particular that there is an $i_1 \in C_1$ such that $1 \in D_{i_1}$ and D_{i_1} has cardinality ≥ 2 . If D_{i_1} is not equal to the whole set S , then choose i_2 such that $D_{i_1} \cap D_{i_2} \neq \emptyset$ and $D_{i_2} \not\subseteq D_{i_1}$. If $D_{i_1} \cup D_{i_2} \subsetneq S$, then choose i_3 such that $D_{i_3} \cap (D_{i_1} \cup D_{i_2}) \neq \emptyset$ and

$D_{i_3} \not\subset D_{i_1} \cup D_{i_2}$. Continuing like this, we obtain sets $D_{i_1}, \dots, D_{i_s}$, such that

$$S = D_{i_1} \cup \dots \cup D_{i_s},$$

$$D_{i_h} \cap (D_{i_1} \cup \dots \cup D_{i_{h-1}}) \neq \emptyset, \quad D_{i_h} \not\subset D_{i_1} \cup \dots \cup D_{i_{h-1}} \quad \text{for } h = 2, \dots, s.$$

Assuming s with this property to be minimal, we have $s \leq |S| - 1 \leq n - 1$ since we started with a set D_{i_1} of cardinality ≥ 2 and each newly chosen set D_{i_h} adds at least one element to the union of the sets chosen previously. Now clearly, $I_h := \{i_h\} \cup D_{i_h}$ is an edge of $\mathcal{H}$ for $h = 1, \dots, s$, and we have

$$S \subset I_1 \cup \dots \cup I_s, \quad I_h \cap (I_1 \cup \dots \cup I_{h-1}) \neq \emptyset \quad \text{for } h = 2, \dots, s.$$

Together with (7.6) this implies $H_S(\mathbf{u}) \leq H_{I_1}(\mathbf{u}) \dots H_{I_s}(\mathbf{u})$. Now this fact and (7.5) imply that there is an edge I of $\mathcal{H}$ such that

$$H_S(\mathbf{u}) \leq H_I(\mathbf{u})^s \leq H_I(\mathbf{u})^{n-1}.$$

This proves Lemma 10. $\square$

Lemma 11. *Suppose that $\mathcal{H}$ has edges. Then for every non-zero $\mathbf{x} \in \mathbb{Z}^n$, there is an edge I of $\mathcal{H}$ such that*

$$(7.8) \quad \|\mathbf{x}\| \leq n! \cdot (2n)^{n^2-1} \cdot 2^{\binom{r}{n}-1} \cdot H^*(F)^{\binom{r}{n}+n-2} \cdot H_I(\mathbf{u})^{n-1}.$$

PROOF. Let S be the set from Lemma 10 and let $\mathbf{x} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. Choose $\sigma \in \text{Gal}(N/\mathbb{Q})$ such that

$$\max_{i \in S} |u_{\sigma^*(i)}|$$

is minimal. Then by Lemma 10, there is an edge I of $\mathcal{H}$ such that

$$(7.9) \quad \max_{i \in S} |u_{\sigma^*(i)}| \leq H_I(\mathbf{u})^{n-1}.$$

We show that $\max_{i \in S} |u_{\sigma^*(i)}| = \max_{i \in S'} |u_i|$ where S' is a set of cardinality n such that $\{L_j : j \in S'\}$ is linearly independent and then we estimate $\|\mathbf{x}\|$ from above in terms of $\max_{i \in S'} |u_i|$.

Define the set S' by $S' := \sigma^*(S)$ if $\mathcal{H}$ is connected and $S' := \sigma^*(S) \cup \iota^* \sigma^*(S)$ if $\mathcal{H}$ has two connected components, where ι denotes the complex conjugation on N . First suppose that $\mathcal{H}$ is connected. Then $\{L_j : j \in \sigma^*(S)\}$ is linearly independent and it spans $\{L_1, \dots, L_r\}$. By (1.11), we

have that $\text{rank}\{L_1, \dots, L_r\} = n$. Hence $S' = \sigma^*(S)$ has cardinality n . Now suppose that $\mathcal{H}$ has two connected components. Then by Lemma 9 these connected components are $\sigma^*(C_1)$ and $\iota^*\sigma^*(C_1)$. We have that $\{L_j : j \in \sigma^*(S)\}$ is linearly independent and spans $\{L_j : j \in \sigma^*(C_1)\}$ and that $\{L_j : j \in \iota^*\sigma^*(S)\}$ is linearly independent and spans $\{L_j : j \in \iota^*\sigma^*(C_1)\}$. Therefore, $\{L_j : j \in S'\}$ spans $\{L_1, \dots, L_r\}$. Suppose that $\{L_j : j \in S'\}$ is linearly dependent. Then S' contains an edge of $\mathcal{H}$. This edge is contained in one of the two connected components, so either in $S' \cap \sigma^*(C_1) = \sigma^*(S)$ or in $S' \cap \iota^*\sigma^*(C_1) = \iota^*\sigma^*(S)$. But this is impossible, since both sets $\{L_j : j \in \sigma^*(S)\}$ and $\{L_j : j \in \iota^*\sigma^*(S)\}$ are linearly independent. It follows that also in the second case, $\{L_j : j \in S'\}$ is linearly independent and S' has cardinality n .

If $\mathcal{H}$ is connected then clearly $\max_{i \in S'} |u_i| = \max_{i \in S} |u_{\sigma^*(i)}|$. If $\mathcal{H}$ has two connected components, then it follows from (4.24) and $u_j = L_j(\mathbf{x})$ for $j = 1, \dots, r$ that $u_{\iota^*\sigma^*(j)} = \overline{u_{\sigma^*(j)}}$ for $j \in S$, hence also $\max_{i \in S'} |u_i| = \max_{i \in S} |u_{\sigma^*(i)}|$. By inserting this into (7.9) we get in both cases,

$$\max_{i \in S'} |u_i| \leq H_I(\mathbf{u})^{n-1}.$$

Therefore, (7.8) follows immediately, once we have shown that

$$(7.10) \quad \|\mathbf{x}\| \leq n! \cdot (2n)^{n^2-1} \cdot 2^{\binom{r}{n}-1} \cdot H^*(F)^{\binom{r}{n}+n-2} \cdot \max_{i \in S'} |u_i|.$$

Suppose that $S' = \{i_1, \dots, i_n\}$. Let A be the matrix, whose j -th column consists of the coefficients of L_{i_j} , for $j = 1, \dots, n$. Then $\mathbf{x} = (u_{i_1}, \dots, u_{i_n})A^{-1}$. The elements of A^{-1} are $\pm \Delta_{ij}/\Delta$, where Δ_{ij} is the determinant of the $(n-1) \times (n-1)$ -matrix obtained by removing the j -th row and i -th column from A , and where $\Delta = \det(L_{i_1}, \dots, L_{i_n})$. Hence

$$(7.11) \quad \|\mathbf{x}\| = \left| \max_{k=1, \dots, n} \sum_{j=1}^n u_{i_j} \cdot \Delta_{jk}/\Delta \right| \leq n \cdot \max_{j,k} |\Delta_{jk}| \cdot |\Delta|^{-1} \cdot \max_{i \in S'} |u_i|.$$

We have

$$\begin{aligned} |\Delta_{jk}| &\leq (n-1)! \cdot \left(\max_k \|L_k\| \right)^{n-1} \\ &\leq (n-1)! \cdot (2n)^{n^2-1} \cdot H^*(F)^{n-1} \quad \text{by (4.18),} \\ |\Delta|^{-1} &\leq (2H^*(F))^{\binom{r}{n}-1} \quad \text{by Lemma 8, (ii)} \end{aligned}$$

By inserting these inequalities into (7.11) we obtain (7.10). This proves Lemma 11. $\square$

We finally arrive at:

Proposition 3. *Suppose that $\mathcal{H}$ has edges. Then for every solution $\mathbf{x} \in \mathbb{Z}^n$ of (1.5) with $\mathbf{x} \neq \mathbf{0}$, there are linearly independent linear forms $L_{i_1}, \dots, L_{i_n}$ among $L_1, \dots, L_r$ such that*

$$(7.12) \quad |L_{i_1}(\mathbf{x}) \dots L_{i_n}(\mathbf{x})| \leq C \cdot \|\mathbf{x}\|^{-1/(n-1)},$$

with

$$(7.13) \quad C := \left(n! \cdot (2n)^{n^2-1} \cdot 2^{\binom{r}{n}-1} \cdot H^*(F)^{\binom{r}{n}+n-2} \right)^{\frac{1}{n-1}} \cdot h^{(n+1)/r}.$$

PROOF. Fix a non-zero solution $\mathbf{x} \in \mathbb{Z}^n$ of (1.5). Choose linearly independent linear forms $L_{i_1}, \dots, L_{i_n}$ from $L_1, \dots, L_r$ such that the quantity

$$U := |L_{i_1}(\mathbf{x}) \dots L_{i_n}(\mathbf{x})|$$

is minimal.

Let I be the edge from Lemma 11. Suppose that I has cardinality t . Each linear form from $\{L_j : j \in I\}$ is linearly dependent on the other forms in this set, and these other forms are linearly independent. Hence $\{L_j : j \in I\}$ has rank $t - 1$. Choose a subset T of $\{1, \dots, r\}$ of cardinality $n - t + 1$ such that $\{L_j : j \in T \cup I\}$ has rank n . Then for each $i \in I$, the set of linear forms $\{L_j : j \in T \cup I \setminus \{i\}\}$ is linearly independent and has cardinality n .

Pick $\sigma \in \text{Gal}(N/\mathbb{Q})$. Choose $i_\sigma \in I$ such that $|u_{\sigma^*(i_\sigma)}| = \max_{i \in I} |u_{\sigma^*(i)}|$. Then the set $\{L_j : j \in \sigma^*(T \cup I \setminus \{i_\sigma\})\}$ is linearly independent and has cardinality n . So by the definition of U we have

$$U \leq \prod_{j \in \sigma^*(T \cup I \setminus \{i_\sigma\})} |u_j| = \prod_{j \in T \cup I} |u_{\sigma^*(j)}| \cdot \left(\max_{i \in I} |u_{\sigma^*(i)}| \right)^{-1}.$$

It follows that U is bounded above by the geometric mean of the terms at the right-hand side for all $\sigma \in \text{Gal}(N/\mathbb{Q})$. By (7.3), the fact that $T \cup I$ has cardinality $n + 1$ and the definition of $H_I(\mathbf{u})$, this geometric mean is equal to $|F(\mathbf{x})|^{(n+1)/r} H_I(\mathbf{u})^{-1}$. Hence

$$U \leq h^{(n+1)/r} \cdot H_I(\mathbf{u})^{-1}.$$

By inserting (7.8) we get Proposition 3. $\square$

8. Proof of Theorem 2

We combine Propositions 1, 2 and 3. We recall that $n \geq 2$. It clearly suffices to show that the set of *primitive* solutions of (1.5) is contained in the union of not more than

$$(8.1) \quad A := (16r)^{(n+10)^2} \cdot \max \left(1, \left(\frac{h^{n/r} P}{H^*(F)} \right)^{\frac{1}{n-1}} \right) \\ \cdot \left(1 + \frac{\log(eh \cdot H^*(F))}{\log eP} \right)^{n-1}$$

proper linear subspaces of $\mathbb{Q}^n$. We divide the primitive solutions $\mathbf{x} \in \mathbb{Z}^n$ of (1.5) into

$$\begin{aligned} &\text{large solutions, i.e., with } \|\mathbf{x}\| \geq e^{-1} (eh \cdot H^*(F))^{(4r)^{n+2}}, \\ &\text{small solutions, i.e., with } \|\mathbf{x}\| < e^{-1} (eh \cdot H^*(F))^{(4r)^{n+2}}. \end{aligned}$$

We first deal with the large solutions. First suppose that the hypergraph $\mathcal{H}$ defined in Section 7 has no edges. Then by Lemma 9, the hypergraph $\mathcal{H}$ has two connected components $\{1\}$ and $\{2\}$ with $2 = \iota^*(1)$. This means that $n = 2$, $r = 2$, that the linear forms L_1, L_2 are linearly independent and that $L_2 = \overline{L_1}$ in view of (4.24). Let $\mathbf{x}$ be a solution of (1.5). Then $|u_1| = |u_2| \leq h^{1/2}$ where $u_i = L_i(\mathbf{x})$. Further, we have $\mathbf{x} = (u_1, u_2)A^{-1}$, where A is the 2×2 -matrix whose i -th column consists of the coefficients of L_i . Now by (4.18) and (4.17), the elements of A^{-1} have absolute values at most

$$\frac{\max(\|L_1\|, \|L_2\|)}{|\det(L_1, L_2)|} \leq 4^3 H^*(F)/H^*(F) = 64.$$

Hence $\|\mathbf{x}\| \leq 128h^{1/2}$. So (1.5) does not have large solutions.

Now assume that $\mathcal{H}$ does have edges. Let C be the quantity defined by (7.13). Then by Proposition 3, for every large solution $\mathbf{x}$ of (1.5) there are linearly independent linear forms $L_{i_1}, \dots, L_{i_n}$ among $L_1, \dots, L_r$ with

$$(8.2) \quad |L_{i_1}(\mathbf{x}) \dots L_{i_n}(\mathbf{x})| \leq C \cdot \|\mathbf{x}\|^{-1/(n-1)} \leq \|\mathbf{x}\|^{-1/n}.$$

We apply Proposition 2 in Section 6 (with $\delta = 1/n$) to (8.2). Note that the large primitive solutions of (1.5) satisfy (6.6). Thus, on observing that

for the set $\{i_1, \dots, i_n\}$ we have at most $\binom{r}{n}$ possibilities, we obtain that the set of large primitive solutions of (1.5) is contained in the union not more than

$$\binom{r}{n} \cdot 16^{(n+10)^2} \cdot n^{2n+4} \log 4r \cdot \log(n \log 4r) < \frac{1}{2}A$$

proper linear subspaces of $\mathbb{Q}^n$, where A is given by (8.1).

We now deal with the small primitive solutions of (1.5) and to this end we apply Proposition 1 in Section 5. Taking $B := e^{-1} (eh \cdot H^*(F))^{(4r)^{n+2}}$ and observing that

$$\left(1 + \frac{\log eB}{\log eP}\right)^{n-1} \leq (4r)^{(n+2)(n-1)} \left(1 + \frac{\log(eh \cdot H^*(F))}{\log eP}\right)^{n-1},$$

we infer that the set of small primitive solutions of (1.5) is contained in the union of not more than

$$(300rn)^n \cdot (4r)^{(n+2)(n-1)} \cdot \max\left(1, \left(\frac{h^{n/r} P}{H^*(F)}\right)^{\frac{1}{n-1}}\right) \cdot \left(1 + \frac{\log(eh \cdot H^*(F))}{\log eP}\right)^{n-1} < \frac{1}{2}A$$

proper linear subspaces of $\mathbb{Q}^n$. It follows that indeed, the set of all primitive solutions of (1.5) is contained in the union of not more than A proper linear subspaces of $\mathbb{Q}^n$. This proves Theorem 2. $\square$

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JAN-HENDRIK EVERTSE
 UNIVERSITEIT LEIDEN
 MATHEMATISCH INSTITUUT
 POSTBUS 9512, 2300 RA LEIDEN
 THE NETHERLANDS
E-mail: evertse@math.leidenuniv.nl

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