

On Finsler spaces of Douglas type IV: Projectively flat Kropina spaces

By S. BÁCSÓ (Debrecen) and M. MATSUMOTO (Kyoto)

1. Introduction

We have a remarkably interesting two (α, β) -metrics; one is the Randers metric $L = \alpha + \beta$ and the other is the Kropina metric $L = \alpha^2/\beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$ ([1], [6]). A Randers space is a Finsler space with the Randers metric. It is projectively flat, if and only if its associated Riemannian space with the metric α is projectively flat and the change of the metric $\alpha \rightarrow \alpha + \beta$ is projective ([4, Theorem 4.3]; [7, Theorem 2]). On the other hand, the condition for a Kropina space (i.e. for a Finsler space with the Kropina metric), to be projectively flat has been given until now only in an unsatisfactory form, asserting the existence of some coordinate systems ([7, Theorem 3]).

The purpose of the present paper is to give this condition for a Kropina space in a completely tensorial form. We obtain it as an application of a new formulation of the theorem on the projective flatness ([3, II]).

We enumerate here some symbols for the later use. $K^n = (M^n, L = \alpha^2/\beta)$ is a Kropina space on a smooth n -manifold M^n , $R^n = (M^n, \alpha)$ is the Riemannian space associated with K^n . Let (a^{ij}) and $(\gamma_j{}^i{}_k)$ be the inverse matrix of (a_{ij}) and the Christoffel symbols constructed from (a_{ij}) .

Mathematics Subject Classification: 53B40.

Key words and phrases: Finsler space, Douglas space, Kropina space, projectively flat space.

Supported by FKFP 0690/99 and by OTKA 32058.

The semi-colon denotes the covariant differentiation with respect to $(\gamma_j^{i_k})$.

$$\begin{aligned} y_i &= a_{ir} y^r, & b^i &= a^{ir} b_r, & b^2 &= b_r b^r, \\ 2r_{ij} &= b_{i;j} + b_{j;i}, & 2s_{ij} &= b_{i;j} - b_{j;i}, & s^i_j &= a^{ir} s_{rj}, \\ s_i &= b^r s_{rj}, & s^i &= a^{ir} s_r. \end{aligned}$$

The change $\alpha \rightarrow \alpha + \beta$ is projective if and only if $s_{ij} = 0$, that is, the covariant vector field b_i is gradient. The transvection by y^i is denoted by subscript 0.

2. Kropina spaces of Berwald type and of Douglas type

It is well-known that if a Finsler space is projectively flat, then it must be of Douglas type ([3, II]). As it has been recently shown [10], a Kropina space K^n is of Douglas type, if and only if

$$\sigma_{ij} = s_{ij} - (b_i s_j - b_j s_i)/b^2$$

vanishes identically.

It is natural to assume $b^2 \neq 0$ for K^n . In fact, K^n has the fundamental tensor ([12], [3, I]),

$$g_{ij} = 2\tau a_{ij} + 3\tau^2 b_i b_j - (4\tau/\beta)(b_i y_j + b_j y_i) + (4/\beta^2) y_i y_j,$$

where $\tau = (\alpha/\beta)^2$, and $\det(g_{ij}) = 2^{n-1} \tau^{n+2} b^2 \det(a_{ij})$ [6]. Thus, on the assumption $b^2 \neq 0$ we get the inverse (g^{ij}) of (g_{ij}) as

$$\begin{aligned} g^{ij} &= (\rho/2) a^{ij} - (\rho/2b^2) b^i b^j + (\rho^2/b^2 \beta) (b^i y^j + b^j y^i) \\ &\quad + (\rho/\beta)^2 (1 - 2\rho^2/b^2) y^i y^j, \end{aligned}$$

where $\rho = (\beta/\alpha)^2$.

We consider the Berwald connection $B\Gamma = \{G^i_j, G_j^{i_k}, 0\}$ of K^n , where $G^i_j = \dot{\partial}_j G^i$ and $G_j^{i_k} = \dot{\partial}_k G^i_j$. If we put $G^i = (\gamma_0^{i_0}/2) + B^i$ for K^n then we have [9]

$$\begin{aligned} (1) \quad B^i &= B_*^i + (r_{00}/2b^2) b^i - (s_0/b^2) y^i, \\ (2) \quad B_*^i &= (\alpha^2/2\beta) \{ (s_0/b_2) b^i - s^i_0 \} - (r_{00}\beta/b^2 \alpha^2) y^i. \end{aligned} \tag{2.1}$$

Let now K^n be a Berwald space, that is let, G_j^i be functions of (x^i) alone: This means that G^i are homogeneous polynomials in (y^i) of degree two. Thus we must take account of only the terms B_*^i of G^i . (2) of (2.1) may be written as

$$2r_{00}\beta^2 y^i = \alpha^2 \{ \alpha^2 (s_0 b^i - b^2 s^i_0) - 2b^2 \beta B_*^i \}.$$

Assume that K^n is of Berwald type. As it has been shown in [2], $b^2 \neq 0$ implies $\alpha^2 \not\equiv 0 \pmod{\beta}$, and hence the above equation demands the existence of $u^i_0 = u^i_r(x)y^r$ satisfying

$$(2.2) \quad \begin{aligned} (a) \quad & \alpha^2 (s_0 b^i - b^2 s^i_0) - 2b^2 \beta B_*^i = \beta^2 u^i_0, \\ (b) \quad & 2r_{00} y^i = \alpha^2 u^i_0. \end{aligned}$$

The latter gives $2r_{00}\beta = \alpha^2 b_r u^r_0$, which implies $b_r u^r_0 = 2u\beta$ for some $u(x)$. Thus (2.2b) yields

$$(2.3) \quad (1) \quad r_{00} = u\alpha^2, \quad (2) \quad u^i_0 = 2uy^i.$$

Then (2.2a) can be written as

$$\alpha^2 (s_0 b^i - b^2 s^i_0) = 2\beta (u\beta y^i + b^2 B_*^i),$$

which shows the existence of $v^i(x)$ such that

$$s_0 b^i - b^2 s^i_0 = \beta v^i, \quad 2(u\beta y^i + b^2 B_*^i) = \alpha^2 v^i.$$

The latter proves that B_*^i are now homogeneous polynomials in (y^i) of degree two. The former is equivalent to $s_j b_i - b^2 s_{ij} = b_j v_i$, and the skew-symmetry of s_{ij} gives $b_i(s_j - v_j) = -b_j(s_i - v_i)$, which easily implies $v_i = s_i$. Consequently we obtain

$$(2.4) \quad (1) \quad r_{ij} = ua_{ij}, \quad (2) \quad s_{ij} = (b_i s_j - b_j s_i)/b^2.$$

Therefore we have

Theorem 1. *A Kropina space is of Berwald type, if and only if (2.4) holds, that is,*

$$(2.4') \quad b_{i;j} = ua_{ij} + (b_i s_j - b_j s_i)/b^2.$$

Remark. The condition for a Kropina space to be of Berwald type has been given first by C. SHIBATA [12] and then S. KIKUCHI [5] and one of the present authors [8]. Though they obtained their results in somewhat different forms, all of them can be expressed in the form

$$(2.5) \quad b_{i;j} = (f_r b^r) a_{ij} + b_i f_j - b_j f_i,$$

where f_i are some functions of (x^i) , that is, $r_{ij} = (f_r b^r) a_{ij}$ and $s_{ij} = b_i f_j - b_j f_i$. From (2.5) we get $s_j = (b^r f_r) b_j - b^2 f_j$ and hence (2) of (2.4). If we put $u = b^r f_r$, then (1) of (2.4) is obtained. Conversely, if we put $f_i = (s_i + u b_i)/b^2$, then (2.4') is rewritten in the form (2.5).

Therefore a Kropina space K^n is of Douglas type, if and only if $\sigma_{ij} = 0$, and further it is of Berwald type, if and only if r_{00} is proportional to α^2 and $\sigma_{ij} = 0$.

We shall restrict our discussions to the two-dimensional case. If $s_{ij} = 0$, then $s_i = 0$ and hence $\sigma_{ij} = 0$. If $s_{ij} \neq 0$, that is, $s_{12} \neq 0$, then we put $(b_1 s_2 - b_2 s_1)/s_{12} = k$, which is nothing but $b_i s_j - b_j s_i = k s_{ij}$, $i, j = 1, 2$, and $k = 1/b^2$ is easily obtained. Thus we have

Theorem 2. *Any Kropina space of dimension two has $\sigma_{ij} = 0$ and is of Douglas type.*

Remark. The result that “any Kropina space of dimension two is a Douglas space” has been proved in ([3, I]) in a completely different way. This fact is also shown by the differential equation of geodesics [11], (3.3), in conformity with the definition of a Douglas space.

3. An outline of the theory of projective flatness

We shall sketch the theory of projective flatness which has been developed by the present authors in ([3, II]), and we apply it here to the Riemannian space $R^n = (M^n, \alpha)$ associated with a Kropina space $K^n = (M^n, L = \alpha^2/\beta)$. All the quantities of R^n will be denoted by putting the superscript $^\circ$.

We start our theory with the projective invariant, Q^0 :

$$Q^h = G^h - G y^h / (n + 1), \quad G = G^r_{r}.$$

Since G^h of R^n is equal to $\gamma_0^h/2$, the Q^0 -invariant of R^n is written as

$$(3.1) \quad Q^{\circ h} = \frac{1}{2}\gamma_0^h - \gamma_0^r y^h / (n+1).$$

From Q^h we construct the Q^1 -invariant $Q^h_i = \dot{\partial}_i Q^h$ and the Q^2 -invariant $Q_i^h_j = \dot{\partial}_j Q^h_i$. In R^n the latter is given by

$$(3.2) \quad Q^{\circ}_i{}^h{}_j = \gamma_i^h{}_j - \gamma_i \delta^h{}_j - \gamma_j \delta^h{}_i, \quad \gamma_i = \gamma_i^r / (n+1).$$

Further we obtain the Q^3 -invariant

$$Q_i^h{}_{jk} = \partial_k Q_i^h{}_j - (\dot{\partial}_r Q_i^h{}_j) Q^r{}_k + Q_i^r{}_j Q_r^h{}_k - (j/k),$$

where (j/k) denotes the interchange of subscripts j, k of the preceding terms. The Q^3 -invariant of R^n is given by

$$(3.3) \quad \begin{aligned} Q^{\circ}_i{}^h{}_{jk} &= R^{\circ}_i{}^h{}_{jk} - \delta^h{}_j \gamma_{ik} + \delta^h{}_k \gamma_{ij}, \\ \gamma_{ij} &= \partial_j \gamma_i - \gamma_i^r \gamma_r + \gamma_i \gamma_j, \end{aligned}$$

where $R^{\circ}_i{}^h{}_{jk}$ is the curvature tensor of R^n . The symmetry of the Ricci tensor $R^{\circ}_{ij} (= R^{\circ}_i{}^r{}_{jr})$ implies the symmetry of γ_{ij} . We get another invariant $Q_{ij} = Q_i^r{}_{jr}$. In R^n we have

$$(3.4) \quad Q^{\circ}_{ij} = R^{\circ}_{ij} + (n-1)\gamma_{ij}.$$

Then the Π -tensor is constructed as

$$\Pi_i^h{}_{jk} = Q_i^h{}_{jk} + \{\delta^h{}_j Q_{ik} - (j/k)\} / (n-1),$$

which is nothing but the Weyl projective curvature tensor W . The Π -tensor of R^n is given in the well-known form

$$(3.5) \quad \Pi^{\circ}_i{}^h{}_{jk} = R^{\circ}_i{}^h{}_{jk} + \{\delta^h{}_j R^{\circ}_{ik} - (j/k)\} / (n-1).$$

Next, for the use in the two-dimensional case, we define

$$\Pi_{ijk} = \delta_k Q_{ij} + Q_i^r{}_j Q_{rk} - (j/k),$$

where $\delta_k = \partial_k - G^r{}_k \dot{\partial}_r$. For R^n we have

$$(3.6) \quad \Pi^{\circ}_{ijk} = R^{\circ}_{ij;k} - R^{\circ}_{ik;j} - (n-1)\gamma_r \Pi^{\circ}_i{}^r{}_{jk}.$$

Since the Weyl projective curvature tensor vanishes identically in the two-dimensional case, in the case of $n = 2$ we obtain for (3.6) the following

$$(3.6_2) \quad \Pi^\circ_{ijk} = R^\circ_{ij;k} - R^\circ_{ik;j}.$$

Now we give a new formulation to the fundamental theorem of projective flatness considered already in ([3, II]).

Theorem BM. *A Finsler space F^n of dimension n is projectively flat if and only if F^n is a Douglas space and*

$$(1) \quad n > 2 : \text{ the } \Pi\text{-tensor} = 0, \quad (2) \quad n = 2 : \Pi_{ijk} = 0.$$

4. Projectively flat Kropina spaces

We consider now an n -dimensional Kropina space $K^n = (M^n, L = \alpha^2/\beta)$. Let $R^n = (M^n, \alpha)$ be the Riemannian space associated with K^n . All the quantities of R^n will be denoted by putting the superscript $^\circ$ as in the last section. Since our purpose is to find the condition for K^n to be projectively flat, it should be assumed first that K^n be a Douglas space, and hence $s_{ij} = (b_i s_j - b_j s_i)/b^2$ according to Theorem BM. Then $s^h_0 = (s_0 b^h - \beta s^h)/b^2$ and B^h of (2.1) is reduced to

$$B^h = (\alpha^2 s^h + r_{00} b^h)/2b^2 - (\alpha^2 s_0 + \beta r_{00})y^h/b^2 \alpha^2.$$

If we define the tensor

$$(4.1) \quad k_i^h{}_j = (a_{ij} s^h + r_{ij} b^h)/b^2,$$

then $k_0^h{}_0 = (\alpha^2 s^h + r_{00} b^h)/b^2$ and $k_{000} (= k_0^h{}_0 y_h) = (\alpha^2 s_0 + \beta r_{00})/b^2$. Hence B^h can be simply written as

$$B^h = k_0^h{}_0/2 - k_{000} y^h/\alpha^2,$$

from which we get $B^h{}_i = \dot{\partial}_i B^h$ and $B = B^r{}_r$ in the forms

$$\begin{aligned} B^h{}_i &= k_0^h{}_i - \{(k_{0i0} + 2k_{00i})y^h - (2k_{000}/\alpha^2)y_i y^h + k_{000}\delta^h{}_i\}/\alpha^2, \\ B &= k_0^r{}_r - (n+1)k_{000}/\alpha^2. \end{aligned}$$

Thus (3.1) leads to

$$(4.2) \quad Q^h = Q^{\circ h} + k_0^h/2 - k_0^r y^h/(n+1).$$

As it was remarked in ([3, II]), $Q^h(x, y)$ are certainly homogeneous polynomials in (y^i) of degree two for the Kropina space K^n of Douglas type.

From (4.2) and (3.2) we obtain

$$(4.3) \quad Q_i^h{}^j = Q^{\circ}{}_i^h{}^j + K_i^h{}^j,$$

where $K_i^h{}^j$ is the tensor defined by

$$(4.3a) \quad K_i^h{}^j = k_i^h{}^j - (k_i^r{}_r \delta^h{}_j + k_j^r{}_r \delta^h{}_i)/(n+1).$$

We remark that $K_i^r{}_r = 0$.

Then (4.3) and (3.3) yield the Q^3 -invariants of K^n in the form

$$(4.4) \quad Q_i^h{}^{jk} = Q^{\circ}{}_i^h{}^{jk} + K_i^h{}^{jk} + (\delta^h{}_j K_i^r{}_k - \delta^h{}_k K_i^r{}_j) \gamma_r,$$

where $K_i^h{}^{jk}$ is the (1,3)-type tensor defined by

$$(4.4a) \quad K_i^h{}^{jk} = K_i^h{}_{j;k} + K_i^r{}_j K_r^h{}_k - (j/k).$$

(4.4) and (3.4) lead to

$$(4.5) \quad Q_{ij} = Q^{\circ}{}_{ij} + K_{ij} - (n-1)K_i^r{}_j \gamma_r,$$

where $K_{ij} = K_i^r{}_{jr}$ is the symmetric tensor given by

$$(4.5a) \quad K_{ij} = K_i^r{}_{j;r} - K_i^r{}_s K_j^s{}_r.$$

Finally the Π -tensor, that is the Weyl projective curvature tensor W of K^n is obtained from (4.4) and (4.5) together with (3.3) and (3.4) as

$$(4.6) \quad \Pi_i^h{}^{jk} = \Pi^{\circ}{}_i^h{}^{jk} + K_i^h{}^{jk} + (\delta^h{}_j K_{ik} - \delta^h{}_k K_{ij})/(n-1).$$

Therefore Theorem BM leads to our main result as follows:

Theorem 3. A Kropina space K^n of dimension $n > 2$ is projectively flat, if and only if $s_{ij} = (b_i s_j - b_j s_i)/b^2$ and

$$\Pi^\circ_i{}^h{}_{jk} + K_i{}^h{}_{jk} + (\delta^h{}_j K_{ik} - \delta^h{}_k K_{ij})/(n-1) = 0,$$

where Π° is the Weyl projective curvature tensor of the associated Riemannian space and $K_i{}^h{}_{jk}$ and K_{ik} are defined by (4.1), (4.3a), (4.4a) and (4.5a).

Next (4.5) and (3.6) give

$$(4.7) \quad \begin{aligned} \Pi_{ijk} = & \Pi^\circ_{ijk} + K_{ijk} + \{K_i{}^r{}_j R^\circ_{rk} - (j/k)\} \\ & - [(n-1)K_i{}^r{}_{jk} + \{\delta^r{}_j K_{ik} - (j/k)\}]\gamma_r, \end{aligned}$$

where the tensor K_{ijk} is defined by

$$(4.7a) \quad K_{ijk} = K_{ij;k} + K_i{}^r{}_j K_{rk} - (j/k).$$

Lemma. In the two-dimensional case

$$\tau_i{}^h{}_{jk} = T_i{}^h{}_{jk} + \delta^h{}_j T_{ik} - \delta^h{}_k T_{ij}, \quad T_{ij} = T_i{}^r{}_{jr},$$

vanishes identically for every tensor $T_i{}^h{}_{jk}$ which is skew-symmetric in j, k .

For instance

$$\begin{aligned} \tau_2{}^1{}_{12} &= T_2{}^1{}_{12} + T_{22} = T_2{}^1{}_{12} + (T_2{}^1{}_{21} + T_2{}^2{}_{22}) \\ &= T_2{}^1{}_{12} + (-T_2{}^1{}_{12} + 0) = 0. \end{aligned}$$

Hence, for $n = 2$ (4.7) reduces to

$$(4.7_2) \quad \Pi_{ijk} = \Pi^\circ_{ijk} + K_{ijk} + K_i{}^r{}_j R^\circ_{rk} - K_i{}^r{}_k R^\circ_{rj}.$$

Therefore Theorem BM together with Theorem 2 leads to

Theorem 4. A Kropina space K^2 of dimension two is projectively flat, if and only if

$$\Pi^\circ_{ijk} + K_{ijk} + K_i{}^r{}_j R^\circ_{rk} - K_i{}^r{}_k R^\circ_{rj} = 0,$$

where R° is the Ricci tensor of the associated Riemannian space, Π° is given by (3.6₂) and K_{ijk} and $K_j{}^i{}_k$ are defined by (4.7a) and (4.3a).

References

- [1] P. L. ANTONELLI, R. S. INGARDEN and M. MATSUMOTO, The theory of Sprays and Finsler spaces with applications in physics and biology, *Kluwer Acad. Publishers, Dordrecht, Boston, London*, 1993.
- [2] S. BÁCSÓ and M. MATSUMOTO, Projective changes between Finsler space with (α, β) -metric, *Tensor, N. S.* **55** (1994), 252–257.
- [3] S. BÁCSÓ and M. MATSUMOTO, On Finsler spaces of Douglas type. I: A generalization of the notion of Berwald space, *Publ. Math. Debrecen* **51** (1997), 385–406; II: Projectively flat spaces, *Publ. Math. Debrecen* **53** (1998), 423–438.
- [4] M. HASHIGUCHI and Y. ICHIJYO, Randers spaces with rectilinear geodesics, *Rep. Fac. Sci., Kagoshima Univ. (Math. Phys. & Chem.)* **13** (1980), 33–40.
- [5] S. KIKUCHI, On the condition that a space of (α, β) -metric be locally Minkowskian, *Tensor, N. S.* **33** (1979), 242–246.
- [6] M. MATSUMOTO, Foundations of Finsler geometry and special Finsler spaces, *Kaiseisha Press, Saikawa, Otsu, Japan*, 1986.
- [7] M. MATSUMOTO, Projectively flat Finsler spaces with (α, β) -metric, *Rep. on Math. Phys.* **30** (1991), 15–20.
- [8] M. MATSUMOTO, The Berwald connection of a Finsler space with an (α, β) -metric, *Tensor, N. S.* **50** (1991), 18–21.
- [9] M. MATSUMOTO, Finsler spaces of constant curvature with Kropina metric, *Tensor, N. S.* **50** (1991), 194–201.
- [10] M. MATSUMOTO, Finsler spaces with (α, β) -metric of Douglas type, *Tensor, N. S.* (to appear).
- [11] M. MATSUMOTO and H. S. PARK, Equations of geodesics in two-dimensional Finsler spaces with (α, β) -metric, *Rev. Roum. Math. Pures Appl.* **42** (1997), 787–793.
- [12] C. SHIBATA, On Finsler spaces with Kropina metric, *Rep. on Math. Phys.* **13** (1978), 117–128.

S. BÁCSÓ
 INSTITUTE OF MATHEMATICS AND INFORMATICS
 LAJOS KOSSUTH UNIVERSITY
 H-4010 DEBRECEN, P.O. BOX 12
 HUNGARY

E-mail: bacsos@math.klte.hu

M. MATSUMOTO
 15, ZENBU-CHO, SHIMOGAMO
 SAKYO-KU, KYOTO, 606-0815
 JAPAN

(Received September 22, 1999)