

Geometry of Riemannian manifolds and their unit tangent sphere bundles

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Abstract. We review some aspects of the geometry of the tangent bundle and the unit tangent sphere bundle of a Riemannian manifold and focus on the relationship between this geometry and that of the manifold.

1. Introduction

It is an approved method in Riemannian geometry to study the geometry of a Riemannian manifold (M, g) via geometric objects naturally associated to it. As an example, one considers the family of small geodesic spheres or tubes and investigates how the geometry of the ambient space (M, g) influences the geometric properties of these hypersurfaces, and conversely, how geometric conditions on the geodesic spheres or tubes are reflected in the geometry of (M, g) . The reference [24] gives but one example of this procedure. See also [58], [59] and [60] for more information and further examples and references. Within this framework, we now take the tangent bundle TM and the unit tangent sphere bundle T_1M of a Riemannian manifold (M, g) , equipped with a particular metric, as the major object of study, i.e., we investigate *to what extent the geometry of the tangent bundle or the unit tangent sphere bundle influences or even determines the geometry of (M, g) .*

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The use of bundles over (M, g) to study its geometric properties is far from new. As an example, the geodesics on (M, g) can be studied via the geodesic flow on T_1M . In the same vein, A. L. BESSE uses the unit tangent sphere bundle with this flow as his basic tool in [4]. Here we report on recent results in the study of both the tangent and the unit tangent sphere bundle, equipped with natural metrics and other “natural” structures (such as an almost complex structure on TM and a contact metric structure on T_1M). By “natural” we mean that these structures are canonically determined by the geometric structure of M (i.e., the metric g and possibly other structural tensors). The reader can consult the original papers [14]–[17] and [20] and the earlier survey [18] for additional results and further references.

One of the best known Riemannian metrics on the tangent bundle TM is the Sasaki metric g_S . As a metric space however, (TM, g_S) is not very interesting for our purposes. For instance, the fairly weak hypothesis to have constant scalar curvature already implies that (M, g) must be flat ([44]). Other natural metrics on TM have been introduced and studied. We mention in particular the Cheeger-Gromoll metric (see [41], [44] and [50]). Other examples can be found in [65].

The unit tangent sphere bundle T_1M with the metric induced from the Sasaki metric g_S is more interesting. A result in [62] and also mentioned by MUSSO and TRICERRI ([44]) says that the unit tangent sphere bundle of a two-point homogeneous space equipped with this metric is homogeneous. A fascinating open problem is whether the converse is also true. A first step towards an answer is to consider when the scalar curvature on T_1M is constant. We give several classes of examples and classify all two- and three-dimensional and all conformally flat manifolds whose unit tangent sphere bundle enjoys this property. Next, we investigate when the unit tangent sphere bundles are Ricci-curvature homogeneous, i.e., they have the same Ricci curvature tensor at each point. Starting from the earlier classification results for constant scalar curvature, we only find two-point homogeneous spaces. This strengthens us in our belief that the converse of the above result is valid too.

It is well-known that T_1M admits a contact metric structure $(\xi, \eta, \varphi, \bar{g})$, where the metric $\bar{g}$ is homothetic to the metric induced by the Sasaki metric g_S ([6]). In particular, the results mentioned above still hold for $(T_1M, \bar{g})$. It turns out that, for a two-point homogeneous space, the unit

tangent sphere bundle is not only homogeneous as a metric space, but even as a contact metric space. Again, it is not known whether the converse holds. Under various conditions on (M, g) we show the converse statement to be true. In the proofs, the theory of Osserman spaces plays a major role.

The final two sections deal with the geodesic flow vector field ξ' on T_1M which is proportional to the characteristic vector field ξ of the contact metric structure. First, we study the reflections with respect to the integral curves of ξ' ; secondly, we consider ξ' as an immersion $\xi' : T_1M \rightarrow T_1(T_1M) : (x, u) \mapsto (x, u, \xi')$ and study when ξ' is a minimal or harmonic vector field, i.e., when the map ξ' satisfies the critical point conditions for the volume or the energy functional if we equip the respective bundles with the Sasaki metric. Two-point homogeneous spaces again play a distinguished role.

Apart from the unit tangent sphere bundle, one could also study tangent sphere bundles of arbitrary radius r equipped with the induced Sasaki metric. Most of our results have analogues in this setting, but some differences do occur, especially as far as the curvature is concerned. We refer to [42], [43] and [21] for the details and for references to other related work.

2. The tangent bundle and the unit tangent sphere bundle

First, we collect the basic facts about the tangent bundle and the unit tangent sphere bundle of a Riemannian manifold and give the necessary formulas. For a more elaborate exposition, we refer to [15], [26], [38], [44], [49], [64] and [65].

Let (M, g) be an n -dimensional ($n \geq 2$) connected, smooth Riemannian manifold and ∇ the associated Levi Civita connection. We take the Riemann curvature tensor R with the sign convention $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ for vector fields X and Y on M . The tangent bundle of (M, g) , denoted by TM , consists of pairs (x, u) where x is a point in M and u a tangent vector to M at x . The mapping $\pi : TM \rightarrow M : (x, u) \mapsto x$ is the natural projection from TM onto M .

Using the Levi Civita connection ∇ on (M, g) , one can define a splitting of the tangent space to TM at (x, u) into the direct sum of the vertical subspace $VTM_{(x, u)} = \ker \pi_{*|(x, u)}$ and the horizontal subspace $HTM_{(x, u)}$:

$$T_{(x, u)}TM = VTM_{(x, u)} \oplus HTM_{(x, u)}.$$

Vertical vectors are tangent to curves $\tilde{\gamma}(t) = (\gamma(t), V(t))$ with $\gamma(t) = x$ for each t , whereas horizontal vectors are tangent to curves $\tilde{\gamma}(t) = (\gamma(t), V(t))$ for which $\nabla_{\dot{\gamma}(t)} V(t) = 0$ holds.

For $X \in T_x M$, there exists a unique vector X^h at the point $(x, u) \in TM$ such that $X^h \in HTM_{(x,u)}$ and $\pi_*(X^h) = X$. X^h is called the *horizontal lift* of X to (x, u) . It is tangent at (x, u) to the curve $(\gamma(t), V(t))$ with $\gamma(0) = x$ and $\dot{\gamma}(0) = X$ and with $V(t)$ the parallel translate of u along γ . Similarly, there is a unique vector X^v at the point (x, u) such that $X^v \in VTM_{(x,u)}$ and $X^v(df) = Xf$ for all functions f on M . (Here we consider df as a function on M .) X^v is called the *vertical lift* of X to (x, u) . It is tangent at (x, u) to the curve $(x, u + tX)$. The map $X \mapsto X^h$, respectively $X \mapsto X^v$, is an isomorphism between $T_x M$ and $HTM_{(x,u)}$, respectively $T_x M$ and $VTM_{(x,u)}$. In a similar way, one lifts vector fields on M to horizontal or vertical vector fields on TM . The expressions in local coordinates for these lifts are given in [15], for example.

Further, if T is a tensor field of type $(1, s)$ on M and $X_1, \dots, X_{s-1}$ are vector fields on M , then we denote by $(T(X_1, \dots, u, \dots, X_{s-1}))^v$ (respectively, $(T(X_1, \dots, u, \dots, X_{s-1}))^h$) the vertical (respectively, horizontal) vector field on TM which, at a point (y, w) takes the value $T(X_{1y}, \dots, w, \dots, X_{s-1y})^v$ (respectively, $T(X_{1y}, \dots, w, \dots, X_{s-1y})^h$). Note that this is *not* the vertical (respectively, horizontal) lift of a vector field on M .

The tangent bundle TM of a Riemannian manifold (M, g) can be equipped with a natural Riemannian metric g_S , the *Sasaki metric*, which depends only on the metric structure g of the base manifold M . It is determined explicitly by

$$g_S(X^h, Y^h) = g_S(X^v, Y^v) = g(X, Y) \circ \pi, \quad g_S(X^h, Y^v) = 0$$

for vector fields X and Y on M .

As far as curvature theory is concerned, TM with the Sasaki metric g_S is not very interesting unfortunately. Indeed, E. MUSSO and F. TRICERRI proved

Theorem 2.1 ([44]). *The tangent bundle (TM, g_S) has constant scalar curvature if and only if (M, g) is flat.*

This result shows that the Sasaki metric, though a very natural Riemannian metric on TM , is “extremely rigid” ([44]). There are at least two possible alternatives: either one studies other interesting metrics on TM

(see, e.g., [41], [44], [50], [65]) or one considers the unit tangent sphere bundle T_1M with the metric induced from g_S . We concentrate on the second option.

The hypersurface T_1M of TM consists of the unit tangent vectors to (M, g) and is given implicitly by the equation $g_x(u, u) = 1$. A unit normal vector N to T_1M at $(x, u) \in T_1M$ is given by the vertical lift of u to (x, u) : $N|_{(x, u)} = u^v$.

As the vertical lift of a vector (field) is not tangent to T_1M in general, we define the *tangential lift* of $X \in T_xM$ to $(x, u) \in T_1M$ by

$$X_{(x, u)}^t = (X - g(X, u)u)^v.$$

Clearly, the tangent space to T_1M at (x, u) is spanned by vectors of the form X^h and X^t where $X \in T_xM$. For the sake of notational clarity, we will use $\bar{X}$ as a shorthand for $X - g(X, u)u$. Then $X^t = \bar{X}^v$.

If we denote the metric on T_1M induced from the Sasaki metric g_S on TM also by g_S , then we have

$$\bar{g}_S(X^t, Y^t) = g(\bar{X}, \bar{Y}) = g(X, Y) - g(X, u)g(Y, u),$$

$$\bar{g}_S(X^t, Y^h) = 0,$$

$$\bar{g}_S(X^h, Y^h) = g(X, Y).$$

With the metric g_S on T_1M in place, it is a fairly routine exercise to calculate the associated Levi Civita connection $\bar{\nabla}$, the Riemann curvature tensor $\bar{R}$, the Ricci tensor $\bar{\rho}$ and the scalar curvature $\bar{\tau}$ in terms of the curvature of the base manifold (M, g) (see [15], [16] or [64], for instance). In the sequel, we will need the expressions for $\bar{R}$, $\bar{\rho}$ and $\bar{\tau}$. The Riemann curvature tensor $\bar{R}$ is given explicitly by

$$\begin{aligned} (1) \quad \bar{R}(X^t, Y^t)Z^t &= -g(\bar{X}, \bar{Z})Y^t + g(\bar{Z}, \bar{Y})X^t, \\ \bar{R}(X^t, Y^t)Z^h &= (R(\bar{X}, \bar{Y})Z)^h + \frac{1}{4}([R(u, X), R(u, Y)]Z)^h, \\ \bar{R}(X^h, Y^t)Z^t &= -\frac{1}{2}(R(\bar{Y}, \bar{Z})X)^h - \frac{1}{4}(R(u, Y)R(u, Z)X)^h, \end{aligned}$$

$$\begin{aligned}
\bar{R}(X^h, Y^t)Z^h &= \frac{1}{2}(R(X, Z)\bar{Y})^t - \frac{1}{4}(R(X, R(u, Y)Z)u)^t \\
&\quad + \frac{1}{2}((\nabla_X R)(u, Y)Z)^h, \\
\bar{R}(X^h, Y^h)Z^t &= (R(X, Y)\bar{Z})^t + \frac{1}{4}(R(Y, R(u, Z)X)u \\
&\quad - R(X, R(u, Z)Y)u)^t \\
&\quad + \frac{1}{2}((\nabla_X R)(u, Z)Y - (\nabla_Y R)(u, Z)X)^h, \\
\bar{R}(X^h, Y^h)Z^h &= (R(X, Y)Z)^h + \frac{1}{2}(R(u, R(X, Y)u)Z)^h \\
&\quad - \frac{1}{4}(R(u, R(Y, Z)u)X - R(u, R(X, Z)u)Y)^h \\
&\quad + \frac{1}{2}((\nabla_Z R)(X, Y)u)^t;
\end{aligned}$$

the Ricci tensor $\bar{\rho}$ by

$$\begin{aligned}
(2) \quad \bar{\rho}(X^t, Y^t) &= (n-2)(g(X, Y) - g(X, u)g(Y, u)) \\
&\quad + \frac{1}{4} \sum_{i=1}^n g(R(u, X)E_i, R(u, Y)E_i), \\
\bar{\rho}(X^t, Y^h) &= \frac{1}{2}((\nabla_u \rho)(X, Y) - (\nabla_X \rho)(u, Y)), \\
\bar{\rho}(X^h, Y^h) &= \rho_x(X, Y) - \frac{1}{2} \sum_{i=1}^n g(R(u, E_i)X, R(u, E_i)Y)
\end{aligned}$$

where $\{E_1, \dots, E_n\}$ is a (local) orthonormal frame on (M, g) and the scalar curvature $\bar{\tau}$ by

$$(3) \quad \bar{\tau}_{(x,u)} = \tau_x + (n-1)(n-2) - \xi_x(u, u)/4.$$

Here, as in [4] and [24], $\xi(u, v) = \sum_{i,j=1}^n g(R(u, E_i)E_j, R(v, E_i)E_j)$. Note that the natural mapping $\pi : (T_1 M, g_S) \rightarrow (M, g)$ is a Riemannian submersion, hence these curvature formulas may also be derived using O'Neill's formalism. (See, e.g., [5].)

3. Homogeneous unit tangent sphere bundles

A mapping $f : M \rightarrow M$ can always be lifted to a mapping $\tilde{f} : TM \rightarrow TM$ by putting $\tilde{f}(x, u) = (f(x), f_*u)$. A lift to a mapping of T_1M into itself is only possible if f_* maps unit vectors to unit vectors, i.e., f is a local isometry. Then the lift of f to T_1M is itself an isometry for the Sasaki metric g_S on T_1M . In particular, we have

Theorem 3.1 ([44], [62]). *If (M, g) is a two-point homogeneous space, then its unit tangent sphere bundle (T_1M, g_S) is a homogeneous Riemannian manifold.*

It is intriguing that the converse question: “if (T_1M, g_S) is a (locally) homogeneous space, is (M, g) then necessarily (locally) isometric to a two-point homogeneous space?” has, to our knowledge, not yet been answered in its full generality. The present authors have made some first steps towards a definitive answer in [16] and [17]. We give a survey of these results in this section.

First we note that a (locally) homogeneous space has constant scalar curvature. From the formula (3) for the scalar curvature on T_1M it follows easily

Theorem 3.2. *The unit tangent sphere bundle (T_1M, g_S) has constant scalar curvature $\bar{\tau}$ if and only if on (M, g) it holds*

$$(4) \quad \xi = \frac{|R|^2}{n} g,$$

$$(5) \quad 4n\tau - |R|^2 = \text{constant}.$$

Remark 1. The algebraic condition (4) has appeared in the literature before (see, e.g., [4], [24], [36]), but without a clear geometric meaning. An analytic interpretation is given in [5, p. 134]: an Einstein metric on a compact manifold is critical for the functional $SR(g) = \int_M |R_g|^2 d\text{vol}$ restricted to those metrics g such that $\text{vol}(M) = 1$ if and only if $\xi = (|R|^2/n)g$. We can now give a nice geometric interpretation for Riemannian manifolds satisfying (5): on such manifolds, (4) holds if and only if their unit tangent sphere bundle has constant scalar curvature.

The case of (locally) reducible manifolds is easy to deal with:

Corollary 3.3. *The unit tangent sphere bundle (T_1M, g_S) of a (local) product manifold $(M, g) = (M_1^{n_1}, g_1) \times (M_2^{n_2}, g_2)$ has constant scalar curvature if and only if the unit tangent sphere bundles of both (M_1, g_1) and (M_2, g_2) have constant scalar curvature and, additionally,*

$$(6) \quad \frac{|R_1|^2}{n_1} = \frac{|R_2|^2}{n_2}.$$

As immediate examples of Riemannian spaces whose unit tangent sphere bundles have constant scalar curvature, we have

0. Spaces of constant curvature.
1. Irreducible symmetric spaces and, more generally, isotropy irreducible homogeneous spaces.
2. Reducible symmetric spaces $(M, g) = (M_1, g_1) \times \cdots \times (M_k, g_k)$ with irreducible components (M_i, g_i) such that $|R_1|^2/n_1 = \cdots = |R_k|^2/n_k$.
3. Super-Einstein spaces ([36]): these are Einstein manifolds satisfying condition (4) with $|R|^2$ constant.
4. Harmonic spaces: as every harmonic space is super-Einstein (see, e.g., [4], [24]).
5. Four-dimensional orientable Einstein manifolds which are self-dual or anti-self-dual. For this result and more in the same direction, we refer to [16].

In low dimensions, we can give a complete classification.

Proposition 3.4. *The unit tangent sphere bundle (T_1M, g_S) of a two-dimensional manifold (M, g) has constant scalar curvature $\bar{\tau}$ if and only if (M, g) has constant curvature.*

Proposition 3.5. *The unit tangent sphere bundle (T_1M, g_S) of a three-dimensional manifold (M, g) has constant scalar curvature $\bar{\tau}$ if and only if (M, g) has constant curvature or (M, g) is a curvature homogeneous space with constant Ricci roots $\rho_1 = \rho_2 = 0 \neq \rho_3$.*

(We refer to [16] for explicit examples and more information about this last class of spaces.)

The proofs of these propositions use the explicit expressions for the curvature tensor R in terms of the scalar curvature τ and the Ricci tensor ρ , namely

$$R = \frac{\tau}{4} g \otimes g$$

in dimension two and

$$R = \rho \oslash g - \frac{\tau}{4} g \oslash g$$

in dimension three. Here $\oslash$ is the Kulkarni–Nomizu product of symmetric two-tensors defined as follows:

$$\begin{aligned} (h \oslash k)(X, Y, Z, V) &= h(X, Z)k(Y, V) + h(Y, V)k(X, Z) \\ &\quad - h(X, V)k(Y, Z) - h(Y, Z)k(X, V). \end{aligned}$$

There is another class of Riemannian manifolds where a similar curvature expression exists: for conformally flat manifolds, it holds

$$R = \frac{1}{n-2} \rho \oslash g - \frac{\tau}{2(n-1)(n-2)} g \oslash g.$$

As before, we use this formula for R to express the conditions (4) and (5). Further, we also use H. TAKAGI's classification of conformally flat locally homogeneous spaces ([51]) which is also valid for curvature homogeneous manifolds ([39]) to obtain

Theorem 3.6. *Let (M^n, g) be conformally flat and $n \geq 4$. The unit tangent sphere bundle (T_1M, g_S) has constant scalar curvature if and only if (M, g) has constant curvature or n is even, say $n = 2k$, and (M, g) is locally isometric to the product $M^k(\kappa) \times M^k(-\kappa)$, $\kappa \neq 0$, or $n = 4$, $|\rho|^2$ is constant and $\tau = 0$.*

Next, for a (locally) homogeneous space (M, g) , the Ricci curvature is the “same” at each point. More precisely, the manifold is *Ricci-curvature homogeneous*, i.e., for every pair of points $x, y \in M$, there exists a linear isometry $F : T_xM \rightarrow T_yM$ such that $F^*\rho_y = \rho_x$. As ρ is a symmetric $(0, 2)$ -tensor field and as such diagonalizable at each point, one can say equivalently that the matrices for ρ_x , respectively ρ_y , with respect to an orthonormal basis of T_xM , respectively of T_yM , must have the same eigenvalues (i.e., the same Ricci roots) with the same multiplicities.

Starting from the above classification results for unit tangent sphere bundles with constant scalar curvature, we find

Proposition 3.7. *Let (M, g) be a two- or three-dimensional Riemannian manifold. Its unit tangent sphere bundle (T_1M, g_S) is (Ricci-)curvature homogeneous if and only if (M, g) has constant curvature. In that case, (T_1M, g_S) is even locally homogeneous.*

Proposition 3.8. *Let (M, g) be conformally flat. Then (T_1M, g_S) is (Ricci-)curvature homogeneous if and only if (M, g) has constant curvature. In that case, (T_1M, g_S) is even locally homogeneous.*

Another result in this framework deals with harmonic spaces. Up to local isometries, the only known examples so far, apart from the two-point homogeneous spaces, are the *Damek–Ricci spaces*, that have only been discovered fairly recently. These are solvable Lie groups whose Lie algebras are solvable extensions of generalized Heisenberg algebras, equipped with a special left-invariant metric. We refer to [1] for the precise definitions, some geometric properties of these remarkable spaces and further references. As harmonic spaces, every Damek–Ricci space has a unit tangent sphere bundle with constant scalar curvature. Moreover, we have

Proposition 3.9. *The unit tangent sphere bundle (T_1S, g_S) of a Damek–Ricci space S is (Ricci-)curvature homogeneous if and only if S is a symmetric space. In that case, S is two-point homogeneous and (T_1S, g_S) is homogeneous.*

A final result deals with the case of product manifolds where at least one of the factors has a Codazzi Ricci tensor (i.e., $(\nabla_X\rho)(Y, Z) = (\nabla_Y\rho)(X, Z)$ for all vectors X, Y and Z). In particular, it settles the case of reducible symmetric manifolds.

Proposition 3.10. *Let (M, g) be locally isometric to the Riemannian product of (M_1, g_1) and (M_2, g_2) and suppose that the Ricci tensor ρ_1 of (M_1, g_1) is a Codazzi tensor. If (T_1M, g_S) is (Ricci-)curvature homogeneous, then (M, g) is flat.*

The case of irreducible symmetric spaces of rank greater than one is as yet undecided. (The symmetric spaces of rank one are two-point homogeneous.)

The proofs of these propositions consist typically in comparing the matrices for $\bar{\rho}$ at different points $(x, u) \in T_1M$ in the same fiber, using the formulas (2), and requiring that they have the same characteristic polynomial.

To the authors' knowledge, this is the state of things at the present time. A logical next step would be to express that (locally) homogeneous spaces are curvature homogeneous and use the constancy of the curvature

invariants. However, a quick glimpse at the formulas (1) for the Riemann curvature $\bar{R}$ of (T_1M, g_S) shows that this is a very complicated task indeed.

4. The natural contact metric structure on the unit tangent sphere bundle

Apart from the naturally defined metric g_S on T_1M , there exists a contact metric structure on T_1M which also only depends on the metric g on the base manifold M . (We refer to [6] for the basic concepts of contact geometry.) In order to define it, we consider first the almost complex structure J on TM given by

$$JX^h = X^v, \quad JX^v = -X^h$$

for vector fields X on M . For this structure, we have

Theorem 4.1 ([26]). *The tangent bundle (TM, g_S, J) is almost Kählerian. It is a Kähler manifold only when (M, g) is flat.*

Using the almost complex structure J on TM , we define a unit vector field ξ' , a one-form η' and a $(1, 1)$ -tensor field φ' on T_1M by

$$\xi' = -JN, \quad JX = \varphi'X + \eta'(X)N.$$

It is easily checked that $(T_1M, \xi', \eta', \varphi', g_S)$ is an almost contact metric manifold. However, $g_S(X, \varphi'Y) = 2d\eta'(X, Y)$, so $(\xi', \eta', \varphi', g_S)$ is not a contact metric structure. This defect can be rectified by taking

$$\xi = 2\xi', \quad 2\eta = \eta', \quad \varphi = \varphi', \quad 4\bar{g} = g_S.$$

Explicitly, the structure tensors ξ , η and φ are given by

$$\begin{aligned} \xi &= 2u^h, \\ \eta(X^t) &= 0, \quad \eta(X^h) = \frac{1}{2}g(X, u), \\ \varphi X^t &= -\bar{X}^h, \quad \varphi X^h = X^t \end{aligned}$$

for vector fields X on M .

Note that the metric $\bar{g}$ is obtained from the one induced from the Sasaki metric g_S on TM by a homothetic change. In particular, the Riemann curvature tensor in its $(1, 3)$ -form and the Ricci tensor of $(T_1M, \bar{g})$ are the same as those of (T_1M, g_S) , while the scalar curvature functions for both metric spaces differ by a factor 4.

The integral curves of the characteristic vector field ξ are geodesics. In what follows, we refer to them as *characteristic curves*. Note also that ξ' describes the *geodesic flow* on the unit tangent sphere bundle (see [4]).

Two other symmetric operators play an important role in contact geometry, namely $h = (1/2)\mathcal{L}_\xi\varphi$ and $\ell = R(\cdot, \xi)\xi$ where $\mathcal{L}$ denotes Lie differentiation. For the contact structure of the unit tangent sphere bundle, these are given explicitly by

$$\begin{aligned}
 hX^t &= X^t - (R_u X)^t, \\
 hX^h &= -\bar{X}^h + (R_u X)^h, \\
 \ell X^t &= (R_u^2 X)^t + 2(R'_u X)^h, \\
 \ell X^h &= 4(R_u X)^h - 3(R_u^2 X)^h + 2(R'_u X)^t
 \end{aligned}
 \tag{7}$$

for vector fields X on M . Here, $R_u = R(\cdot, u)u$ is the Jacobi operator for the unit vector u and $R'_u = (\nabla_u R)(\cdot, u)u$.

It is well-known that a contact metric manifold is a K -contact manifold, i.e., ξ is a Killing vector field, if and only if $h = 0$. From the above expressions for the operator h we then easily deduce the following standard result:

Theorem 4.2 ([54]). *The natural contact metric structure on T_1M is K -contact if and only if (M, g) has constant curvature 1, in which case the structure on T_1M is Sasakian.*

5. Contact homogeneous unit tangent sphere bundles

We start with a strengthening of Theorem 3.1.

Theorem 5.1. *If (M, g) is a two-point homogeneous space, then its unit tangent sphere bundle $(T_1M, \xi, \eta, \varphi, \bar{g})$ is a homogeneous contact metric Riemannian manifold, i.e., the automorphisms of the contact metric structure $(\xi, \eta, \varphi, \bar{g})$ act transitively on T_1M .*

PROOF. It suffices to show that the lift $\tilde{f}$ of an isometry f of (M, g) preserves the characteristic vector field $\xi = 2u^h$. Let $f : M \rightarrow M$ map a unit vector u at x to a unit vector v at y . The vector ξ at (x, u) is tangent to the curve $\tilde{\gamma}(t) = (\gamma(t), \dot{\gamma}(t))$ where $\gamma(t)$ is the geodesic through $x = \gamma(0)$ tangent to $2u$. The isometry f maps the geodesic γ to the geodesic $f \circ \gamma$ through y tangent to $2v$. Hence, $\tilde{f}_*\xi_{(x,u)} = \tilde{f}_*2u^h = 2v^h = \xi_{(y,v)}$. $\square$

In [14], the authors have proved the converse of this theorem under various conditions on the base manifold (M, g) , though not for a general base space. That problem is still open. A first approach leads via Osserman spaces; in a second approach we study the isometries of the unit tangent sphere bundle in some more detail.

Recall that a Riemannian manifold (M, g) is *pointwise Osserman* if the eigenvalues of the Jacobi operator R_u only depend on x and not on the choice of unit vector u at x . It is *globally Osserman* if the eigenvalues do not depend on x either, but are global constants. The *Osserman conjecture* claims that a globally Osserman space is locally isometric to a two-point homogeneous space. Q.-S. CHI ([25]) has shown this conjecture to hold in dimensions $n = 2m+1$, $n = 4m+2$ and $n = 4$. It also holds when (M, g) is locally reducible (in which case (M, g) is flat), or locally symmetric, or a $\mathfrak{P}$ -space. (In a $\mathfrak{P}$ -space, the eigenspaces of the Jacobi operator are parallel along each geodesic, see [2].) Moreover, in [27], it is shown that it holds for homogeneous manifolds with negative curvature. Finally, in [25] it is proved that the conjecture also holds when (M, g, J) is a Kähler manifold with non-positive or non-negative sectional curvature. To prove the conjecture, one still needs to consider the case of $4k$ -dimensional spaces for $k > 1$. But up to now, this remains an open problem. We refer to [1], [3] and [40] for more details and further information.

Using the expression (7) for the operator h on T_1M , we see that Osserman spaces can be easily recognized from their unit tangent sphere bundle.

Proposition 5.2. *Let (M, g) be an n -dimensional Riemannian manifold. Then (M, g) is a pointwise (resp. globally) Osserman space if and only if the eigenvalues of h on T_1M are constant on the fibers (resp. are global constants).*

Clearly, for a homogeneous contact metric manifold, the eigenvalues of h are constant. Combining this with results on Osserman spaces, we get some first results concerning the converse of Theorem 5.1.

Theorem 5.3. *Let (M, g) be a Riemannian manifold of dimension $n = 2m+1$, $n = 4m+2$ or $n = 4$, or a $\mathfrak{P}$ -space. Then (M, g) is locally isometric to a two-point homogeneous space if and only if $(T_1M, \xi, \eta, \varphi, \bar{g})$ is a locally homogeneous contact metric manifold.*

Proposition 5.4. *Let (M, g) be locally reducible. Then $(T_1M, \xi, \eta, \varphi, \bar{g})$ is a locally homogeneous contact metric space if and only if (M, g) is locally flat.*

Finally, the work on the Osserman conjecture leads to a characterization of two-point homogeneous spaces using the fundamental tensors h and ℓ on the unit tangent sphere bundle.

Theorem 5.5. *A Riemannian manifold (M, g) is locally isometric to a two-point homogeneous space if and only if on its unit tangent sphere bundle we have both*

- (1) *the eigenvalues of the operator h are constant along the fibers;*
- (2) *the operator ℓ sends vertical vectors into vertical vectors (or equivalently, horizontal vectors into horizontal vectors).*

Note that the second condition, by the expression (7), is equivalent to (M, g) being locally symmetric.

In a second approach we find necessary conditions for an isometry of the unit tangent sphere bundle to be the lift of an isometry on the base manifold.

Proposition 5.6. *A (local) isometry $F : T_1M \rightarrow T_1M$ is the lift of a (local) isometry $f : M \rightarrow M$ if and only if*

- (a) *F maps fibers into fibers; and*
- (b) *F preserves the geodesic flow, i.e., $F_*\xi = \xi$.*

Remark 2. An isometry of (T_1M, g) need not satisfy these two conditions. If we consider the unit tangent sphere bundle of a two-dimensional sphere of radius 1, then the rotation over a right angle around a characteristic curve preserves the geodesic flow, but not the fibers. On the other hand, for $T_1\mathbb{R}^n = \mathbb{R}^n \times S^{n-1}$, an isometry of the second factor gives rise to an isometry of $T_1\mathbb{R}^n$ which preserves fibers, but not the geodesic flow.

With this characterization we can now prove

Theorem 5.7. *Let (M, g) be a Riemannian manifold and suppose that $(T_1M, \xi, \eta, \varphi, \bar{g})$ is a (locally) homogeneous contact metric manifold. If one of the following conditions holds:*

- (1) *the Ricci tensor of (M, g) is a Codazzi tensor and non-positive, or*
 - (2) *the sectional curvature K of (M, g) satisfies either $K > 1$ or $K < 1$,*
- then (M, g) is (locally) isometric to a two-point homogeneous space.*

PROOF. Consider a (local) isometry $F : T_1M \rightarrow T_1M$ mapping (p, v) to (q, w) and preserving the geodesic flow. We will show further on that each of the curvature conditions above implies that F preserves fibers. But then the previous proposition gives a (local) isometry $f : M \rightarrow M$ such that $F(x, u) = (f(x), f_*u)$. In particular, $f(p) = q$ and $f_*v = w$. So, (M, g) is (locally) isometric to a two-point homogeneous space.

It remains to show that any isometry of (T_1M, g) preserves the fibers under each of the additional conditions stated above. F. PODESTÀ proves in [47]: *if (M, g) is a Riemannian manifold whose Ricci tensor is parallel and non-positive, then every isometry of $(T_1M, \bar{g})$ preserves the horizontal and vertical distribution, and in particular the fibers.* His proof goes through unaltered when the Ricci tensor is only a Codazzi tensor and non-positive. (See the formulas (2) for the Ricci tensor of $(T_1M, \bar{g})$.) This takes care of the first case.

For the second case, we consider the eigenvalues of h on T_1M . As $(T_1M, \xi, \eta, \varphi, \bar{g})$ is a locally homogeneous contact metric space, the eigenvalues of h are global constants and by Proposition 5.2 (M, g) is a global Osserman space. Let $\lambda_1, \dots, \lambda_{n-1}$ be the eigenvalues of (any) Jacobi operator R_u for a unit vector u . Then, from (7), h has eigenvalues $\lambda_1 - 1, \dots, \lambda_{n-1} - 1$ on $H(T_1M) \cap \xi^\perp$ and eigenvalues $1 - \lambda_1, \dots, 1 - \lambda_{n-1}$ on $V(T_1M)$.

Now consider an isometry F of T_1M preserving ξ . Then F_* maps eigenspaces of h into eigenspaces of h with the same eigenvalue. In particular, if for all $i, j \in \{1, \dots, n-1\}$ it holds $\lambda_i - 1 \neq 1 - \lambda_j$, i.e., $\lambda_i + \lambda_j \neq 2$, then F_* will map both $V(T_1M)$ and $H(T_1M)$ into itself. So, F preserves fibers. The conditions $\lambda_i + \lambda_j \neq 2$ are obviously satisfied when all sectional curvatures are either strictly smaller or strictly greater than 1. $\square$

6. Characteristic reflections on unit tangent sphere bundles

In this section, we review some results about symmetry properties of unit tangent sphere bundles. We start with some observations about Sasakian manifolds.

Local symmetry is a very strong condition for the class of K -contact or Sasakian manifolds. Indeed, such spaces must have constant curvature equal to 1 ([45], [53]). For this reason, T. TAKAHASHI introduced the weaker notion of a *(locally) φ -symmetric space* in the context of Sasakian geometry ([52]): a Sasakian space is locally φ -symmetric if it satisfies the curvature condition

$$(8) \quad g((\nabla_X R)(Y, Z)V, W) = 0$$

for all vector fields X, Y, Z, V and W orthogonal to the characteristic vector field ξ . Takahashi proves that this condition is equivalent to having *characteristic reflections* (i.e., reflections with respect to the integral curves of ξ) which are local automorphisms of the Sasakian structure. In [10], the authors prove that the isometry property of the reflections is already sufficient. (For the case of K -contact spaces, see [22].)

An analogous situation presents itself for unit tangent sphere bundles, which in general are only contact metric spaces. Also here, local symmetry is a very restrictive property.

Theorem 6.1 ([7], [16]). *The unit tangent sphere bundle $(T_1M, \bar{g})$ of a Riemannian manifold (M, g) is locally symmetric if and only if (M, g) is flat or is a surface of constant curvature 1.*

(In [16], this theorem is obtained as a consequence of the analogous result for the case of a parallel Ricci tensor on $(T_1M, \bar{g})$. The proof uses only metric information. The proof of the above theorem in [7] on the other hand uses the contact metric structure in an essential way.)

For this reason, the present authors generalized the notion of a locally φ -symmetric space to the class of contact metric manifolds.

Definition 6.2. A contact metric manifold $(M, \xi, \eta, \varphi, g)$ will be called a *locally φ -symmetric space* if and only if all characteristic reflections are (local) isometries.

We note that in [9], the authors define a contact metric space to be locally φ -symmetric if it satisfies the curvature condition (8). This notion is strictly weaker than ours: explicit examples of three-dimensional homogeneous contact metric spaces satisfying (8) but for which the characteristic reflections are not isometric are presented in [13].

Definition 6.2 gives rise to an infinite number of conditions on the Riemann curvature tensor R and its covariant derivatives.

Theorem 6.3. *Let $(M, \xi, \eta, \varphi, g)$ be a contact metric manifold. If it is a locally φ -symmetric space, then the following hold:*

- 1) $g((\nabla_{\bar{X} \dots \bar{X}}^{2k} R)(X, Y)X, \xi) = 0,$
- 2) $g((\nabla_{\bar{X} \dots \bar{X}}^{2k+1} R)(X, Y)X, Z) = 0,$
- 3) $g((\nabla_{\bar{X} \dots \bar{X}}^{2k+1} R)(X, \xi)X, \xi) = 0$

for all vectors X, Y and Z orthogonal to ξ and $k = 0, 1, 2, \dots$. Moreover, if (M, g) is analytic, these conditions are also sufficient for the contact metric manifold to be a locally φ -symmetric space.

With this criterion, we can show

Theorem 6.4. *The unit tangent sphere bundle $(T_1M, \xi, \eta, \varphi, \bar{g})$ is locally φ -symmetric if and only if (M, g) has constant curvature.*

PROOF. A complete proof can be found in [15]. Here we only outline the major steps. We use the explicit expressions for the curvature tensor $\bar{R}$ of $(T_1M, \bar{g})$ in terms of the curvature tensor R of (M, g) and its covariant derivatives (see (1)).

If we suppose that $(T_1M, \xi, \eta, \varphi, \bar{g})$ is locally φ -symmetric, it follows already from the condition $\bar{g}(\bar{R}(\bar{X}, \bar{Y})\bar{X}, \xi) = 0$ for $\bar{X}$ and $\bar{Y}$ orthogonal to ξ that (M, g) has constant curvature (via Cartan's criterion, see [23]).

Conversely, for a space of constant curvature, the tangent unit sphere bundle is analytic and the expressions (1) simplify considerably. By an induction argument, we then show that the infinite list of curvature conditions holds. $\square$

Recently, another subclass of the contact metric spaces has attracted quite some attention. The (κ, μ) -spaces were introduced in [8] as contact

metric manifolds for which the characteristic vector field ξ belongs to the (κ, μ) -nullity distribution for some real numbers κ and μ , i.e.,

$$(9) \quad R(X, Y)\xi = \kappa (\eta(Y)X - \eta(X)Y) + \mu (\eta(Y)hX - \eta(X)hY)$$

for all vector fields X and Y . Sasakian spaces satisfy this condition for $\kappa = 1$ and μ arbitrary. For unit tangent sphere bundles, we have

Proposition 6.5 ([8]). *A unit tangent sphere bundle $(T_1M, \xi, \eta, \varphi, \bar{g})$ is a (κ, μ) -space if and only if (M, g) is a space of constant curvature c . In that case $\kappa = c(2 - c)$ and $\mu = -2c$.*

Moreover, several of the properties of unit tangent sphere bundles of spaces of constant curvature also hold for the broader class of (κ, μ) -spaces.

Theorem 6.6 ([11]). *A non-Sasakian (κ, μ) -space is a locally homogeneous contact metric space.*

Theorem 6.7 ([11]). *A non-Sasakian (κ, μ) -space is locally φ -symmetric.*

The first author has recently succeeded in fully classifying the non-Sasakian contact metric (κ, μ) -spaces in [12]. The unit tangent sphere bundles of spaces of constant curvature feature prominently in this classification, together with specific Lie groups equipped with a left-invariant contact metric structure.

7. Unit vector fields on the unit tangent sphere bundle

On the unit tangent sphere bundle of any Riemannian manifold, there is a distinguished vector field, the geodesic flow vector field $\xi' = u^h$. If we consider the Sasaki metric g_S on T_1M , ξ' is unit and its integral curves are geodesics. In this section, we comment on some special properties of ξ' .

For that purpose, we first need to introduce some additional concepts. Consider a unit vector field U on a Riemannian manifold (M, g) . U can be regarded as the immersion $U : M \rightarrow T_1M : x \mapsto U_x$ of M into its unit tangent sphere bundle. As the pull-back metric U^*g_S is given by

$$(U^*g_S)(X, Y) = g(X, Y) + g(\nabla_X U, \nabla_Y U),$$

the mapping U is an isometry if and only if the vector field U is parallel.

If the manifold M is compact and orientable, we can define the *energy* of U as the energy of the corresponding map ([63]) and the *volume* of U as the volume of the immersion. If we define operators A_U and L_U as

$$A_U X = -\nabla_X U, \quad L_U X = X + A_U^t(A_U X),$$

then the energy $E(U)$ and the volume $\text{Vol}(U)$ are given by

$$E(U) = \frac{1}{2} \int_M \text{tr } L_U \, d\text{vol} = \frac{n}{2} \text{Vol}(M) + \frac{1}{2} \int_M |\nabla U|^2 \, d\text{vol},$$

$$\text{Vol}(U) = \int_M \sqrt{\det L_U} \, d\text{vol}.$$

Note that $E(U)$ is equal, up to constants, to the quantity $\int_M |\nabla U|^2 \, d\text{vol}$, which is known as the *total bending* of U ([61]).

In this way, two functionals are defined on the space $\mathfrak{X}^1(M)$ of unit vector fields on (M, g) , which we suppose to be non-empty. In analogy with the notions of harmonic maps and minimal immersions, we define

Definition 7.1. A unit vector field which is critical for the energy functional E is called a *harmonic vector field*; one which is critical for the volume functional Vol is called a *minimal vector field*.

The critical point conditions for the functionals E and Vol have been derived in [61] and [30], respectively. For the energy (or total bending), we have: U is a harmonic unit vector field on (M, g) if and only if the one-form ν_U , given as

$$(10) \quad \nu_U(X) = \text{tr}(Z \mapsto (\nabla_Z A_U^t)X),$$

vanishes on $U^\perp$. It is worth pointing out that a *harmonic vector field need not correspond to a harmonic map*, i.e., a unit vector field which is critical for the energy functional on vector fields is not necessarily critical for the energy functional on *all* maps from (M, g) to $(T_1 M, g_S)$ ([29]). U determines a *harmonic map* from (M, g) to $(T_1 M, g_S)$ if and only if it is a harmonic vector field and in addition the one-form $\tilde{\nu}_U$, given as

$$(11) \quad \tilde{\nu}_U(X) = \text{tr}(Z \mapsto R(A_U Z, U)X),$$

vanishes for *all* vectors X .

To describe the critical point condition for the volume functional, we first define the function f_U on M by $f_U = (\det L_U)^{1/2}$ and the operator K_U by $K_U = -f_U L_U^{-1} \circ A_U^t$. Then U is a minimal vector field on M if and only if the one-form ω_U given as

$$(12) \quad \omega_U(X) = \text{tr}(Z \mapsto (\nabla_Z K_U)X)$$

vanishes on $U^\perp$. Here we mention that *a minimal vector field does correspond to a minimal submanifold of the unit tangent sphere bundle* (T_1M, g_S) , i.e., minimal vector fields are also critical points for the volume functional on the larger space of *all* immersions of M into T_1M ([30]).

Clearly the three critical conditions above make sense also if M is non-orientable or non-compact. Therefore, the definitions of harmonic and minimal vector fields have been extended to include those unit vector fields on possibly non-compact or non-orientable manifolds which satisfy the respective critical point conditions.

For more information and further references on minimal and harmonic vector fields, we refer to [29]–[32], [37], [46], [48], [61] and [63]. We specifically mention that the Hopf vector fields on odd-dimensional spheres S^{2n+1} , $n \geq 1$, are minimal, but they do not have minimal volume except for the Hopf vector fields on S^3 . Recently, whole families of new minimal and harmonic vector fields have been found. See [19], [20], [28], [33]–[35], [55]–[57]. Here, we state the major results of [20] about minimal and harmonic vector fields on the unit tangent sphere bundle.

The geodesic flow vector field ξ' is the primary candidate on the unit tangent sphere bundle (T_1M, g_S) . For this vector field, the associated operator $A_{\xi'}$ can be calculated easily:

$$A_{\xi'}X^h = \frac{1}{2}(R_uX)^t, \quad A_{\xi'}X^t = -\bar{X}^h + \frac{1}{2}(R_uX)^h,$$

where, as before, R_u denotes the Jacobi operator associated to the unit vector u . We can then explicitly compute the critical conditions for minimality and harmonicity and use these to show

Theorem 7.2. *Let (M, g) be a two-point homogeneous space. Then the geodesic flow vector field ξ' on the unit tangent sphere bundle (T_1M, g_S) is both minimal and harmonic and determines a harmonic map.*

An essential ingredient of the proof is the following curvature property of two-point homogeneous spaces: if u and v are orthogonal unit vectors

such that $R_u v = \lambda v$, then also $R_v u = \lambda u$. (See, e.g., [25].) In view of the expressions above for $A_{\xi'}$, it is not surprising that properties of the Jacobi operator and earlier work on the Osserman conjecture are useful here.

For low dimensions, we can prove the converse.

Proposition 7.3. *Suppose that the geodesic flow vector field ξ' on $(T_1 M, g_S)$ is either harmonic or minimal or determines a harmonic map. If the dimension of M equals two or three, then (M, g) has constant curvature.*

The proof goes by explicit calculation, which for the three-dimensional case is rather tedious.

Theorem 7.2 and Proposition 7.3 naturally lead to the (as yet unanswered) question: *are the two-point homogeneous spaces the only Riemannian manifolds for which the geodesic flow vector field on the unit tangent sphere bundle is minimal or harmonic, or determines a harmonic map?*

For general manifolds (M, g) , the geodesic flow vector field ξ' is (up to sign) the only naturally distinguished unit vector field on $(T_1 M, g_S)$. When (M, g) has more structure, this may give rise to additional naturally defined vector fields on $T_1 M$. If (M, g, J) is an almost Hermitian manifold for instance, we can consider the vector fields $\xi_1 = (Ju)^t$ and $\xi_2 = (Ju)^h$. For these, we have

Theorem 7.4. *On the unit tangent sphere bundle $(T_1 M, g_S)$ of a complex space form (M, g, J) , the unit vector fields $\xi_1 = (Ju)^t$ and $\xi_2 = (Ju)^h$ are both harmonic and minimal and determine harmonic maps.*

Actually, we have a one-dimensional family of special horizontal unit vector fields.

Theorem 7.5. *On the unit tangent sphere bundle $(T_1 M, g_S)$ of a complex space form (M, g, J) , for every $\alpha \in \mathbb{R}$ the unit vector field $\xi_\alpha = \cos \alpha u^h + \sin \alpha (Ju)^h$ is both harmonic and minimal and it determines a harmonic map.*

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