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Binomial coefficients in arithmetic progressions

By CS. RAKACZKI (Debrecen)

Dedicated to Professor Kálmán Győry on his 60th birthday

Abstract. In this paper we consider arithmetic progressions of polynomial values and binomial coefficients. Under certain conditions we obtain effective finiteness results concerning such progressions. Moreover, in some special cases we determine all the arithmetic progressions in question.

1. Introduction

In this paper we consider three-term arithmetic progressions of binomial coefficients and polynomial values. More precisely, we consider arithmetic progressions of the form

$$f(x)+g(x), \binom{y}{m}, \binom{x}{n}, \tag{1}$$

where m, n are given positive integers, $f(x) \in \mathbb{Q}[x]$ is an integer-valued polynomial with $\deg f(x) \leq n-1$, and $g(x) \in \mathbb{Z}[x]$. Our purpose is to investigate those integers $x \geq n$, $y \geq m$, for which the corresponding values in (1) form an arithmetic progression in some order.

For $m \in \{2,4\}$ we will provide (cf. Theorem 1) an effective upper bound for these solutions x, y of (1). For these values of m the study of such

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arithmetic progressions can be reduced to the investigation of equations of the form

$$u^2 = a\binom{x}{n} + b\,(f(x) + g(x)) + c \text{ in integers } x, u, \tag{2}$$

where a, b, c are given integers. This is a special hyperelliptic equation. In 1993 Pintér [12] showed that under certain conditions equation (2) has only finitely many solutions which can be effectively determined. Further, in the special case when the polynomial $f(x) + g(x) = k$ is constant, a result of Ping Zhi [11] implies the same assertion. In the latter case, we have to examine the equation

$$2\binom{x}{n} = \binom{y}{m} + k. \tag{3}$$

A recent theorem of Beukers, Shorey and Tijdeman [3] implies that apart from some particular cases (e.g. $m = n = 2$) this equation has only finitely many solutions when $k = 0$. Further, in the case $(n, m, k) = (4, 4, 0)$, Cohn [5] gave all the solutions.

An important equation of similar type is

$$\binom{x}{n} = \binom{y}{m} \text{ in integers } x, y \text{ with } x \geq n,\ y \geq m \tag{4}$$

where $n, m \geq 2$. There are many effective and numerical results concerning this equation. In 1988 Kiss [9] proved that if p is a given odd prime, then the equation

$$\binom{x}{p} = \binom{y}{2}$$

has only finitely many positive integer solutions which can be effectively determined. In 1991 Brindza [4] using Baker's method proved that for any integer n with $n \geq 3$, the hyperelliptic equation

$$\binom{x}{n} = \binom{y}{2}$$

has only finitely many solutions. There are also several numerical re-

sults concerning the equation (4). For $(n,m) = (3,2)$ Avanesov [1], for $(n,m) = (2,4)$ de Weger [21] and independently Pintér [13], for $(n,m) = (3,4)$ de Weger [22], for $(n,m) = (6,2)$ and $(n,m) = (6,4)$ Stroeker and de Weger [19] and independently Hajdu and Pintér [8], for $(n,m) = (3,6)$, $(n,m) = (2,8)$ and $(n,m) = (4,8)$ Stroeker and de Weger [19] determined all the integer solutions of equation (4).

In our Theorem 2 we give all the integer solutions x, y of equation (3) in the case when $0 \le k \le 10$ and

$$(n,m) \in \{(2,3),(3,2),(3,4),(4,3),(2,6),(6,2),(4,6),(6,4)\}.$$

For each pair (n,m) in the above set we can transform equation (3) to an elliptic equation of the form

$$u^2 = v^3 + rv + s \text{ in integers } u, v, \tag{5}$$

where r, s are given integers depending on n, m and k. In 1994 Gebel, Pethő and Zimmer [17], and independetly Stroeker and Tzanakis [17] worked out an efficient algorithm for the computation of all solutions of a concrete elliptic equation. This algorithm was implemented in the program package Simath [16]. We used this program package to solve our transformed elliptic equations (5), and hence equation (3), too.

2. Results

Our first result is an effective finiteness theorem concerning arithmetic progressions, whose terms are binomial coefficients and polynomial values. The main tool of the proof will be a result of Pintér [12], which ultimately relies on Baker's method.

Theorem 1. *Let $n \ge 5$ be an integer and $m \in \{2,4\}$. Further, let $f(x)$ be an integer-valued polynomial with $\deg f(x) \le n-1$, and let $g(x) \in \mathbb{Z}[x]$. Then there exists an effectively computable constant C depending only on n and the polynomials $f(x)$ and $g(x)$ such that if for the integers x, y with $x \ge n$, $y \ge m$ the numbers*

$$f(x) + g(x), \ \binom{y}{m}, \ \binom{x}{n}$$

in some order form an arithmetic progression, then

$$\max\{x, y\} \le C. \tag{6}$$

In particular, our theorem applies to the case when $f(x) = \binom{x}{k}$ with $1 \le k \le n-1$ and $g(x) \equiv 0$. Then we get that if $n \ge 5$, $m \in \{2, 4\}$ and $\binom{x}{k}$, $\binom{y}{m}$, $\binom{x}{n}$ form an arithmetic progression, so (6) follows with an effective constant C depending only on n.

We note that for $n = m = 2$ our Theorem 1 does not remain valid. In this case 0, $\binom{y}{2}$ and $\binom{x}{2}$ form an arithmetic progression, which is equivalent to the Pell equation $2(2y-1)^2-(2x-1)^2 = 1$ and this equation has infinity many solutions in integers $x, y \ge 2$. We do not know whether Theorem 1 is true or not for $n = 3$ and 4.

Now we give the complete list of solutions of equation (3), corresponding to some fixed values of m, n and k. The following table contains the examined equations. All these equations turn out to be elliptic equations after suitable substitutions. We also provide the transformations which were used.

	equation	transformed elliptic equation	transformations
(3.1)	$2\binom{x}{2} = \binom{y}{3} + k$	$u^2 = v^3 - 36v + 324(4k+1)$	$u = 36x - 18$, $v = 6y - 6$
(3.2)	$2\binom{x}{3} = \binom{y}{2} + k$	$u^2 = v^3 - 36v - 81(8k-1)$	$u = 18y - 9$, $v = 6x - 6$
(3.3)	$2\binom{x}{3} = \binom{y}{4} + k$	$u^2 = v^3 - 64v - 64(24k+1)$	$u = 2(2y-3)^2 - 10$, $v = 8x - 8$
(3.4)	$2\binom{x}{4} = \binom{y}{3} + k$	$u^2 = v^3 - 64v + 256(12k+1)$	$u = 4(2x-3)^2 - 20$, $v = 8y - 8$
(3.5)	$2\binom{x}{2} = \binom{y}{6} + k$	$u^2 = v^3 - 302400v + 4320000(972k+235)$	$u = 64800x - 32400$, $v = 45(2y-5)^2 - 525$
(3.6)	$2\binom{x}{6} = \binom{y}{2} + k$	$u^2 = v^3 - 302400v - 1080000(1944k-211)$	$u = 32400y - 16200$, $v = 45(2x-5)^2 - 525$
(3.7)	$2\binom{x}{4} = \binom{y}{6} + k$	$u^2 = v^3 - 33600v + 160000(972k+73)$	$u = 900(2x-3)^2 - 4500$, $v = 15(2y-5) - 175$
(3.8)	$2\binom{x}{6} = \binom{y}{4} + k$	$u^2 = v^3 - 33600v - 40000(1944k-49)$	$u = 450(2y-3)^2 - 2250$, $v = 15(2x-5)^2 - 175$

Table 1.

Theorem 2. *Let k be an integer with $0 \le k \le 10$. Then all the solutions $(x, y) \in \mathbb{N}^2$, $x \ge n$, $y \ge m$, of the eight equations occurring in the first column of Table 1 are listed in the following Tables 2–9. (We indicate only those values of k for which there is at least one solution.)*

(3.1)	$2\binom{x}{2} = \binom{y}{3} + k$
k	(x, y)
0	$(85, 36)$; $(5, 6)$; $(8, 8)$; $(1190, 205)$
1	$(2, 3)$; $(970, 179)$
2	$(158, 54)$; $(167, 56)$; $(25482929, 157357)$; $(4, 5)$; $(37, 21)$; $(3, 4)$; $(1234, 210)$
5	$(3, 3)$
6	$(743, 150)$; $(758, 152)$; $(2530912508, 3374701)$; $(61, 29)$; $(10, 9)$
7	$(7, 7)$; $(5209, 547)$; $(22, 15)$
8	$(4, 4)$
10	$(195, 62)$; $(71, 32)$; $(5, 5)$; $(6, 6)$; $(360311, 9202)$; $(5866, 592)$

Table 2.

(3.2)	$2\binom{x}{3} = \binom{y}{2} + k$
k	(x, y)
1	$(3, 2)$
2	$(20, 68)$; $(4, 4)$
4	$(6, 9)$; $(7, 12)$; $(590, 11672)$
5	$(4, 3)$; $(90, 686)$; $(5, 6)$; $(12, 30)$; $(11, 26)$; $(166, 1731)$
7	$(4, 2)$; $(15, 43)$; $(8, 15)$
9	$(10, 22)$
10	$(5, 5)$

Table 3.

(3.3)	$2\binom{x}{3} = \binom{y}{4} + k$
k	(x, y)
0	$(7, 8);\ (11, 11)$
1	$(3, 4)$
3	$(4, 5)$
5	$(5, 6);\ (6, 7)$
7	$(4, 4)$

Table 4.

(3.4)	$2\binom{x}{4} = \binom{y}{3} + k$
k	(x, y)
0	$(5, 5)$
1	$(4, 3)$
6	$(5, 4)$
9	$(5, 3)$
10	$(6, 6)$

Table 5.

(3.5)	$2\binom{x}{2} = \binom{y}{6} + k$
k	(x, y)
0	$(15, 10);\ (22, 11)$
1	$(2, 6)$
2	$(90, 16);\ (6, 8)$
5	$(3, 6);\ (4, 7)$
6	$(10, 9);\ (31, 12);\ (42, 13)$

Table 6.

(3.6)	$2\binom{x}{6} = \binom{y}{2} + k$
k	(x, y)
0	$(18, 273)$
1	$(6, 2);\ (8, 11)$
4	$(7, 5)$
8	$(7, 4)$

Table 7.

(3.7)	$2\binom{x}{4} = \binom{y}{6} + k$
k	(x, y)
1	$(4, 6)$
2	$(6, 8)$
3	$(5, 7)$
9	$(5, 6)$

Table 8.

(3.8)	$2\binom{x}{6} = \binom{y}{4} + k$
k	(x, y)
1	$(6, 4)$
9	$(7, 5)$

Table 9.

3. Proofs

To prove Theorem 1 we need the following lemmas.

Lemma 1. *Let $n \geq 5$ be an integer. Then there is a prime number p which satisfies*

$$n \geq p \geq \frac{n+3}{2}.$$

PROOF. As usual, let $\pi(x)$ denote the number of primes not exceeding x. Then by a Theorem of ROSSER and SCHOENFELD [15]

$$\frac{3x}{5\log x} < \pi(2x) - \pi(x) \quad \text{for } x \geq 20.5.$$

This implies our assertion for $20.5 \leq \frac{n+3}{2}$. A simple direct computation shows that the statement of Lemma 1 is also true for $\frac{n+3}{2} < 20.5$. □

The next lemma is a slightly modified version of a result of PINTÉR [12].

Lemma 2. *Let $n \geq 5$ be an integer, $\tilde{f}(x) \in \mathbb{Q}[x]$ an integer-valued polynomial with $\deg \tilde{f}(x) \leq n-1$ and $\tilde{g}(x) \in \mathbb{Z}[x]$. Let a be an integer for which there exists a prime p with*

$$n \geq p \geq \frac{n+3}{2}$$

such that $(a, p) = 1$. Then the polynomial

$$F(x) = a\binom{x}{n} + \tilde{f}(x) + \tilde{g}(x)$$

has at least three simple zeros.

PROOF. To prove the statement one can repeat Pintér's argument with a slight modification. However, for convenience of the reader we give the full proof here.

Put $f_i(x) = x(x-1)\cdots(x-i+1)$ for $i = 1, \ldots, n$ and $f_0(x) = 1$. As $\tilde{f}(x)$ is an integer-valued polynomial,

$$\tilde{f}(x) = a_{n-1}\binom{x}{n-1} + \ldots + a_1\binom{x}{1} + a_0$$

(cf. [14]), where the coefficients $a_{n-1}, \ldots, a_1, a_0$ are rational integers. We get

$$n!\,F(x) = af_n(x) + \cdots + a_p n(n-1)\cdots(p+1)f_p(x) + \cdots \\ \cdots + n!\,a_0 + n!\,\tilde{g}(x) \in \mathbb{Z}[x].$$

For $S(x) \in \mathbb{Z}[x]$, we denote by $(S(x))_p$ the image of $S(x)$ in $\mathbb{Z}_p[x]$ under the canonical homomorphism $\mathbb{Z} \to \mathbb{Z}_p$. There is a $h(x) \in \mathbb{Z}[x]$ such that

$$(n!F(x))_p = (f_p(x))_p(h(x))_p$$

and $\deg(h(x))_p = n - p$. Since all the zeros of $(f_p(x))_p$ are simple, the polynomial $(n!F(x))_p$ as well as the polynomial $n!F(x)$ has at least $p - (n-p) = 2p - n \geq 3$ simple zeros. □

The next lemma is a famous result of Baker [2], which provides a bound for the solutions of superelliptic equations.

Lemma 3. *Let $f(x) \in \mathbb{Z}[x]$ be a polynomial having at least three simple zeros and let b be an integer. Then all the integer solutions x, y of the equation*

$$f(x) = by^2$$

satisfy

$$\max\{|x|, |y|\} \leq C,$$

where C is an effectively computable constant depending only on b and f.

Pproof of Theorem 1. Suppose that for the integers x, y with $x \geq n$, $y \geq m$, the numbers

$$f(x) + g(x), \ \binom{y}{m}, \ \binom{x}{n}$$

in some order form an arithmetic progression. It is easy to verify that in this case (x, y) is a solution to one of the following equations:

(a) $u^2 = 4\binom{x}{n} + 4f(x) + 4g(x) + 1$ where $u = 2y - 1$

(b) $u^2 = 192\binom{x}{n} + 192f(x) + 192g(x) + 16$ where $u = (2y-3)^2 - 5$

(c) $u^2 = 16\binom{x}{n} - 8f(x) - 8g(x) + 1$ where $u = 2y - 1$

(d) $u^2 = 768\binom{x}{n} - 384f(x) - 384g(x) + 16$ where $u = (2y-3)^2 - 5$

(e) $u^2 = -8\binom{x}{n} + 16f(x) + 16g(x) + 1 \qquad$ where $u = 2y - 1$

(f) $u^2 = -384\binom{x}{n} + 768f(x) + 768g(x) + 16 \qquad$ where $u = (2y-3)^2 - 5$.

Each of the above equations is of the form

$$u^2 = a\binom{x}{n} + b\,(f(x) + g(x)) + c, \tag{2}$$

where a, b, c are integers, and a is of the form $2^t 3^s$ with some $t, s \geq 0$. Thus by Lemma 1 there exists a prime number p satisfying

$$n \geq p \geq \frac{n+3}{2} \quad \text{and } p \nmid a.$$

Applying now Lemma 2 and Lemma 3 to equations (a) to (f), we immediately obtain our statement. □

To prove Theorem 2, we need some further notation. Let E be an elliptic curve defined by

$$E : y^2 = x^3 + ax + b = p(x) \qquad (a, b \in \mathbb{Z}). \tag{7}$$

By a famous theorem of Mordell [10] we know that the group of the rational points on E is finitely generated. We denote by $E(\mathbb{Q})$ the group of the the rational points on E. Let r denote the rank, g the number of torsion points of E. For any point P of E denote by $\hat{h}(P)$ the canonical height of P. $\hat{h}$ is a positive definite quadratic form; denote by λ its smallest eigenvalue. Let Δ_0 denote the discriminant of E then

$$\Delta_0 = 4a^3 + 27b^2.$$

Let $P_1, \ldots, P_r \in E(\mathbb{Q})$ denote a basis of the Mordell–Weil group of E, and let u_i be the elliptic logarithm of P_i for $i = 1, \ldots, r$. We put $u_i' = gu/\omega_1$ where ω_1 is the real period of E. Each rational point $P \in E(\mathbb{Q})$ has a unique representation of the form

$$P = \sum_{i=1}^{r} n_i P_i + P_{r+1} \qquad (n_i \in \mathbb{Z}),$$

where P_{r+1} is some torsion point. Put

$$N = \max_{1\le i\le r}\{|n_i|\}.$$

The following result of Pethő, Zimmer, Gebel and Herrmann [6] gives an upper bound for the value N, if P is an integral point of $E(\mathbb{Q})$.

Lemma 4. *Let the elliptic curve E be defined by equation (7). Assume that the integral point $P=(x,y)\in E(\mathbb{Q})$ has the representation*

$$P=\sum_{i=1}^{r} n_iP_i + P_{r+1}$$

where P_{r+1} is some torsion point. For $N=\max\limits_{1\le i\le r}\{|n_i|\}$ we have

$$N\le N_0=\sqrt{(k_1/2+k_2)/\lambda}$$

with

$$k_2=\log\max\Big\{|2a|^{\frac12},|4b|^{\frac13}\Big\},$$

$$k_1=5\times 10^{64}k_3\log(k_3(k_3+\log k_4))$$

where

$$k_3=\frac{32}{3}\sqrt{|\Delta_0|}\Big(8+\frac12\log|\Delta_0|\Big)^4,$$

$$k_4=10^4\max\Big\{16a^2,256\sqrt{|\Delta_0|}^3\Big\}.$$

Moreover

$$\left|\sum_{i=1}^{r} n_iu_i^{'}+n_{r+1}\right|\le k_5\exp\{-\lambda N^2+k_2\}$$

with $k_5=2g/(3\omega_1)$.

We note that the above upper bound for N in general is too large for computing all integral points on a concrete elliptic curve. Thus in practice some reduction method is also needed.

Proof of Theorem 2. We detail the proof of Theorem 2 only for the equation $2\binom{x}{3}=\binom{y}{2}$. The other equations can be solved in a similar manner. In this case, according to Table 1, we have to examine the elliptic curve

$$E:=\big\{(u,v)\mid u^2=v^3-36v+81\big\}\cup\{\mathcal{O}\}.$$

In the sequel we determine some parameters of $E(\mathbb{Q})$ using Simath [16]. The rank of $E(\mathbb{Q})$ is $r=1$, the discriminant $\Delta_0=-9477$. $E(\mathbb{Q})$ has two

torsion points: $\mathcal{O}$ and $P_2 = (3, 0)$. Hence $g = 2$. We obtain

$$P_1 = (6, 9),\ \hat{h}(P_1) = 0.1801176633\ldots\ .$$

The single positive eigenvalue of $\hat{h}$ is

$$\lambda = 0.1801176633\ldots\ .$$

The real period is

$$\omega_1 = 1.6179172250\ldots\ .$$

The elliptic logarithm of the point P_1 is $u_1 = 0.2793301565\ldots$. Hence we have

$$k_2 \le 2.14, \qquad k_3 \le 2.6 \times 10^7, \qquad k_4 \le 2.37 \times 10^{12}$$

and so

$$k_1 \le 4.44 \times 10^{73}.$$

Thus applying Lemma 4, we obtain

$$N \le 1.1 \times 10^{37}.$$

Now using a result of DE WEGER (see [20], Lemma 3.2), this initial bound can be reduced. After the first reduction we obtain the new upper bound $N \le 80$. After repeating the reduction, we get $N \le 45$. A third reduction step leads to the same upper bound for N. Thus we have to test for integrality all the points

$$P = n_1 P_1 + n_2 P_2, \quad |n_1| \le 45,\ n_2 \in \{0, 1\}.$$

After the above computations we obtain all the integral points on E:

$$(v, \pm u) = (3, 0),\ (6, 9),\ (0, 9),\ (4, 1),\ (-6, 9),\ (15, 54),\ (28, 145).$$

Hence from Table 1 we get that the equation $2\binom{x}{3} = \binom{y}{2}$ has no solution with $x \ge 3, y \ge 2$. □

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CS. RAKACZKI
INSTITUTE OF MATHEMATICS AND INFORMATICS
UNIVERSITY OF DEBRECEN
4010 DEBRECEN, P.O. BOX 12
HUNGARY

E-mail: rcsaba@math.klte.hu

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