

An existence theorem for the commutative neutrix product of distributions

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Abstract. In this paper we prove that the commutative neutrix product of the distributions x_+^{-r} and x_+^{-s} exists for $r, s = 1, 2, \dots$

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The distribution x_+^{-r} is defined by the equation

$$x_+^{-r} = \frac{(-1)^{r-1}(\ln x_+)^{(r)}}{(r-1)!}$$

for $r = 1, 2, \dots$ and not as in GEL'FAND and SHILOV [6]. If we denote GEL'FAND and SHILOV's definition of x_+^{-r} by $F(x_+, -r)$, it was proved in [4] that

$$x_+^{-r} = F(x_+, -r) + \frac{(-1)^r \phi(r-1)}{(r-1)!} \delta^{(r-1)}(x)$$

for $r = 1, 2, \dots$, where

$$\phi(r) = \begin{cases} \sum_{i=1}^r 1/i, & r \geq 1, \\ 0, & r = 0. \end{cases}$$

Mathematics Subject Classification: 46F10.

Key words and phrases: distribution, delta-function, neutrix, neutrix limit, neutrix product.

Our definition of x_+^{-r} is more convenient to use because it satisfies the equation

$$(x_+^{-r})' = -rx_+^{-r-1}$$

for $r = 1, 2, \dots$

Further, the distribution $x_+^{-1} \ln x_+$ is defined by

$$x_+^{-1} \ln x_+ = \frac{1}{2}(\ln^2 x_+)'$$

and in general, the distribution $x_+^{-r} \ln x_+$ is defined inductively by the equation

$$x_+^{-r} \ln x_+ = \frac{x_+^{-r} - (x_+^{-r+1} \ln x_+)'}{r-1}$$

for $r = 2, 3, \dots$

Now let $\rho(x)$ be a function in $\mathcal{D}$ having the following properties:

- (i) $\rho(x) = 0$ for $|x| \geq 1$,
- (ii) $\rho(x) \geq 0$,
- (iii) $\rho(x) = \rho(-x)$,
- (iv) $\int_{-1}^1 \rho(x) dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

If now f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \dots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition for the commutative neutrix product of two distributions was given in [3].

Definition 1. Let f and g be distributions in $\mathcal{D}'$ and let $f_n(x) = (f * \delta_n)(x)$, $g_n(x) = (g * \delta_n)(x)$. We say that the neutrix product $f \square g$ of f and g exists and is equal to the distribution h on the interval (a, b) if

$$\text{N-}\lim_{n \rightarrow \infty} \langle f_n(x)g_n(x), \varphi(x) \rangle = \langle h(x), \varphi(x) \rangle$$

for all functions φ in $\mathcal{D}$ with support contained in the interval (a, b) , where N is the neutrix, see van der CORPUT [1], having domain $N' = \{1, 2, \dots, n, \dots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \quad \lambda > 0, \quad r = 1, 2, \dots$$

and all functions which converge to zero in the normal sense as n tends to infinity. Further, if

$$\lim_{n \rightarrow \infty} \langle f_n(x) g_n(x), \varphi(x) \rangle = \langle h(x), \varphi(x) \rangle$$

we simply say that the product $f.g$ exists and equals h , see [2].

Before proving our main result, we note the following lemmas which are easily proved by induction.

Lemma 1. *If φ is an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$, then*

$$(1) \quad \langle x_+^{-r}, \varphi(x) \rangle = \int_0^1 x^{-r} \left[\varphi(x) - \sum_{i=0}^{r-1} \frac{x^i}{i!} \varphi^{(i)}(0) \right] dx \\ - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!(r-i-1)} - \frac{\phi(r-1)}{(r-1)!} \varphi^{(r-1)}(0),$$

for $r = 1, 2, \dots$.

Lemma 2.

$$(2) \quad \int_{-1}^1 v^i \rho^{(r)}(v) dv = \begin{cases} 0, & 0 \leq i < r, \\ (-1)^r r!, & i = r \end{cases}$$

for $r = 0, 1, 2, \dots$.

The following theorem was proved in [5].

Theorem 1. *The neutrix product $x^{-r} \square x^{-s}$ exists and*

$$x^{-r} \square x^{-s} = x^{-r-s}$$

for $r, s = 1, 2, \dots$.

The limits involved in the proof of Theorem 1 were easily evaluated. However, in the following, we are going to consider the neutrix product

$x_+^{-r} \square x_+^{-s}$. For this neutrix product, the limits are more complicated and so we only prove the existence of the limits and thus the existence of the neutrix product $x_+^{-r} \square x_+^{-s}$.

We now prove the following theorem.

Theorem 2. *The neutrix product $x_+^{-r} \square x_+^{-s}$ exists for $r, s = 1, 2, \dots$.*

PROOF. We first of all consider the case $s = 1$ and put

$$(x_+^{-r})_n = x_+^{-r} * \delta_n(x) = \frac{(-1)^{r-1}}{(r-1)!} \int_{-1/n}^{1/n} \ln(x-t)_+ \delta_n^{(r)}(t) dt,$$

for $r = 1, 2, \dots$. Then

$$\begin{aligned} (3) \quad & (-1)^{r-1} (r-1)! \int_{-1}^1 (x_+^{-r})_n (x_+^{-1})_n x^k dx \\ &= \int_{-1/n}^{1/n} \delta_n^{(r)}(t) \int_t^{1/n} \delta_n'(s) \int_s^{1/n} x^k \ln(x-t) \ln(x-s) dx ds dt \\ &\quad + \int_{-1/n}^{1/n} \delta_n^{(r)}(t) \int_{-1/n}^t \delta_n'(s) \int_t^{1/n} x^k \ln(x-t) \ln(x-s) dx ds dt \\ &\quad + \int_{-1/n}^{1/n} \delta_n^{(r)}(t) \int_{-1/n}^{1/n} \delta_n'(s) \int_{1/n}^1 x^k \ln(x-t) \ln(x-s) dx ds dt \\ &= n^{r-k} \int_{-1}^1 \rho^{(r)}(v) \int_v^1 \rho'(u) \\ &\quad \times \int_u^1 w^k \ln[(w-v)/n] \ln[(w-u)/n] dw du dv \\ &\quad + n^{r-k} \int_{-1}^1 \rho^{(r)}(v) \int_{-1}^v \rho'(u) \\ &\quad \times \int_v^1 w^k \ln[(w-v)/n] \ln[(w-u)/n] dw du dv \\ &\quad + n^{r-k} \int_{-1}^1 \rho^{(r)}(v) \int_{-1}^1 \rho'(u) \\ &\quad \times \int_1^n w^k \ln[(w-v)/n] \ln[(w-u)/n] dw du dv = I_1 + I_2 + I_3, \end{aligned}$$

where the substitutions $ns = u$, $nt = v$ and $nx = w$ have been made.

It follows immediately that

$$(4) \quad \text{N-lim}_{n \rightarrow \infty} I_1 = \text{N-lim}_{n \rightarrow \infty} I_2 = 0,$$

for $k = 0, 1, 2, \dots, r-1$.

Now

$$(5) \quad \begin{aligned} & \int_1^n w^k \ln[(w-v)/n] \ln[(w-u)/n] dw \\ &= \int_1^n w^k [\ln(w-v) - \ln n] [\ln(w-u) - \ln n] dw \\ &= \ln^2 n \int_1^n w^k dw - 2 \ln n \int_1^n w^k \ln(w-v) dw \\ & \quad + \int_1^n w^k \ln(w-v) \ln(w-u) dw \end{aligned}$$

and it follows immediately that

$$(6) \quad \text{N-lim}_{n \rightarrow \infty} n^{r-k} \ln^2 n \int_1^n w^k dw = 0$$

for $k = 0, 1, 2, \dots$.

Further, by expanding $\ln(w-v)$ in powers of v/w , it also follows that

$$(7) \quad \text{N-lim}_{n \rightarrow \infty} n^{r-k} \ln n \int_1^n w^k \ln(w-v) dw = 0$$

for $k = 0, 1, 2, \dots$.

Finally, we have

$$(8) \quad \begin{aligned} & \int_1^n w^k \ln(w-v) \ln(w-u) dw \\ &= \int_1^n w^k \left[\ln w - \sum_{i=1}^{\infty} \frac{v^i}{i w^i} \right] \left[\ln w - \sum_{j=1}^{\infty} \frac{u^j}{j w^j} \right] dw \\ &= \int_1^n w^k \ln^2 w dw - 2 \sum_{i=1}^{\infty} \frac{v^i}{i} \int_1^n w^{k-i} \ln w dw \\ & \quad + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{v^i u^j}{i j} \int_1^n w^{k-i-j} dw \end{aligned}$$

and it follows that

$$\begin{aligned} \text{N-lim}_{n \rightarrow \infty} n^{r-k} \int_1^n w^k \ln(w-v) \ln(w-u) dw \\ = - \sum_{j=1}^r \frac{v^{r-j+1} u^j}{j(r-k)(r-j+1)}. \end{aligned}$$

Hence

$$\begin{aligned} \text{N-lim}_{n \rightarrow \infty} n^{r-k} \int_{-1}^1 \rho^{(r)}(v) \int_{-1}^1 \rho'(u) \int_1^n w^k \ln(w-v) \ln(w-u) dw du dv \\ (9) \quad = \frac{(-1)^r (r-1)!}{r-k}, \end{aligned}$$

for $k = 0, 1, 2, \dots, r-1$ on using equation (2). It follows from equations (5), (6), (7) and (9) that

$$(10) \quad \text{N-lim}_{n \rightarrow \infty} I_3 = \frac{(-1)^r (r-1)!}{r-k}.$$

It now follows from equations (3), (4) and (10) that

$$(11) \quad \text{N-lim}_{n \rightarrow \infty} \int_{-1}^1 (x_+^{-r})_n (x_+^{-1})_n x^k dx = -\frac{1}{r-k},$$

for $k = 0, 1, 2, \dots, r-1$.

We now deal with the case $k = r$. Equation (3) still holds but this time it follows that

$$(12) \quad \text{N-lim}_{n \rightarrow \infty} I_1 = \int_{-1}^1 \rho^{(r)}(v) \int_v^1 \rho'(u) \int_u^1 w^r \ln |(w-v)| \\ \times \ln |(w-u)| dw du dv,$$

$$(13) \quad \text{N-lim}_{n \rightarrow \infty} I_2 = \int_{-1}^1 \rho^{(r)}(v) \int_{-1}^v \rho'(u) \int_v^1 w^r \ln |(w-v)| \\ \times \ln |(w-u)| dw du dv.$$

Further, equation (8) is replaced by the equation

$$\begin{aligned} \int_1^n w^r \ln(w-v) \ln(w-u) dw &= \int_1^n w^r \ln^2 w dw \\ -2 \sum_{i=1}^{\infty} \frac{v^i}{i} \int_1^n w^{r-i} \ln w dw &+ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{v^i u^j}{ij} \int_1^n w^{r-i-j} dw. \end{aligned}$$

It follows that

$$\text{N-lim}_{n \rightarrow \infty} \int_1^n w^r \ln(w-v) \ln(w-u) dw = g_r(u, v),$$

say and so

$$(14) \quad \text{N-lim}_{n \rightarrow \infty} I_3 = \int_{-1}^1 \rho^{(r)}(v) \int_{-1}^1 \rho'(u) g_r(u, v) du dv.$$

We therefore see from equations (3), (12), (13) and (14) that

$$\text{N-lim}_{n \rightarrow \infty} \int_{-1}^1 (x_+^{-r})_n (x_+^{-1})_n x^r dx$$

exists and we put

$$(15) \quad \text{N-lim}_{n \rightarrow \infty} \int_{-1}^1 (x_+^{-r})_n (x_+^{-1})_n x^r dx = L_{r,1}.$$

When $k = r + 1$, it follows as for equation (3) that for any continuous function ψ

$$\begin{aligned} &\int_{-1/n}^{1/n} (x_+^{-r})_n (x_+^{-1})_n x^{r+1} \psi(x) dx \\ &= n^{-1} \int_{-1}^1 \rho^{(r)}(v) \int_v^1 \rho'(u) \int_u^1 w^{r+1} \psi(w/n) \ln |(w-v)/n| \\ &\quad \times \ln |(w-u)/n| dw du dv \\ &+ n^{-1} \int_{-1}^1 \rho^{(r)}(v) \int_{-1}^v \rho'(u) \int_u^1 w^{r+1} \psi(w/n) \ln |(w-v)/n| \\ &\quad \times \ln |(w-u)/n| dw du dv \end{aligned}$$

and it follows that

$$(16) \quad \lim_{n \rightarrow \infty} \int_{-1/n}^{1/n} (x_+^{-r})_n (x_+^{-1})_n x^{r+1} \psi(x) dx = 0.$$

Next, when $x \geq 1/n$, we have

$$\begin{aligned} (-1)^{r-1} (r-1)! (x_+^{-r})_n &= \int_{-1/n}^{1/n} \ln |x-t| \delta_n^{(r)}(t) dt \\ &= n^r \int_{-1}^1 \ln |x-v/n| \rho^{(r)}(v) dv \\ &= n^r \int_{-1}^1 \left[\ln x - \sum_{i=1}^{\infty} \frac{v^i}{i n^i x^i} \right] \rho^{(r)}(v) dv \\ &= - \sum_{i=r}^{\infty} \int_{-1}^1 \frac{v^i}{i n^{i-r} x^i} \rho^{(r)}(v) dv. \end{aligned}$$

It follows that

$$|(r-1)! (x_+^{-r})_n| \leq \sum_{i=r}^{\infty} \int_{-1}^1 \frac{|v|^i}{i n^{i-r} x^i} |\rho^{(r)}(v)| dv \leq \sum_{i=r}^{\infty} \frac{K_r}{i n^{i-r} x^i},$$

where

$$K_r = \int_{-1}^1 |\rho^{(r)}(v)| dv$$

for $r = 1, 2, \dots$.

If now $n^{-1} < \eta < 1$, then

$$\begin{aligned} (r-1)! \int_{1/n}^{\eta} |(x_+^{-r})_n (x_+^{-1})_n x^{r+1}| dx \\ \leq K_1 K_r \sum_{i=r}^{\infty} \sum_{j=1}^{\infty} \int_{1/n}^{\eta} \frac{n^{r-i-j+1} x^{r-i-j-1}}{ij} dx \\ = \frac{K_1 K_r}{n} \sum_{i=r}^{\infty} \sum_{j=1}^{\infty} \int_1^{n\eta} \frac{w^{r-i-j+1}}{ij} dw \end{aligned}$$

$$= \frac{K_1 K_r}{n} \left[\frac{\ln w}{r+1} + \frac{\ln w}{2r} + \sum_{i=r}^{\infty} \sum_{\substack{j=1 \\ i+j \neq r+2}}^{\infty} \frac{w^{r-i-j+2}}{ij(r-i-j+2)} \right]_1^{n\eta}$$

and it follows that

$$\lim_{n \rightarrow \infty} \int_{1/n}^{\eta} |(x_+^{-r})_n (x_+^{-1})_n x^{m+r}| dx \leq \frac{K_1 K_r \eta}{r!}$$

for $r = 1, 2, \dots$

Thus, if ψ is a continuous function

$$(17) \quad \lim_{n \rightarrow \infty} \left| \int_{1/n}^{\eta} (x_+^{-r})_n (x_+^{-1})_n x^{r+1} \psi(x) dx \right| = O(\eta)$$

for $r = 1, 2, \dots$

Now let φ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$. By Taylor's Theorem we have

$$\varphi(x) = \sum_{k=0}^r \frac{x^k \varphi^{(k)}(0)}{k!} + \frac{x^{r+1} \varphi^{(r+1)}(\xi x)}{(r+1)!},$$

where $0 < \xi < 1$. Thus

$$\begin{aligned} \langle (x_+^{-r})_n (x_+^{-1})_n, \varphi(x) \rangle &= \int_{-1}^1 (x_+^{-r})_n (x_+^{-1})_n \varphi(x) dx \\ &= \sum_{k=0}^r \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^1 (x_+^{-r})_n (x_+^{-1})_n x^k dx \\ &\quad + \int_{-1/n}^{1/n} \frac{(x_+^{-r})_n (x_+^{-1})_n x^{r+1} \varphi^{(r+1)}(\xi x)}{(r+1)!} dx \\ &\quad + \int_{1/n}^{\eta} \frac{(x_+^{-r})_n (x_+^{-1})_n x^{r+1} \varphi^{(r+1)}(\xi x)}{(r+1)!} dx \\ &\quad + \int_{\eta}^1 \frac{(x_+^{-r})_n (x_+^{-1})_n x^{r+1} \varphi^{(r+1)}(\xi x)}{(r+1)!} dx. \end{aligned}$$

On using the equations (11), (15), (16) and (17) and noting that the sequence $\{(x_+^{-r})_n (x_+^{-1})_n\}$ converges uniformly to x^{-r-1} on the interval

$[\eta, 1]$, it follows that

$$\begin{aligned} & \text{N-lim}_{n \rightarrow \infty} \langle (x_+^{-r})_n (x_+^{-1})_n, \varphi(x) \rangle \\ &= - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{(r-k)k!} + L_{r,1} \frac{\varphi^{(r)}(0)}{r!} + \int_{\eta}^1 \frac{\varphi^{(r+1)}(\xi x)}{(r+1)!} dx + O(\eta), \end{aligned}$$

but since η can be made arbitrarily small, it follows that

$$\begin{aligned} & \text{N-lim}_{n \rightarrow \infty} \langle (x_+^{-r})_n (x_+^{-1})_n, \varphi(x) \rangle \\ &= - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{(r-k)k!} + L_{r,1} \frac{\varphi^{(r)}(0)}{r!} + \int_0^1 \frac{\varphi^{(r+1)}(\xi x)}{(r+1)!} dx \\ &= \int_0^1 x^{-r-1} \left[\varphi(x) - \sum_{k=0}^r \frac{x^k \varphi^{(k)}(0)}{k!} \right] dx - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{(r-k)k!} + L_{r,1} \frac{\varphi^{(r)}(0)}{r!} \\ &= \langle x_+^{-r-1}, \varphi(x) \rangle + \frac{(-1)^r}{r!} [L_{r,1} + \phi(r)] \langle \delta^{(r)}(x), \varphi(x) \rangle \end{aligned}$$

on using equation (1).

The neutrix product $x_+^{-r} \square x_+^{-1}$ therefore exists and

$$x_+^{-r} \square x_+^{-1} = x_+^{-r-1} + \frac{(-1)^r}{r!} [L_{r,1} + \phi(r)] \delta^{(r)}(x)$$

on the interval $[-1, 1]$. However, the product $x_+^{-r} \cdot x_+^{-1}$ obviously exists on any interval not containing the origin, and so the neutrix product $x_+^{-r} \square x_+^{-1}$ exists on the real line for $r = 1, 2, \dots$

Suppose now that $x_+^{-r} \square x_+^{-s}$ exists and is of the form

$$x_+^{-r} \square x_+^{-s} = x_+^{-r-s} + a_{r,s} \delta^{(r+s-1)}(x)$$

for $r = 1, 2, \dots$ and for some positive integer s . Then the derivative of $x_+^{-r} \square x_+^{-s}$ exists, and

$$\begin{aligned} (x_+^{-r} \square x_+^{-s})' &= -(r+s) x_+^{-r-s-1} + a_{r,s} \delta^{(r+s)}(x) \\ &= -s x_+^{-r} \square x_+^{-s-1} - r x_+^{-r-1} \square x_+^{-s} \\ &= -s x_+^{-r} \square x_+^{-s-1} - r x_+^{-r-s-1} - r a_{r+1,s} \delta^{(r+s)}(x). \end{aligned}$$

The product $x_+^{-r} \square x_+^{-s-1}$ therefore exists and

$$\begin{aligned} x_+^{-r} \square x_+^{-s-1} &= x_+^{-r-s-1} - \frac{ra_{r+1,s} + a_{r,s}}{s} \delta^{(r+s)}(x) \\ &= x_+^{-r-s-1} + a_{r,s+1} \delta^{(r+s)}(x). \end{aligned}$$

It follows by induction that the product $x_+^{-r} \square x_+^{-s}$ exists for $r, s = 1, 2, \dots$

Defining the distribution x_-^{-r} by

$$x_-^{-r} = (-x)_+^{-r}$$

for $r = 1, 2, \dots$, we have □

Corollary 2.1. *The neutrix product $x_-^{-r} \square x_-^{-s}$ exists for $r, s = 1, 2, \dots$*

PROOF. With the above notation we have

$$(18) \quad x_+^{-r} \square x_+^{-s} = x_+^{-r-s} + a_{r,s} \delta^{(r+s-1)}(x).$$

Replacing x in this equation by $-x$, we get

$$(19) \quad x_-^{-r} \square x_-^{-s} = x_-^{-r-s} - (-1)^{r+s} a_{r,s} \delta^{(r+s-1)}(x),$$

proving the existence of neutrix product $x_-^{-r} \square x_-^{-s}$. □

Corollary 2.2.

$$(20) \quad x_+^{-r} \square x_+^{-s} + (-1)^{r+s} x_-^{-r} \square x_-^{-s} = x^{-r-s}$$

for $r, s = 1, 2, \dots$

PROOF. Equation (20) follows immediately from equations (18) and (19). □

Theorem 3. *The neutrix product $x_+^{-r} \square \ln x_+$ exists for $r = 1, 2, \dots$. In particular, the product $x_+^{-1} \cdot \ln x_+$ exists and*

$$(21) \quad x_+^{-1} \cdot \ln x_+ = x_+^{-1} \ln x_+.$$

PROOF. We put

$$(\ln x_+)_n = \ln x_+ * \delta_n(x) = \int_{-1/n}^{1/n} \ln(x-t)_+ \delta_n(t) dt$$

and

$$(x_+^{-1})_n = x_+^{-1} * \delta_n(x) = \int_{-1/n}^{1/n} \ln(x-t)_+ \delta'_n(t) dt.$$

Since $\ln x_+$ and $\ln^2 x_+$ are locally summable functions, it follows that

$$\lim_{n \rightarrow \infty} (\ln x_+)_n^2 = \ln^2 x_+.$$

Thus, for arbitrary φ in $\mathcal{D}$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle [(\ln x_+)_n^2]', \varphi(x) \rangle &= 2 \lim_{n \rightarrow \infty} \langle (\ln x_+)_n (x_+^{-1})_n, \varphi(x) \rangle \\ &= \langle (\ln x_+^2)', \varphi(x) \rangle = 2 \langle x_+^{-1} \ln x_+, \varphi(x) \rangle \end{aligned}$$

and equation (21) follows.

Now suppose that the neutrix product $x_+^{-r} \square \ln x_+$ exists and is of the form

$$x_+^{-r} \square \ln x_+ = x_+^{-r} \ln x_+ + a_{r,0} \delta^{(r-1)}(x)$$

for some positive integer r . Then the derivative of $x_+^{-r} \square \ln x_+$ exists and

$$\begin{aligned} (x_+^{-r} \square \ln x_+)' &= -r x_+^{-r-1} \ln x_+ + x_+^{-r-1} + a_{r,0} \delta^{(r)}(x) \\ &= -r x_+^{-r-1} \square \ln x_+ + x_+^{-r} \square x_+^{-1} \\ &= -r x_+^{-r-1} \square \ln x_+ + x_+^{-r-1} + a_{r,1} \delta^{(r)}(x). \end{aligned}$$

The product $x_+^{-r-1} \square \ln x_+$ therefore exists and

$$\begin{aligned} x_+^{-r-1} \square \ln x_+ &= x_+^{-r-1} \ln x_+ + \frac{a_{r,1} - a_{r,0}}{r} \delta^{(r)}(x) \\ &= x_+^{-r-1} \ln x_+ + a_{r+1,1} \delta^{(r)}(x). \end{aligned}$$

It follows by induction that the product $x_+^{-r} \square \ln x_+$ exists for $r = 1, 2, \dots$

□

Corollary 3.1. *The neutrix product $x_-^{-r} \square \ln x_-$ exists for $r = 1, 2, \dots$. In particular, the product $x_-^{-1} \cdot \ln x_-$ exists and*

$$(22) \quad x_-^{-1} \cdot \ln x_- = x_-^{-1} \ln x_-.$$

PROOF. With the above notation we have

$$(23) \quad x_+^{-r} \square \ln x_+ = x_+^{-r} \ln x_+ + a_{r,0} \delta^{(r-1)}(x).$$

Replacing x in this equation by $-x$, we get

$$(24) \quad x_-^{-r} \square \ln x_- = x_-^{-r} \ln x_- - (-1)^r a_{r,0} \delta^{(r-1)}(x),$$

proving the existence of neutrix product $x_-^{-r} \square \ln x_-$. The particular case $r = 1$ of course reduces to the product

$$x_-^{-1} \cdot \ln x_- = x_-^{-1} \ln x_- \quad \square$$

Corollary 3.2.

$$(25) \quad x_+^{-r} \square \ln x_+ + (-1)^r x_-^{-r} \square \ln x_- = x^{-r} \ln |x|$$

for $r = 1, 2, \dots$

PROOF. Equation (25) follows immediately from equations (23) and (24). $\square$

Acknowledgments. The second author is very grateful to the Turkish Scientific and Research Council (Tübitak) for its support and he also wishes to thank the staff of the Department of Mathematics and Computer Science in Leicester University for their hospitality.

References

- [1] J. G. VAN DER CORPUT, Introduction to the neutrix calculus, *J. Analyse Math.* **7** (1959-60), 291–398.
- [2] B. FISHER, The product of distributions, *Quart. J. Math. Oxford* (2), **22** (1971), 291–298.
- [3] B. FISHER, Neutrices and the product of distributions, *Studia Math.* **57** (1976), 263–274.
- [4] B. FISHER, Some notes on distributions, *Math. Student* **48** (3) (1980), 269–281.

- [5] B. FISHER and J. D. NICHOLAS, Some results on the commutative neutrix product of distributions, *J. Analysis* **6** (1998), 33–41.
- [6] I. M. GEL'FAND and G. E. SHILOV, Generalized Functions, Vol. I, *Academic Press*, 1964.

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(Received August 3, 1998; revised March 4, 2000)