

Differentiable solutions of a polynomial-like iterative equation with variable coefficients

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Abstract. This paper is concerned with a polynomial-like iterative equation with variable coefficients $\sum_{i=1}^n \lambda_i(x) \varphi^{[i]}(x) = F(x)$, where $\varphi^{[i]}(x)$ is the i^{th} iterate of the function $\varphi(x)$. Using the fixed point theorems of Schauder and Banach we discuss the existence, uniqueness and stability of Lip C^1 -solutions of the equation.

1. Introduction

Let $\varphi^{[k]}$ denote the k -th iterate of a function φ , and $\varphi^{[0]}$ the identify function. To find a function φ such that its k -th iterate $\varphi^{[k]}$ is equal to a give function F plays an important role in the theory of dynamical systems [1], [2]. As a natural generalization, the polynomial-like iterative functional equations in the following form

$$(*) \quad \lambda_1 \varphi(x) + \lambda_2 \varphi^{[2]}(x) + \cdots + \lambda_n \varphi^{[n]}(x) = F(x)$$

for $x \in R$, $\lambda_i \in R$, $i = 1, 2, \dots, n$, or some special cases were considered recently [3–10]. In particular, W. ZHANG [6] considered the existence, uniqueness and stability of differentiable solutions of equation (*). However, conditions for the existence of differentiable solutions are not known in the case of variable coefficients. In this paper, we will consider a polynomial-like iterative equation with variable coefficients:

$$(1) \quad \lambda_1(x) \varphi(x) + \lambda_2(x) \varphi^{[2]}(x) + \cdots + \lambda_n(x) \varphi^{[n]}(x) = F(x),$$

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where n is a positive integer greater than or equal to 2. By means of the fixed point theorems of Banach and Schauder, we discuss the existence, uniqueness and stability of Lip C^1 -solutions of equation (1).

We write $\varphi \in C^1$ if φ, φ' are continuous. The set of all C^1 function each of which maps a closed interval I into I will be denoted by $C^1(I, I)$. It is well known that when endowed with the norm $\|\cdot\|_{C^1}$, where

$$\|\varphi\|_{C^1} = \|\varphi\|_{C^0} + \|\varphi'\|_{C^0}, \quad \|\varphi\|_{C^0} = \max_{x \in I} \{|\varphi(x)|\},$$

$C^1(I, I)$ is a Banach space (see also [6]).

We write $\varphi \in \text{Lip } C^1$ if $\varphi \in C^1(I, I)$ and φ' is Lipschitzian. Let $I = [a, b] \subset R$, for given constants $M > 0$, $M^* > 0$, we will denote by $\Omega(M, M^*; I)$ the subset of all $\varphi \in \text{Lip } C^1$ each of which satisfies

$$\begin{aligned} \varphi(a) = a, \quad \varphi(b) = b, \quad 0 \leq \varphi'(x) \leq M, \\ |\varphi'(x_1) - \varphi'(x_2)| \leq M^*|x_1 - x_2|, \quad \forall x, x_1, x_2 \in I. \end{aligned}$$

2. Preparatory lemmas

Our discussion depends on the following several preparatory lemmas the proof of which can be found in [6].

Lemma 1. *Suppose that $\varphi \in \Omega(M, M^*; I)$. Then*

$$(2) \quad |(\varphi^{[i]})'(x_1) - (\varphi^{[i]})'(x_2)| \leq M^* \left(\sum_{j=i-1}^{2i-2} M^j \right) |x_1 - x_2|.$$

Lemma 2. *Suppose that $\varphi_1, \varphi_2 \in \Omega(M, M^*; I)$. Then*

$$(3) \quad \|\varphi_1^{[i]} - \varphi_2^{[i]}\|_{C^0} \leq \left(\sum_{j=1}^i M^{j-1} \right) \|\varphi_1 - \varphi_2\|_{C^0}.$$

Lemma 3. *Suppose that $\varphi_1, \varphi_2 \in \Omega(M, M^*; I)$. Then*

$$(4) \quad \begin{aligned} \|(\varphi_1^{[k+1]})' - (\varphi_2^{[k+1]})'\|_{C^0} &\leq (k+1)M^k \|\varphi_1' - \varphi_2'\|_{C^0} \\ &+ Q(k+1)M^* \left(\sum_{i=1}^k (k-i+1)M^{k+i-1} \right) \|\varphi_1 - \varphi_2\|_{C^0}, \end{aligned}$$

for $k = 0, 1, 2, \dots$, where $Q(s) = 0$ when $s = 1$ and $Q(s) = 1$ when $s = 2, 3, \dots$.

Lemma 4. *If φ_1, φ_2 are homeomorphisms from I onto itself and*

$$|\varphi_i(x_1) - \varphi_i(x_2)| \leq L|x_1 - x_2|, \quad \forall x_1, x_2 \in I,$$

where L is a positive constant and $i = 1, 2$, then

$$(5) \quad \|\varphi_1 - \varphi_2\|_{C^0} \leq L\|\varphi_1^{-1} - \varphi_2^{-1}\|_{C^0}.$$

3. The existence and uniqueness of Lip C^1 -solutions for equation (1)

In this section we give the existence and uniqueness theorems of Lip C^1 -solutions for equation (1).

Theorem 1. *Let $I = [a, b]$, $\lambda_1, \lambda_2, \dots, \lambda_n : I \rightarrow [0, 1]$ be continuous, $\lambda_1(x) \geq \alpha$, $\sum_{i=1}^n \lambda_i(x) = 1$ for all $x \in I$, and*

$$|\lambda_k(x_1) - \lambda_k(x_2)| \leq \beta_k|x_1 - x_2|, \quad \forall x_1, x_2 \in I, \quad k = 1, 2, \dots, n,$$

where $\alpha > (1 - \alpha)(M + \frac{\kappa}{\alpha} \sum_{i=1}^n \beta_i) \sum_{j=0}^{2n-4} M^{j+1}$, β_k ($k = 1, 2, \dots, n$) are positive constants. Suppose that $F \in \Omega(\alpha M, M'; I)$. Then (1) has a solution in $\Omega(M, M^*; I)$. Here

$$M^* \geq \frac{M' + \sum_{i=1}^n \beta_i M^i}{\alpha - (1 - \alpha)(M + \frac{\kappa}{\alpha} \sum_{i=1}^n \beta_i) \sum_{j=0}^{2n-4} M^{j+1}}, \quad \kappa = \max\{|\alpha|, |\beta|\}.$$

PROOF. We will seek of a solution (1) in $\Omega(M, M^*; I)$. To this end, for each $\varphi \in \Omega(M, M^*; I)$, let us define

$$(6) \quad \varphi_x(t) = \sum_{i=1}^n \lambda_i(x) \varphi^{[i-1]}(t), \quad \forall t \in I.$$

It is easy to see that $\varphi_x(a) = a$, $\varphi_x(b) = b$, and $\varphi_x \in C^1(I, I)$. Since

$$(7) \quad \varphi'_x(t) = \sum_{i=1}^n \lambda_i(x) (\varphi^{[i-1]})'(t),$$

$$(8) \quad 0 < \alpha \leq \lambda_1(x) \leq \varphi'_x(t) \leq \sum_{i=1}^n M^{i-1} := K_1.$$

So

$$(9) \quad 0 < \frac{1}{K_1} \leq (\varphi_x^{-1})'(t) = \frac{1}{\varphi'_x(\varphi_x^{-1}(t))} \leq \frac{1}{\alpha}.$$

Thus $\varphi_x : I \rightarrow I$ is a self-diffeomorphism. $\square$

First, we prove the following lemma.

Lemma 5. *Let $\varphi, g, h \in \Omega(M, M^*; I)$, and $x_1, x_2, t_1, t_2, t \in I$. Then*

$$(10) \quad |\varphi'_x(t_1) - \varphi'_x(t_2)| \leq K_2 |t_1 - t_2|,$$

where $K_2 = (1 - \alpha)M^* \sum_{j=0}^{2n-4} M^j$.

$$(11) \quad |(\varphi_x^{-1})'(t_1) - (\varphi_x^{-1})'(t_2)| \leq \frac{K_2}{\alpha^3} |t_1 - t_2|.$$

$$(12) \quad |\varphi'_{x_1}(t) - \varphi'_{x_2}(t)| \leq \left(\sum_{i=1}^n \beta_i M^{i-1} \right) |x_1 - x_2|.$$

$$(13) \quad |\varphi_{x_1}(t) - \varphi_{x_2}(t)| \leq \left(\kappa \sum_{i=1}^n \beta_i \right) |x_1 - x_2|.$$

$$(14) \quad |\varphi_{x_1}^{-1}(t) - \varphi_{x_2}^{-1}(t)| \leq \frac{\kappa}{\alpha} \left(\sum_{i=1}^n \beta_i \right) |x_1 - x_2|.$$

$$(15) \quad |\varphi_{x_1}^{-1}(t_1) - \varphi_{x_2}^{-1}(t_2)| \leq \frac{1}{\alpha} |t_1 - t_2| + \frac{\kappa}{\alpha} \left(\sum_{i=1}^n \beta_i \right) |x_1 - x_2|.$$

$$(16) \quad |(\varphi_{x_1}^{-1})'(t_1) - (\varphi_{x_2}^{-1})'(t_2)| \\ \leq \frac{1}{\alpha^2} \left(\frac{K_2 \kappa}{\alpha} \sum_{i=1}^n \beta_i + \sum_{i=1}^n \beta_i M^{i-1} \right) |x_1 - x_2| + \frac{K_2}{\alpha^3} |t_1 - t_2|.$$

$$(17) \quad |(\varphi_{x_1}^{-1})'(t) - (\varphi_{x_2}^{-1})'(t)| \\ \leq \frac{1}{\alpha^2} \left(\frac{K_2 \kappa}{\alpha} \sum_{i=1}^n \beta_i + \sum_{i=1}^n \beta_i M^{i-1} \right) |x_1 - x_2|.$$

$$(18) \quad \|g_x - h_x\|_{C^0} \leq \left(\sum_{i=2}^n \sum_{j=1}^{i-1} M^{j-1} \right) \|g - h\|_{C^0}.$$

$$(19) \quad \|g'_x - h'_x\|_{C^0} \leq \sum_{i=2}^n (i-1)M^{i-2}\|g' - h'\|_{C^0} \\ + M^* \sum_{i=2}^n \sum_{j=1}^{i-2} Q(i-1)(i-j-1)M^{i+j-3}\|g - h\|_{C^0}.$$

$$(20) \quad \|g_x^{-1} - h_x^{-1}\|_{C^0} \leq \frac{1}{\alpha} \left(\sum_{i=2}^n \sum_{j=1}^{i-1} M^{j-1} \right) \|g - h\|_{C^0}.$$

$$(21) \quad \|(g_x^{-1})' - (h_x^{-1})'\|_{C^0} \\ \leq \left[\frac{K_2}{\alpha^3} \sum_{i=2}^n \sum_{j=1}^{i-1} M^{j-1} + \frac{M^*}{\alpha^2} \sum_{i=2}^n \sum_{j=1}^{i-2} Q(i-1)(i-j-1)M^{i+j-3} \right] \\ \times \|g - h\|_{C^0} + \frac{1}{\alpha^2} \sum_{i=2}^n (i-1)M^{i-2}\|g' - h'\|_{C^0}.$$

PROOF of Lemma 5. By Lemma 1 we see that

$$|\varphi'_x(t_1) - \varphi'_x(t_2)| = \left| \sum_{i=1}^n \lambda_i(x) \left[(\varphi^{[i-1]})'(t_1) - (\varphi^{[i-1]})'(t_2) \right] \right| \\ \leq \sum_{i=2}^n \lambda_i(x) \left(M^* \sum_{j=i-2}^{2i-4} M^j \right) |t_1 - t_2| \leq \sum_{i=2}^n \lambda_i(x) \left(M^* \sum_{j=0}^{2n-4} M^j \right) |t_1 - t_2| \\ = (1 - \lambda_1(x)) M^* \sum_{j=0}^{2n-4} M^j |t_1 - t_2| \leq (1 - \alpha) M^* \sum_{j=0}^{2n-4} M^j |t_1 - t_2| = K_2 |t_1 - t_2|.$$

This proves (10).

From (8)–(10) we have

$$|(\varphi_x^{-1})'(t_1) - (\varphi_x^{-1})'(t_2)| = \left| \frac{1}{\varphi'_x(\varphi_x^{-1}(t_1))} - \frac{1}{\varphi'_x(\varphi_x^{-1}(t_2))} \right| \\ = \left| \frac{\varphi'_x(\varphi_x^{-1}(t_1)) - \varphi'_x(\varphi_x^{-1}(t_2))}{\varphi'_x(\varphi_x^{-1}(t_1))\varphi'_x(\varphi_x^{-1}(t_2))} \right| \leq \frac{K_2}{\alpha^2} |\varphi_x^{-1}(t_1) - \varphi_x^{-1}(t_2)| \leq \frac{K_2}{\alpha^3} |t_1 - t_2|.$$

This proves (11).

From (7) it follows that

$$\begin{aligned} |\varphi'_{x_1}(t) - \varphi'_{x_2}(t)| &= \left| \sum_{i=1}^n (\lambda_i(x_1) - \lambda_i(x_2)) (\varphi^{[i-1]})'(t) \right| \\ &\leq \sum_{i=1}^n |\lambda_i(x_1) - \lambda_i(x_2)| \left| (\varphi^{[i-1]})'(t) \right| \leq \left(\sum_{i=1}^n \beta_i M^{i-1} \right) |x_1 - x_2|. \end{aligned}$$

This proves (12).

By (6) we have

$$\begin{aligned} |\varphi_{x_1}(t) - \varphi_{x_2}(t)| &= \left| \sum_{i=1}^n (\lambda_i(x_1) - \lambda_i(x_2)) \varphi^{[i-1]}(t) \right| \\ &\leq \sum_{i=1}^n |\lambda_i(x_1) - \lambda_i(x_2)| |\varphi^{[i-1]}(t)| \leq \kappa \sum_{i=1}^n \beta_i |x_1 - x_2|. \end{aligned}$$

This proves (13).

(14) follows from

$$\begin{aligned} |\varphi_{x_1}^{-1}(t) - \varphi_{x_2}^{-1}(t)| &= |\varphi_{x_1}^{-1}(\varphi_{x_2}(\varphi_{x_2}^{-1}(t))) - \varphi_{x_1}^{-1}(\varphi_{x_1}(\varphi_{x_2}^{-1}(t)))| \\ &\stackrel{(9)}{\leq} \frac{1}{\alpha} |\varphi_{x_2}(\varphi_{x_2}^{-1}(t)) - \varphi_{x_1}(\varphi_{x_2}^{-1}(t))| \stackrel{(13)}{\leq} \left(\frac{\kappa}{\alpha} \sum_{i=1}^n \beta_i \right) |x_1 - x_2|. \end{aligned}$$

From (9) and (14) we have

$$\begin{aligned} |\varphi_{x_1}^{-1}(t_1) - \varphi_{x_2}^{-1}(t_2)| &\leq |\varphi_{x_1}^{-1}(t_1) - \varphi_{x_1}^{-1}(t_2)| + |\varphi_{x_1}^{-1}(t_2) - \varphi_{x_2}^{-1}(t_2)| \\ &\leq \frac{1}{\alpha} |t_1 - t_2| + \left(\frac{\kappa}{\alpha} \sum_{i=1}^n \beta_i \right) |x_1 - x_2|. \end{aligned}$$

This proves (15).

In view of (9), (10), (12) and (14), (16) follows from

$$\begin{aligned} |(\varphi_{x_1}^{-1})'(t_1) - (\varphi_{x_2}^{-1})'(t_2)| &= \left| \frac{1}{\varphi'_{x_1}(\varphi_{x_1}^{-1}(t_1))} - \frac{1}{\varphi'_{x_2}(\varphi_{x_2}^{-1}(t_2))} \right| \\ &\leq \frac{1}{\alpha^2} |\varphi'_{x_1}(\varphi_{x_1}^{-1}(t_1)) - \varphi'_{x_2}(\varphi_{x_2}^{-1}(t_2))| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\alpha^2} [|\varphi'_{x_1}(\varphi_{x_1}^{-1}(t_1)) - \varphi'_{x_1}(\varphi_{x_2}^{-1}(t_1))| + |\varphi'_{x_1}(\varphi_{x_2}^{-1}(t_1)) - \varphi'_{x_1}(\varphi_{x_2}^{-1}(t_2))| \\
&+ |\varphi'_{x_1}(\varphi_{x_2}^{-1}(t_2)) - \varphi'_{x_2}(\varphi_{x_2}^{-1}(t_2))|] \leq \frac{1}{\alpha^2} \\
&\times \left[K_2 |\varphi_{x_1}^{-1}(t_1) - \varphi_{x_2}^{-1}(t_1)| + K_2 |\varphi_{x_2}^{-1}(t_1) - \varphi_{x_2}^{-1}(t_2)| + \left(\sum_{i=1}^n \beta_i M^{i-1} \right) |x_1 - x_2| \right] \\
&\leq \frac{1}{\alpha^2} \left[\frac{K_2 \kappa}{\alpha} \sum_{i=1}^n \beta_i |x_1 - x_2| + \frac{K_2}{\alpha} |t_1 - t_2| + \left(\sum_{i=1}^n \beta_i M^{i-1} \right) |x_1 - x_2| \right] \\
&= \frac{1}{\alpha^2} \left(\frac{K_2 \kappa}{\alpha} \sum_{i=1}^n \beta_i + \sum_{i=1}^n \beta_i M^{i-1} \right) |x_1 - x_2| + \frac{K_2}{\alpha^3} |t_1 - t_2|.
\end{aligned}$$

(17) follows from

$$\begin{aligned}
|(\varphi_{x_1}^{-1})'(t) - (\varphi_{x_2}^{-1})'(t)| &= \left| \frac{1}{\varphi'_{x_1}(\varphi_{x_1}^{-1}(t))} - \frac{1}{\varphi'_{x_2}(\varphi_{x_2}^{-1}(t))} \right| \\
&\leq \frac{1}{\alpha^2} |\varphi'_{x_1}(\varphi_{x_1}^{-1}(t)) - \varphi'_{x_2}(\varphi_{x_2}^{-1}(t))| \\
&\leq \frac{1}{\alpha^2} [|\varphi'_{x_1}(\varphi_{x_1}^{-1}(t)) - \varphi'_{x_1}(\varphi_{x_2}^{-1}(t))| + |\varphi'_{x_1}(\varphi_{x_2}^{-1}(t)) - \varphi'_{x_2}(\varphi_{x_2}^{-1}(t))|] \\
&\stackrel{(10),(12)}{\leq} \frac{1}{\alpha^2} \left[K_2 |\varphi_{x_1}^{-1}(t) - \varphi_{x_2}^{-1}(t)| + \left(\sum_{i=1}^n \beta_i M^{i-1} \right) |x_1 - x_2| \right] \\
&\stackrel{(14)}{\leq} \frac{1}{\alpha^2} \left[\frac{K_2 \kappa}{\alpha} \sum_{i=1}^n \beta_i |x_1 - x_2| + \sum_{i=1}^n \beta_i M^{i-1} |x_1 - x_2| \right] \\
&= \frac{1}{\alpha^2} \left(\frac{K_2 \kappa}{\alpha} \sum_{i=1}^n \beta_i + \sum_{i=1}^n \beta_i M^{i-1} \right) |x_1 - x_2|.
\end{aligned}$$

(18) follows from

$$\begin{aligned}
\|g_x - h_x\|_{C^0} &= \max_{t \in I} |g_x(t) - h_x(t)| \\
&= \max_{t \in I} \left| \sum_{i=1}^n \lambda_i(x) (g^{[i-1]}(t) - h^{[i-1]}(t)) \right| \leq \sum_{i=1}^n \|g^{[i-1]} - h^{[i-1]}\|_{C^0}
\end{aligned}$$

$$\stackrel{(3)}{\leq} \left(\sum_{i=2}^n \sum_{j=1}^{i-1} M^{j-1} \right) \|g - h\|_{C^0}.$$

(19) follows from

$$\begin{aligned} \|g'_x - h'_x\|_{C^0} &= \max_{t \in I} \left| \sum_{i=1}^n \lambda_i(x) \left((g^{[i-1]})'(t) - (h^{[i-1]})'(t) \right) \right| \\ &\leq \sum_{i=1}^n \lambda_i(x) \left\| (g^{[i-1]})' - (h^{[i-1]})' \right\|_{C^0} \stackrel{(4)}{\leq} \sum_{i=1}^n \left[(i-1) M^{i-2} \|g'_x - h'_x\|_{C^0} \right. \\ &\quad \left. + Q(i-1) M^* \left(\sum_{j=1}^{i-2} (i-j-1) M^{i+j-3} \right) \|g_x - h_x\|_{C^0} \right] \\ &\leq \sum_{i=2}^n (i-1) M^{i-2} \|g'_x - h'_x\|_{C^0} \\ &\quad + M^* \sum_{i=2}^n \sum_{j=1}^{i-2} Q(i-1)(i-j-1) M^{i+j-3} \|g - h\|_{C^0}. \end{aligned}$$

(20) follows from

$$\|g_x^{-1} - h_x^{-1}\|_{C^0} \stackrel{(5)}{\leq} \frac{1}{\alpha} \|g_x - h_x\|_{C^0} \stackrel{(18)}{\leq} \left(\frac{1}{\alpha} \sum_{i=2}^n \sum_{j=1}^{i-1} M^{j-1} \right) \|g - h\|_{C^0}.$$

Finally, by (9), (10), (19) and (20), we have

$$\begin{aligned} \|(g_x^{-1})' - (h_x^{-1})'\|_{C^0} &= \max_{t \in I} \left| \frac{1}{g'_x(g_x^{-1}(t))} - \frac{1}{h'_x(h_x^{-1}(t))} \right| \\ &\leq \frac{1}{\alpha^2} \max_{t \in I} |g'_x(g_x^{-1}(t)) - h'_x(h_x^{-1}(t))| \\ &\leq \frac{1}{\alpha^2} \max_{t \in I} [|g'_x(g_x^{-1}(t)) - g'_x(h_x^{-1}(t))| + |g'_x(h_x^{-1}(t)) - h'_x(h_x^{-1}(t))|] \\ &\leq \frac{1}{\alpha^2} [K_2 \|g_x^{-1} - h_x^{-1}\|_{C^0} + \|g'_x - h'_x\|_{C^0}] \\ &\leq \left(\frac{K_2}{\alpha^3} \sum_{i=2}^n \sum_{j=1}^{i-1} M^{j-1} \right) \|g - h\|_{C^0} + \frac{1}{\alpha^2} \sum_{i=2}^n (i-1) M^{i-2} \|g' - h'\|_{C^0} \end{aligned}$$

$$\begin{aligned}
& + \frac{M^*}{\alpha^2} \sum_{i=2}^n \sum_{j=1}^{i-2} Q(i-1)(i-j-1)M^{i+j-3} \|g-h\|_{C^0} \\
& = \left[\frac{K_2}{\alpha^3} \sum_{i=2}^n \sum_{j=1}^{i-1} M^{j-1} + \frac{M^*}{\alpha^2} \sum_{i=2}^n \sum_{j=1}^{i-2} Q(i-1)(i-j-1)M^{i+j-3} \right] \|g-h\|_{C^0} \\
& + \frac{1}{\alpha^2} \sum_{i=2}^n (i-1)M^{i-2} \|g'-h'\|_{C^0}.
\end{aligned}$$

This proves (21).

Now we continue to prove Theorem 1. Define an operator T from $\Omega(M, M^*; I)$ into $C^1(I, I)$ by

$$(22) \quad (T\varphi)(x) = \varphi_x^{-1}(F(x)), \quad \varphi \in \Omega(M, M^*; I).$$

Clearly $(T\varphi)(a) = a$, $T(\varphi)(b) = b$, $T\varphi \in C^1(I, I)$, and (9) yields that

$$0 \leq \frac{F'(x)}{K_1} \leq (T\varphi)'(x) = (\varphi_x^{-1})'(F(x))F'(x) \leq \frac{1}{\alpha} \cdot \alpha M = M.$$

Furthermore, by (11), (17), we have

$$\begin{aligned}
& |(T\varphi)'(x_1) - (T\varphi)'(x_2)| = |(\varphi_{x_1}^{-1})'(F(x_1))F'(x_1) - (\varphi_{x_2}^{-1})'(F(x_2))F'(x_2)| \\
& \leq |(\varphi_{x_1}^{-1})'(F(x_1)) - (\varphi_{x_1}^{-1})'(F(x_2))|F'(x_1) + |(\varphi_{x_1}^{-1})'(F(x_2)) - (\varphi_{x_2}^{-1})'(F(x_2))|F'(x_2)| \\
& \quad + |(\varphi_{x_1}^{-1})'(F(x_2)) - (\varphi_{x_2}^{-1})'(F(x_2))|F'(x_2)| \\
& \leq \frac{K_2}{\alpha^3} |F(x_1) - F(x_2)| \cdot \alpha M + \frac{1}{\alpha} M' |x_1 - x_2| \\
& \quad + \frac{1}{\alpha^2} \left(\frac{K_2 \kappa}{\alpha} \sum_{i=1}^n \beta_i + \sum_{i=1}^n \beta_i M^{i-1} \right) |x_1 - x_2| \cdot \alpha M \\
& \leq \left[\frac{K_2 M^2}{\alpha} + \frac{M'}{\alpha} + \frac{M}{\alpha} \left(\frac{K_2 \kappa}{\alpha} \sum_{i=1}^n \beta_i + \sum_{i=1}^n \beta_i M^{i-1} \right) \right] |x_1 - x_2| \\
& \leq M^* |x_1 - x_2|,
\end{aligned}$$

so $(T\varphi)(x) \in \Omega(M, M^*; I)$, that is, T is a operator from $\Omega(M, M^*; I)$ into itself.

Now we will show that T is continuous. Let $g, h \in \Omega(M, M^*; I)$, $(Tg)(x) = g_x^{-1}(F(x))$, $(Th)(x) = h_x^{-1}(F(x))$, then

$$\begin{aligned}
(23) \quad & \|Tg - Th\|_{C^1} = \|Tg - Th\|_{C^0} + \|(Tg)' - (Th)'\|_{C^0} \\
& = \max_{x \in I} \{|g_x^{-1}(F(x)) - h_x^{-1}(F(x))|\} \\
& \quad + \max_{x \in I} \{|(g_x^{-1})'(F(x))F'(x) - (h_x^{-1})'(F(x))F'(x)|\} \\
& \leq \|g_x^{-1} - h_x^{-1}\|_{C^0} + \alpha M \|(g_x^{-1})' - (h_x^{-1})'\|_{C^0} \\
& \leq \left(\frac{1}{\alpha} \sum_{i=2}^n \sum_{j=1}^{i-1} M^{j-1} \right) \|g - h\|_{C^0} + \alpha M \\
& \quad \times \left\{ \left[\frac{K_2}{\alpha^3} \sum_{i=2}^n \sum_{j=1}^{i-1} M^{j-1} + \frac{M^*}{\alpha^2} \sum_{i=2}^n \sum_{j=1}^{i-2} Q(i-1)(i-j-1)M^{i+j-3} \right] \right. \\
& \quad \times \|g - h\|_{C^0} + \frac{1}{\alpha^2} \sum_{i=2}^n (i-1)M^{i-2} \|g' - h'\|_{C^0} \Big\} \\
& = \left[\frac{1}{\alpha} \sum_{i=2}^n \sum_{j=1}^{i-1} M^{j-1} + \frac{K_2 M}{\alpha^2} \sum_{i=2}^n \sum_{j=1}^{i-1} M^{j-1} \right. \\
& \quad \left. + \frac{MM^*}{\alpha} \sum_{i=2}^n \sum_{j=1}^{i-2} Q(i-1)(i-j-1)M^{i+j-3} \right] \|g - h\|_{C^0} \\
& \quad + \frac{M}{\alpha} \sum_{i=2}^n (i-1)M^{i-2} \|g' - h'\|_{C^0} \leq \Theta \|g - h\|_{C^1},
\end{aligned}$$

where

$$\begin{aligned}
(24) \quad & \Theta = \max \left\{ \left(\frac{1}{\alpha} + \frac{K_2 M}{\alpha^2} \right) \sum_{i=2}^n \sum_{j=1}^{i-1} M^{j-1} \right. \\
& \quad \left. + \frac{MM^*}{\alpha} \sum_{i=2}^n \sum_{j=1}^{i-2} Q(i-1)(i-j-1)M^{i+j-3}; \frac{M}{\alpha} \sum_{i=2}^n (i-1)M^{i-2} \right\}.
\end{aligned}$$

This shows that T is continuous.

It is easy to see that $\Omega(M, M^*; I)$ is closed and convex. We now show that $\Omega(M, M^*; I)$ is a relatively compact subset of $C^1(I, I)$. For any

$\varphi = \varphi(x)$ in $\Omega(M, M^*; I)$,

$$\|\varphi\|_{C^1} = \|\varphi\|_{C^0} + \|\varphi'\|_{C^0} \leq \kappa + M.$$

Hence $\Omega(M, M^*; I)$ is bounded in $C^1(I, I)$. Next, for any $\varphi = \varphi(x)$ in $\Omega(M, M^*; I)$ and any $x_1, x_2 \in I$, we have

$$|\varphi(x_1) - \varphi(x_2)| \leq M|x_1 - x_2|.$$

This shows that $\Omega(M, M^*; I)$ is equicontinuous on I . By means of the Arzela–Ascoli theorem, we see that $\Omega(M, M^*; I)$ is relatively compact in $C^1(I, I)$. By Schauder’s fixed point theorem we assert that there is a function $\varphi \in \Omega(M, M^*; I)$ such that

$$\varphi(x) = (T\varphi)(x) = \varphi_x^{-1}(F(x))$$

or

$$\varphi_x(\varphi(x)) = F(x),$$

that is, φ is a solution of equation (1) in $\Omega(M, M^*; I)$. This completes the proof. $\square$

Theorem 2. *Under the hypotheses of Theorem 1, (1) has a unique solution in $\Omega(M, M^*; I)$ if $\Theta < 1$ in (24).*

PROOF. Since $\Theta < 1$, we see that T defined by (22) is contraction mapping on the closed subset $\Omega(M, M^*; I)$ of $C^1(I, I)$. Thus the fixed point φ in the proof of Theorem 1 must be unique. This completes the proof. $\square$

Remark 1. A referee of this paper proposes the following interesting question: Whether it is possible that (under the weaker assumptions) of Theorem 1 there exist indeed more than one differential solution of (1) in the sense that T has at least two fixed points. We do not know how to solve this question.

4. The stability of $\text{Lip } C^1$ -solutions for equation (1)

In this section we consider the problem of the continuous dependence of $\text{Lip } C^1$ -solutions of equation (1) on the given functions. We have the following

Theorem 3. *The unique solution obtained in Theorem 2 depends continuously on the given functions F and λ_i ($i = 1, 2, \dots, n$).*

PROOF. Under the assumptions of Theorem 2, if $G = G(x)$ and $H = H(x)$ are any two functions in $\Omega(\alpha M, M'; I; \alpha_j(x)$ and $\mu_j(x)$ ($j = 1, 2, \dots, n$) are any functions which satisfy the same conditions as $\lambda_j(x)$ ($j = 1, 2, \dots, n$) in Theorem 1. Then there correspond two unique functions $g = g(x)$ and $h = h(x)$ in $\Omega(M, M^*; I)$ such that

$$g(x) = g_x^{-1}(G(x))$$

and

$$h(x) = h_x^{-1}(H(x)),$$

where

$$g_x(t) = \sum_{i=1}^n \alpha_i(x) g^{[i-1]}(t),$$

$$h_x(t) = \sum_{i=1}^n \mu_i(x) h^{[i-1]}(t).$$

First of all, it is easy to see that

$$\begin{aligned} |g_x(t) - h_x(t)| &\leq \sum_{i=1}^n |\alpha_i(x) - \mu_i(x)| |g^{[i-1]}(t)| \\ &\quad + \sum_{i=1}^n |\mu_i(x)| |g^{[i-1]}(t) - h^{[i-1]}(t)| \\ &\leq \kappa \sum_{i=1}^n \|\alpha_i - \mu_i\|_{C^0} + \left(\sum_{i=2}^n \sum_{j=1}^{i-1} M^{j-1} \right) \|g - h\|_{C^0}, \\ |g'_x(t) - h'_x(t)| &= \left| \sum_{i=1}^n \left(\alpha_i(x) (g^{[i-1]})'(t) - \mu_i(x) (h^{[i-1]})'(t) \right) \right| \\ &\leq \sum_{i=1}^n \left[|\alpha_i(x) - \mu_i(x)| (g^{[i-1]})'(t) + \mu_i(x) \left| (g^{[i-1]})'(t) - (h^{[i-1]})'(t) \right| \right] \\ &\leq \sum_{i=1}^n M^{i-1} \|\alpha_i - \mu_i\|_{C^0} + \sum_{i=2}^n \left\| (g^{[i-1]})' - (h^{[i-1]})' \right\|_{C^0} \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=1}^n M^{i-1} \|\alpha_i - \mu_i\|_{C^0} + \sum_{i=2}^n (i-1) M^{i-2} \|g' - h'\|_{C^0} \\
&\quad + M^* \sum_{i=2}^n \sum_{j=1}^{i-2} Q(i-1)(i-j-1) M^{i+j-3} \|g - h\|_{C^0}. \\
\|(g_x^{-1})' - (h_x^{-1})'\|_{C^0} &= \max_{t \in I} \left| \frac{1}{g'_x(g_x^{-1}(t))} - \frac{1}{h'_x(h_x^{-1}(t))} \right| \\
&\leq \frac{1}{\alpha^2} \max_{t \in I} |g'_x(g_x^{-1}(t)) - h'_x(h_x^{-1}(t))| \\
&\leq \frac{1}{\alpha^2} \max_{t \in I} [|g'_x(g_x^{-1}(t)) - g'_x(h_x^{-1}(t))| + |g'_x(h_x^{-1}(t)) - h'_x(h_x^{-1}(t))|] \\
&\leq \frac{1}{\alpha^2} [K_2 \|g_x^{-1} - h_x^{-1}\|_{C^0} + \|g'_x - h'_x\|_{C^0}] \\
&\leq \frac{1}{\alpha^2} \left[\frac{K_2}{\alpha} \|g_x - h_x\|_{C^0} + \|g'_x - h'_x\|_{C^0} \right] \\
&\leq \frac{K_2 \kappa}{\alpha^3} \sum_{i=1}^n \|\alpha_i - \mu_i\|_{C^0} + \frac{1}{\alpha^2} \sum_{i=1}^n M^{i-1} \|\alpha_i - \mu_i\|_{C^0} \\
&\quad + \left[\frac{K_2}{\alpha^3} \sum_{i=2}^n \sum_{j=1}^{i-1} M^{j-1} + \frac{M^*}{\alpha^2} \sum_{i=2}^n \sum_{j=1}^{i-2} Q(i-1)(i-j-1) M^{i+j-3} \right] \|g - h\|_{C^0} \\
&\quad + \frac{1}{\alpha^2} \sum_{i=2}^n (i-1) M^{i-2} \|g' - h'\|_{C^0}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|g - h\|_{C^1} &= \|g - h\|_{C^0} + \|g' - h'\|_{C^0} = \max_{x \in I} \left\{ |g_x^{-1}(G(x)) - h_x^{-1}(H(x))| \right\} \\
&\quad + \max_{x \in I} \left\{ |(g_x^{-1})'(G(x))G'(x) - (h_x^{-1})'(H(x))H'(x)| \right\} \\
&\leq \max_{x \in I} \left\{ |g_x^{-1}(G(x)) - h_x^{-1}(G(x))| + |h_x^{-1}(G(x)) - h_x^{-1}(H(x))| \right\} \\
&\quad + \max_{x \in I} \left\{ |(g_x^{-1})'(G(x)) - (h_x^{-1})'(G(x))|G'(x) \right. \\
&\quad \left. + |(h_x^{-1})'(G(x)) - (h_x^{-1})'(H(x))|G'(x) + (h_x^{-1})'(H(x))|G'(x) - H'(x)| \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \|g_x^{-1} - h_x^{-1}\|_{C^0} + \frac{1}{\alpha} \|G - H\|_{C^0} + \alpha M \|(g_x^{-1})' - (h_x^{-1})'\|_{C^0} \\
&\quad + \alpha M \cdot \frac{K_2}{\alpha^3} \|G - H\|_{C^0} + \frac{1}{\alpha} \|G' - H'\|_{C^0} \\
&\leq \frac{1}{\alpha} \|g_x - h_x\|_{C^0} + \alpha M \left\{ \frac{K_2 \kappa}{\alpha^3} \sum_{i=1}^n \|\alpha_i - \mu_i\|_{C^0} + \frac{1}{\alpha^2} \sum_{i=1}^n M^{i-1} \|\alpha_i - \mu_i\|_{C^0} \right. \\
&\quad \left. + \left[\frac{K_2}{\alpha^3} \sum_{i=2}^n \sum_{j=1}^{i-1} M^{j-1} + \frac{M^*}{\alpha^2} \sum_{i=2}^n \sum_{j=1}^{i-2} Q(i-1)(i-j-1) M^{i+j-3} \right] \|g-h\|_{C^0} \right. \\
&\quad \left. + \frac{1}{\alpha^2} \sum_{i=1}^n (i-1) M^{i-2} \|g' - h'\|_{C^0} \right\} + \frac{\alpha + K_2 M}{\alpha^2} \|G - H\|_{C^1} \\
&\leq \left(\frac{\kappa}{\alpha} + \frac{K_2 \kappa M}{\alpha^2} \right) \sum_{i=1}^n \|\alpha_i - \mu_i\|_{C^0} + \frac{M}{\alpha} \sum_{i=1}^n M^{i-1} \|\alpha_i - \mu_i\|_{C^0} \\
&\quad + \Theta \|g - h\|_{C^1} + \frac{\alpha + K_2 M}{\alpha^2} \|G - H\|_{C^1}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|g - h\|_{C^1} &\leq \frac{\kappa \alpha + K_2 \kappa M}{\alpha^2 (1 - \Theta)} \sum_{i=1}^n \|\alpha_i - \mu_i\|_{C^0} + \frac{1}{\alpha (1 - \Theta)} \sum_{i=1}^n M^i \|\alpha_i - \mu_i\|_{C^0} \\
&\quad + \frac{\alpha + K_2 M}{\alpha^2 (1 - \Theta)} \|G - H\|_{C^1}.
\end{aligned}$$

We now may conclude that the solution of (1) depends continuously on the function F and $\lambda_j (j = 1, 2, \dots, n)$. This completes the proof. $\square$

Remark 2. During the proof of Theorem 1–Theorem 3, the differentiability of $\lambda_i(x) (i = 1, 2, \dots, n)$ are not required.

5. Example

In this section we show the conditions in Theorem 1 do not self-contradict by means of an example. Consider the following equation

$$(25) \quad \lambda_1(x) \varphi(x) + \lambda_2(x) \varphi(\varphi(x)) = F(x), \quad x \in I = [0, 1],$$

where

$$\lambda_1(x) = \frac{2 - (2 - \alpha)x}{2 - x}, \quad \lambda_2(x) = \frac{(1 - \alpha)x}{2 - x},$$

$$0 < \alpha < 1, \quad F(x) = \ln(1 + x) - x \ln \frac{2}{e}.$$

Obviously, $\lambda_1(x) + \lambda_2(x) = 1$. Since

$$0 \leq \lambda_2(x) = \frac{(1 - \alpha)x}{2 - x} \leq 1 - \alpha < 1,$$

$$\alpha \leq \lambda_1(x) = 1 - \lambda_2(x) \leq 1.$$

Moreover, we also have

$$F(0) = 0, F(1) = 1,$$

and

$$0 < F'(x) = \frac{1}{x+1} - \ln \frac{2}{e}$$

$$\leq 2 - \ln 2 = \alpha M \left(M = \frac{1}{\alpha}(2 - \ln 2) \right),$$

$$|F'(x_1) - F'(x_2)| = \left| \frac{1}{x_1+1} - \frac{1}{x_2+1} \right| \leq |x_1 - x_2|,$$

$$|(\lambda_1(x))'| = |(1 - \lambda_2(x))'| = |\lambda_2(x))'| = \frac{2(1 - \alpha)}{(2 - x)^2} \leq 2(1 - \alpha).$$

Choose $M' = 1$, $\beta_1 = 2(1 - \alpha)$, $\beta_2 = 2(1 - \alpha)$.

On the other hand, since

$$\lim_{\alpha \rightarrow 1} \left[(1 - \alpha) \left(M + \frac{\kappa}{\alpha} \sum_{i=1}^2 \beta_i \right) M \right] = \lim_{\alpha \rightarrow 1} \left[(1 - \alpha) \left(M + \frac{4(1 - \alpha)}{\alpha} \right) M \right] = 0,$$

there is a positive constant $\Lambda < 1$ such that

$$\alpha > (1 - \alpha) \left(M + \frac{\kappa}{\alpha} \sum_{i=1}^2 \beta_i \right) M$$

for any $\alpha \in (\Lambda, 1)$. Namely, the conditions of Theorem 1 are satisfied for any $\alpha \in (\Lambda, 1)$.

We have thus shown that there will be a solution of (25) in $\Omega(M, M^*; I)$ $\left(M^* \geq \frac{1+2(1-\alpha)M(1+M)}{\alpha-(1-\alpha)M\left[M+\frac{1}{\alpha}(1-\alpha)\right]} \right)$.

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