

On the characterization of additive functions on Gaussian integers

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Abstract. Let f denote an additive function on the Gaussian integers. We prove some theorems of characterization with linear and quadratic arguments. If e.g. for a completely additive function $f(a\alpha + b) - tf(\alpha) = c$ or $f(\alpha^2 + 1) + f(\alpha^2 - 1) = c$, then $f(\alpha) = 0$ for all nonzero Gaussian integers α .

In 1946 ERDŐS [2] proved the following theorems:

Theorem 1 (ERDŐS). *Let f be a real valued additive function. If $f(n+1) - f(n) \rightarrow 0$, then $f(n) = c \log n$ for all $n \in \mathbb{N}$.*

Theorem 2 (ERDŐS). *If a real valued additive function f is monotonically increasing, then $f(n) = c \log n$ for all $n \in \mathbb{N}$.*

I. KÁTAI [3] generalized Theorem 1 for completely additive functions using a result of E. WIRSING [6]:

Theorem 3 (KÁTAI). *Let f be a completely additive function. If $\sum_{i=1}^m c_i f(n + a_i) = o(\log n)$, then $f(n) = c \log n$ for all $n \in \mathbb{N}$ or $f = 0$.*

The following generalizations are due to P.D.T.A. ELLIOTT [1] and myself ([4], [5]):

Theorem 4 (ELLIOTT, [1]). *Let f be an additive function, $A > 0$, $C > 0$, B, D integers and $\Delta_1 = AC(AD - BC) \neq 0$. If $f(An + B) - f(Cn + D) \rightarrow c$, then $f(n) = c' \log n$ for all (n, Δ_1) .*

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Theorem 5 [5]. *Let f be a completely additive function. If $f(2n + A) - f(n)$ is monotonic from some number on, then $f(n) = c \log n$ for all $n \in \mathbb{N}$.*

Theorem 6 [4]. *Let f denote a completely additive function. If $f(n^2 + 1) = s_1 f(n) + s_2 f(n - 1) + o(\log n)$ (s_1, s_2 are not both zero), then $f(n) = c \log n$.*

In this article we intend to prove some similar results on the set of the Gaussian integers. Let G^* denote the set of the nonzero Gaussian integers. Let α, β be the elements of this set and $N(\alpha) := \alpha \bar{\alpha}$.

Definition 1. The function f is G -additive, if $f(\alpha\beta) = f(\alpha) + f(\beta)$ for all relatively prime $\alpha, \beta \in G^*$.

Definition 2. The function f is completely G -additive, if $f(\alpha\beta) = f(\alpha) + f(\beta)$ for all $\alpha, \beta \in G^*$.

Remark. We can prove easily that $f(\epsilon) = 0$ for any additive f and arbitrary Gaussian unit ϵ .

We prove the following results:

Theorem 7. *Let a, b denote some fixed elements of G^* and let $t \in \mathbb{C} \setminus \{0\}$.*

(i) *If for a G -additive function*

$$(1) \quad f(a\alpha + b) - tf(\alpha) \rightarrow c,$$

then $f(n) = c' \log n$ for all $n \in \mathbb{N}^+$ coprime to $2N(ab)$.

(ii) *If for a completely G -additive function*

$$(1') \quad f(a\alpha + b) - tf(\alpha) = c,$$

then $f(\alpha) = 0$ for all $\alpha \in G^$.*

Theorem 8. *If for a completely G -additive function*

$$f(\alpha^2 + 1) + f(\alpha^2 - 1) = c,$$

then $f(\alpha) = 0$ for all $\alpha \in G^$.*

Theorem 9. (i) *If for a completely G-additive function*

$$f(\alpha^2 + 1) = s_1 f(\alpha) + s_2 f(\alpha - 1) + o(\log N(\alpha)) \quad (s_1, s_2 \text{ are not both zero}),$$

then $f(z) = 0$ for all $z \in \mathbb{Z} \setminus \{0\}$.

(ii) *If for a completely G-additive function*

$$f(\alpha^2 + 1) = s_1 f(\alpha) + s_2 f(\alpha - 1) + c \quad (s_1, s_2 \text{ are not both zero}),$$

then $f(\alpha) = 0$ for all $\alpha \in G^$.*

Proofs

PROOF of Theorem 7. (i) If f is G-additive, then by substituting $\bar{a}bN(b)\alpha$ into (1) we have

$$(2) \quad f(N(ab)\alpha + 1) - tf(\alpha) \rightarrow C',$$

for all α coprime to $N(ab)$ with $C' = c + tf(\bar{a}bN(b)) - f(b)$. By substituting 2α into (2) we have

$$(3) \quad f(2N(ab)\alpha + 1) - tf(\alpha) \rightarrow C''$$

for all α coprime to $2N(ab)$ with $C'' = c + tf(2\bar{a}bN(b)) - f(b)$. The difference of (3) and (2) shows that

$$(4) \quad f(2N(ab)\alpha + 1) - f(N(ab)\alpha + 1) \rightarrow C'''$$

for all α coprime to $2N(ab)$. By substituting $2N(ab)\alpha + 1$ into (4) we have that

$$f(4N^2(ab)\alpha + 2N(ab) + 1) - f(2N^2(ab)\alpha + N(ab) + 1) \rightarrow C''''$$

for all $\alpha \in G^*$. Applying Theorem 4 we get that $f(n) = c' \log n$ for all $n \in \mathbb{N}$ coprime to $2N(ab)$.

(ii) Let f be a completely G-additive function.

By substituting $b\alpha$ into (1') we have

$$(5) \quad f(a\alpha + 1) = tf(\alpha) + c_1$$

with $c_1 = c + (t - 1)f(b)$. Therefore

$$f((a\alpha + 1)^2) = f(a[\alpha(a\alpha + 2)] + 1) = tf(\alpha) + tf(a\alpha + 2) + c_1$$

and

$$f((a\alpha + 1)^2) = 2f(a\alpha + 1) = 2tf(\alpha) + 2c_1,$$

i.e.

$$(6) \quad f(a\alpha + 2) = f(\alpha) + c_1/t.$$

By substituting 2α into (6) we get

$$(7) \quad f(a\alpha + 1) = f(\alpha) + c_1/t.$$

By the comparison of (5) and (7) f is constant, i.e. $f(\alpha) = 0$ for all $\alpha \in G^*$ or $t = 1$. If $t = 1$, then we also prove, that $f(\alpha) = 0$ for all $\alpha \in G^*$. First we prove by induction, that

$$(8) \quad f(a\alpha + s) = f(\alpha) + c_1$$

for all $s \in \mathbb{N}^+$. For $s = 1$ it is true by (7). By the assumption of induction for $s \leq z$ we have

$$f((a\alpha + 1)(a\alpha + z)) = f(a\alpha + 1) + f(a\alpha + z) = 2f(\alpha) + 2c_1$$

and

$$f(a[\alpha(a\alpha + z + 1)] + z) = f(\alpha) + f(a\alpha + z + 1) + c_1,$$

which follow $f(a\alpha + z + 1) = f(\alpha) + c$. By substituting $s = a$ into (8) we have $f(\alpha + 1) = f(\alpha) + c_1 - f(a)$. By restricting α to the natural numbers $f(n) - f(n - 1) = c_1 - f(a)$ for all $n \in \mathbb{N}$. By substituting n^2 here we have $c_1 - f(a) = f(n^2) - f(n^2 - 1) = [f(n) - f(n - 1)] - [f(n + 1) - f(n)] = 0$, i.e. $c_1 = f(a)$. Applying Theorem 1 $f(n + 1) = f(n)$ follows $f(n) = 0$ for all $n \in \mathbb{N}$. Using that $f(\alpha + 1) = f(\alpha)$ for all $\alpha \in G^*$, we prove by induction that $f(\delta) = 0$ also for all not real Gaussian primes δ . By the Remark it is enough to consider the Gaussian primes of form $\pi = 1 + i$ and $\pi = x + yi$ with even number x and odd number y as $f(\pi) + f(\bar{\pi}) = f(N(\pi)) = 0$ and $f(i\pi) = f(-i\pi) = f(-\pi) = f(\pi)$. We have $0 = f(2) = 2f(1 + i)$. For any other π the Gaussian integer $\pi - 1$ is divisible by $1 + i$, i.e. it is not a Gaussian prime. We assume that $f(\gamma) = 0$ for all Gaussian-primes which

norm is less than $f(\pi)$. As $f(\pi) = f(\pi - 1)$ and $f(\beta) = 0$ for all prime divisors β of $\pi - 1$ by the hypothesis of the induction, therefore $f(\pi) = 0$ is also satisfied as f is a completely G-additive function. $\square$

PROOF of Theorem 8. As $\alpha^2 + 1 = (\alpha + i)(\alpha - i)$, we have

$$(9) \quad f(\alpha + i) + f(\alpha - i) + f(\alpha + 1) + f(\alpha - 1) = c.$$

By substituting $\alpha - i$ into (9), $(1 + i)\alpha$ into (10), $2\alpha + i$ and $(1 - i)\alpha + i$ into (11) we have that

$$(10) \quad f(\alpha) + f(\alpha - 2i) + f(\alpha + 1 - i) + f(\alpha - 1 - i) = c,$$

$$(11) \quad f(\alpha) + f(\alpha - 1 - i) + f(\alpha - i) + f(\alpha - 1) = c_1,$$

$$(12) \quad f(2\alpha + i) + f(2\alpha - 1) + f(\alpha) + f(2\alpha - 1 + i) = c_2$$

and

$$(13) \quad f(2\alpha - 1 + i) + f(2\alpha - 1 - i) + f(\alpha) + f(\alpha - 1) = c_3.$$

The difference of (12) and (13) shows that

$$(14) \quad f(2\alpha + i) - f(2\alpha - 1 - i) + f(2\alpha - 1) - f(\alpha - 1) = c_4.$$

By substituting 2α into (11) and (9) we have

$$(15) \quad f(\alpha) + f(2\alpha - 1 - i) + f(2\alpha - i) + f(2\alpha - 1) = c_1$$

and

$$(16) \quad f(2\alpha + i) + f(2\alpha - i) + f(2\alpha + 1) + f(2\alpha - 1) = c.$$

The difference of (15) and (16) shows that

$$(17) \quad f(2\alpha + i) - f(2\alpha - 1 - i) + f(2\alpha + 1) - f(\alpha) = c_5.$$

By the comparison of (14) and (17) we get

$$(18) \quad f(2\alpha + 1) - f(\alpha) = f(2\alpha - 1) - f(\alpha - 1) + c_6.$$

By restricting α to natural numbers (18) follows that $f(2n + 1) - f(n)$ is monotonic. Applying Theorem 5 we obtain $f(n) = 0$ for all $n \in \mathbb{N}$ and also that $c_6 = 0$. We prove by induction that $f(\pi) = 0$ also for all not

real Gaussian primes. As the norm of $2\alpha + 1$ is greater than the norm of any other argument in (18) and the minimal value of $N(2\alpha + 1)$ is 13, it is enough to verify that $f(1 + i) = f(2 + i) = f(2 + 3i) = 0$ using the Remark. It is true as $0 = f(2) = 2f(1 + i)$ and $\alpha = 1 + i$ and $\alpha = i$ in (18) imply $f(1 + 2i) = f(3 + 2i)$ and $f(1 + 2i) = f(1 + i)$. $\square$

PROOF of Theorem 9. (i) is a direct consequence of Theorem 6 and the Remark.

(ii) By induction we prove that $f(x + yi) = 0$ for all $x \in \mathbb{Z}$ and $y \in \mathbb{N}$. For $y = 0$ it is true by (i) (x can be arbitrarily chosen). Let us assume that it is true for all $0 \leq y \leq s$. If we substitute $x + si$ in the condition of the theorem we get

$$f(x + (s + 1)i) + f(x + (s - 1)i) = sf(x + si) + tf(x - 1 + si),$$

i.e. by the assumption of the induction $f(x + (s + 1)i) = 0$. If $y < 0$, then $f(x + iy) = f(-x - iy) = 0$ as $-y \in \mathbb{N}$. $\square$

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