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On a certain application of Patterson's curvature identity

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Abstract. We present curvature properties of four-dimensional semi-Riemannian manifolds satisfying some condition of pseudosymmetry type. We prove that every such manifold with non-zero associated function L is pseudosymmetric, its scalar curvature does not vanish and L must be equal to $\frac{1}{3}$. We also describe non-trivial example of a manifold realizing all these coditions.

1. Introduction

Let (M, g) be a connected n-dimensional, $n \geq 3$, semi-Riemannian manifold of class C^{∞}. We denote by ∇, R, C, S and κ the Levi–Civita connection, the Riemann–Christoffel curvature tensor, the Weyl conformal curvature tensor, the Ricci tensor and the scalar curvature of (M, g), respectively.

E. M. Patterson [13] has given (among others algebraic identities satisfied by the curvature tensor) the following result

Proposition 1.1. *The Weyl conformal curvature tensor C of every 4-dimensional Riemannian manifold satisfies the identity*

$$g_{hm}C_{lijk} + g_{lm}C_{ihjk} + g_{im}C_{hljk} + g_{hj}C_{likm} + g_{lj}C_{ihkm} \\ + g_{ij}C_{hlkm} + g_{hk}C_{limj} + g_{lk}C_{ihmj} + g_{ik}C_{hlmj} = 0.$$

The above identity plays an important role in investigations of the curvature properties of 4-dimensional semi-Riemannian manifolds. Among

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others, the authors of the present paper have used the Patterson's identity during the study of 4-dimensional manifolds of pseudosymmetry type ([3], [8]).

A semi-Riemannian manifold (M, g) is said to be semisymmetric [14] if $R \cdot R = 0$ holds on M. As a proper generalization of locally symmetric spaces ($\nabla R = 0$) semisymmetric manifolds were studied by many authors. In the Riemannian case, Z. I. Szabó obtained in the early eighties a full intrinsic classification of semisymmetric Riemannian manifolds [14]. Very recently a theory of Riemannian semisymmetric manifolds has been presented in the monograph [1].

The profound investigation of several properties of semisymmetric manifolds, gave rise to their next generalization: the pseudosymmetric manifolds.

A semi-Riemannian manifold (M, g) is said to be *pseudosymmetric* ([6], Section 3.1) if at every point of M the following condition is satisfied:

$(*)_1$ the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.

This condition is equivalent to the relation $R \cdot R = L_R Q(g, R)$ on the set $\mathcal{U}_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)} G \neq 0 \,\mathrm{at}\, x\}$, where L_R is some function on $\mathcal{U}_R$. The definitions of the tensors used will be given in Section 2. There exist various examples of pseudosymmetric manifolds which are non-semisymmetric and a review of results on pseudosymmetric manifolds is given in [6] (see also [15]).

It is easy to see that if $(*)_1$ holds on a semi-Riemannian manifold (M, g), then at every point of M the following condition is satisfied:

$(*)_2$ the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent.

The converse statement is not true ([6], Section 8.1). A semi-Riemannian manifold (M, g) is called *Ricci-pseudosymmetric* ([6], Section 4.1) if at every point of M the condition $(*)_2$ is fulfilled. If a manifold (M, g) is Ricci-pseudosymmetric then the relation $R \cdot S = L_S\, Q(g, S)$ holds on the set $\mathcal{U}_S = \{x \in M \mid S \neq \frac{\kappa}{n} g \,\mathrm{at}\, x\}$, where L_S is a certain function on $\mathcal{U}_S$.

It is easy to verify that if $(*)_1$ holds on a semi-Riemannian manifold (M, g), $n \geq 4$, then at every point of M the following condition is satisfied:

$(*)_3$ the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent.

The converse statement is not true (cf. Example 4.1; see also [6], Section 9.3).

A semi-Riemannian manifold (M,g), $n \geq 4$, is called *Weyl-pseudosymmetric* if at every point of M the condition $(*)_3$ is fulfilled. If a manifold (M,g) is Weyl-pseudosymmetric then the relation $R\cdot C = L_C\, Q(g,C)$ holds on the set $\mathcal{U}_C = \{x \in M \mid C \neq 0 \,\text{at}\, x\}$, where L_C is some function on $\mathcal{U}_C$.

Evidently, every semi-Riemannian semisymmetric manifold realizes trivially at every point the following condition ([9])

$(*)$ the tensors $R \cdot C$ and $Q(S,C)$ are linearly dependent.

This condition is equivalent to the relation

$$R \cdot C = L\, Q(S,C) \tag{1}$$

on the set $\mathcal{U} = \{x \in M \mid Q(S,C) \neq 0 \,\text{at}\, x\}$, for a certain function L on $\mathcal{U}$. There exist non-semisymmetric manifolds realizing $(*)$ ([9], [10]).

Semi-Riemannian manifolds realizing $(*)_1$, $(*)_2$, $(*)_3$ and $(*)$ or other conditions of this kind, are called *manifolds of pseudosymmetry type*.

Recently 4-dimensional warped product manifolds $M\times_F N$, $\dim M{=}1$, satisfying the condition $(*)$ have been considered in [10]. In the present paper we investigate arbitrary 4-dimensional semi-Riemannian manifolds fulfilling $(*)$. In Section 2 we fix the notations and present auxiliary lemmas. In Section 3 we consider 4-dimensional manifolds satisfying the equality $Q(S,C) = 0$ and we prove that such manifolds are semisymmetric and their scalar curvature is equal to zero. In Section 4 we investigate 4-dimensional manifolds satisfying (1) with $L \neq 0$. We prove, among others, that every such manifold is pseudosymmetric with $L_R = \frac{\kappa}{12}$, its scalar curvature does not vanish and the associated function L must be equal to $\frac{1}{3}$. Finally, we describe an example of a manifold having all these properties.

2. Preliminaries

Let (M,g) be an n-dimensional, $n \geq 3$, semi-Riemannian manifold. The Ricci operator $\mathcal{S}$ is defined by $g(\mathcal{S}X,Y) = S(X,Y)$, where $X, Y \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M. Next, we define the endomorphisms $\mathcal{R}(X,Y)$, $\mathcal{C}(X,Y)$ and $X \wedge Y$ of $\Xi(M)$ by

$$\mathcal{R}(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$
$$\mathcal{C}(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{n-2}\Big(X \wedge \mathcal{S}Y + \mathcal{S}X \wedge Y - \frac{\kappa}{n-1} X \wedge Y\Big)Z,$$
$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y,$$

respectively, where $X, Y, Z \in \Xi(M)$. Now the Riemann–Christoffel curvature tensor R, the Weyl conformal curvature tensor C and the (0,4)-tensor G of (M,g) are defined by

$$R(X_1,X_2,X_3,X_4) = g(\mathcal{R}(X_1,X_2)X_3,X_4),$$
$$C(X_1,X_2,X_3,X_4) = g(\mathcal{C}(X_1,X_2)X_3,X_4),$$
$$G(X_1,X_2,X_3,X_4) = g((X_1 \wedge X_2)X_3,X_4),$$

respectively. A tensor $\mathcal{B}$ of type $(1,3)$ on M is said to be a generalized curvature tensor if

$$\underset{X_1,X_2,X_3}{\mathfrak{S}} \mathcal{B}(X_1,X_2)X_3 = 0, \qquad \mathcal{B}(X_1,X_2) + \mathcal{B}(X_2,X_1) = 0,$$
$$B(X_1,X_2,X_3,X_4) = B(X_3,X_4,X_1,X_2),$$

where $B(X_1,X_2,X_3,X_4) = g(\mathcal{B}(X_1,X_2)X_3,X_4)$.

For an $(0,2)$-tensor field A on (M,g) we define the endomorphism $X \wedge_A Y$ of $\Xi(M)$ by $(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y$, where $X,Y,Z \in \Xi(M)$. In particular we have $X \wedge_g Y = X \wedge Y$.

For an $(0,k)$-tensor field T, $k \geq 1$, an $(0,2)$-tensor field A and a generalized curvature tensor $\mathcal{B}$ on (M,g) we define the tensors $B \cdot T$ and $Q(A,T)$ by

$$\begin{aligned}(B \cdot T)(X_1,\ldots,X_k;X,Y) &= -T(\mathcal{B}(X,Y)X_1,X_2,\ldots,X_k)\\ &\quad - \cdots - T(X_1,\ldots,X_{k-1},\mathcal{B}(X,Y)X_k),\\ Q(A,T)(X_1,\ldots,X_k;X,Y) &= -T((X \wedge_A Y)X_1,X_2,\ldots,X_k)\\ &\quad - \cdots - T(X_1,\ldots,X_{k-1},(X \wedge_A Y)X_k),\end{aligned}$$

where $X,Y,Z,X_1,X_2,\ldots \in \Xi(M)$. Putting in the above formulas $\mathcal{B} = \mathcal{R}$ or $\mathcal{B} = \mathcal{C}$, $T = R$ or $T = C$ or $T = S$, $A = g$ or $A = S$, we obtain the tensors

$R \cdot R$, $R \cdot C$, $R \cdot S$, $C \cdot S$, $Q(g, R)$, $Q(g, C)$, $Q(g, S)$ and $Q(S, C)$, respectively. Let (M, g) be a semi-Riemannian manifold covered by a system of charts $\{W; x^k\}$.
We denote by $g_{ij}, R_{hijk}, S_{ij}, G_{hijk} = g_{hk}g_{ij} - g_{hj}g_{ik}$ and

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj}) + \frac{\kappa}{(n-1)(n-2)}G_{hijk} \tag{2}$$

the local components of the metric tensor g, the Riemann–Christoffel curvature tensor R, the Ricci tensor S, the tensor G and the Weyl tensor C, respectively. Further, we denote by $S^2_{ij} = S_{ir}S_j{}^r$ and $S_i{}^j = g^{jr}S_{ir}$ the local components of the tensor S^2 defined by $S^2(X, Y) = S(\mathcal{S}X, Y)$, and of the Ricci operator $\mathcal{S}$, respectively.

At the end of this section we present some results which will be used in the next sections.

Lemma 2.1 ([5], Lemma 1). *Let a tensor $A_{lmhs_1\ldots s_N}$ of type $(0, N+3)$ be symmetric in (l, m) and skew-symmetric in (m, h). Then $A_{lmhs_1\ldots s_N} = 0$.*

Lemma 2.2 ([11], Lemma 2). *Let A and D be two symmetric $(0, 2)$-tensors at a point x of a semi-Riemannian manifold (M, g). If the condition $Q(A, D) = 0$ is fulfilled at x, then the tensors A and D are linearly dependent at x.*

Lemma 2.3 ([7], Theorem 1). *Let $\mathcal{B}$ be a generalized curvature tensor at $x \in M$ such that the condition*

$$\underset{X,Y,Z}{\mathfrak{S}} \omega(X)\mathcal{B}(Y, Z) = 0$$

is satisfied for $\mathcal{B}$ and a covector ω at x, where $X, Y, Z \in T_x(M)$ and $\mathfrak{S}$ denotes the cyclic sum. If $\omega \neq 0$ then the following relation holds at $x : B \cdot B = Q(\mathrm{Ric}(\mathcal{B}), B)$.

Lemma 2.4 ([2], Proposition 4.1). *Let (M, g), $\dim M \geq 3$, be a semi-Riemannian manifold. Let A be a non-zero symmetric $(0, 2)$-tensor and $\mathcal{B}$ a generalized curvature tensor at a point x of M satisfying the condition $Q(A, B) = 0$.*

Moreover, let V be a vector at x such that the scalar $\rho = a(V)$ is non-zero, where a is the covector defined by $a(X) = A(X, V)$, $X \in T_x(M)$.

(i) *If the tensor* $A - \frac{1}{\rho} a \otimes a$ *vanishes then the relation*

$$\mathop{\mathfrak{S}}_{X,Y,Z} a(X)\mathcal{B}(Y,Z) = 0$$

holds at x*, where* $X, Y, Z \in T_x(M)$*.*

(ii) *If the tensor* $A - \frac{1}{\rho} a \otimes a$ *is non-zero then the relation*

$$\rho B(X,Y,Z,W) = \lambda(A(X,W)A(Y,Z) - A(X,Z)A(Y,W))$$

holds at x*, where* $\lambda \in \mathbb{R}$ *and* $X, Y, Z, W \in T_x(M)$*.*

Moreover, in both cases the following condition $B \cdot B = Q(\mathrm{Ric}(\mathcal{B}), B)$ *holds at* x*.*

Lemma 2.5 ([12], Theorems 1 and 2)**.** *Let* (M, g) *be a Weyl-pseudo-symmetric semi-Riemannian manifold satisfying the following condition*

$$\mathop{\mathfrak{S}}_{X,Y,Z} a(X)\mathcal{C}(Y,Z) = 0,$$

where a *is a 1-form on* M*.*

If $a \neq 0$ *and* $C \neq 0$ *at a point* $x \in M$*, then the following relations are satisfied at* x*:*

$$L_C = \frac{\kappa}{n(n-1)}, \qquad S(W, \mathcal{C}(X,Y)Z) = \frac{\kappa}{n}\, C(X,Y,Z,W),$$

$$Q\left(S - \frac{\kappa}{n} g, C\right) = 0, \qquad R \cdot R = L_C Q(g, R).$$

3. Manifolds with vanishing tensor field $Q(S, C)$

Theorem 3.1. *Let* (M, g)*,* $\dim M = 4$*, be a semi-Riemannian manifold satisfying at a point* x *of* M *the equality* $Q(S, C) = 0$*. If* $S \neq 0$ *and* $C \neq 0$ *at* x*, then the following relations*

$$\kappa = 0, \quad R \cdot R = 0$$

hold at x*. Moreover, there exists a non-zero covector* a *at* x *such that*

$$\mathop{\mathfrak{S}}_{X,Y,Z} a(X)\mathcal{C}(Y,Z) = 0.$$

Proof. It is easy to verify that the following identity is satisfied on M

$$(C\cdot C)_{hijklm} = (R\cdot C)_{hijklm} + \frac{1}{n-2}\Big(\frac{\kappa}{n-1}Q(g,C)_{hijklm} - Q(S,C)_{hijklm}\Big) - \frac{1}{n-2}\left(g_{hl}S_{mr}C^{r}{}_{ijk} - g_{hm}S_{lr}C^{r}{}_{ijk} - g_{il}S_{mr}C^{r}{}_{hjk} + g_{im}S_{lr}C^{r}{}_{hjk} + g_{jl}S_{mr}C^{r}{}_{khi} - g_{jm}S_{lr}C^{r}{}_{khi} - g_{kl}S_{mr}C^{r}{}_{jhi} + g_{km}S_{lr}C^{r}{}_{jhi}\right). \tag{3}$$

According to Lemma 2.4, we may consider two cases (we will use the notations of the mentioned lemma):

(i) $S = \frac{1}{\rho}a\otimes a$. In this case we have

$$a_l C_{hijk} + a_h C_{iljk} + a_i C_{lhjk} = 0, \tag{4}$$

which implies $a_r C^{r}{}_{ijk} = 0$ and consequently $S_{ir}C^{r}{}_{hjk} = 0$. Thus the equation $C\cdot C = 0$, which follows from Lemma 2.3, and our assumption turn (3) into $R\cdot C = -\frac{\kappa}{(n-1)(n-2)}Q(g,C)$. Applying now Lemma 2.5 we obtain $\kappa = 0$ and $R\cdot R = 0$.

Now we consider the case:

(ii) $S - \frac{1}{\rho}a\otimes a \neq 0$. In this case we have

$$\rho C_{hijk} = \lambda(S_{hk}S_{ij} - S_{hj}S_{ik}). \tag{5}$$

Contracting (5) with g^{ij} we get $S^2_{hk} = \kappa S_{hk}$. On the other hand transvecting (5) with $S_p{}^h$ and using the last equality we obtain

$$S_p{}^r C_{rijk} = \kappa C_{pijk}. \tag{6}$$

Transvecting now the Patterson's identity with S^{hm}, in virtue of (6), we immediately have $\kappa C_{lijk} = 0$. Thus we have $\kappa = 0$, which turns (6) into $S_p{}^r C_{rijk} = 0$. Transvecting the Patterson's identity with $S_p{}^h$, in view of the last equality we get $S_{pm}C_{lijk} + S_{pj}C_{likm} + S_{pk}C_{limj} = 0$, which immediately implies (4). Semisymmetry of M follows in the same manner as in the case (i). This completes the proof. □

4. Manifolds satisfying the condition (1)

First we observe that if $L = 0$ at x then the relation (1) reduces to $R \cdot C = 0$. This shows that (M, g) is a so called Weyl-semisymmetric manifold. Such a manifold need not be semisymmetric, as is shown by the example below.

Example 4.1. Let (M, g) be the 4-dimensional manifold defined in [4] (Lemme 1.1). As was shown in [4] (see Lemme 1.1 and Remarqué 1.5), (M, g) is a non-conformally flat and non-semisymmetric, Weyl-semisymmetric manifold, i.e. the tensors C and $R \cdot R$ are non-zero and the condition $R \cdot C = 0$ holds on M.

Thus we restrict our considerations in this section to the open subset $\mathcal{U}_L \subset \mathcal{U}$ on which the associated function L does not vanish.

Lemma 4.1. *Let (M, g) be a 4-dimensional semi-Riemannian manifold satisfying the condition* (1)*. If $L \neq 0$ at x, then the following relations are fulfilled at x:*

$$S_m{}^r C_{rijk} + S_j{}^r C_{rikm} + S_k{}^r C_{rimj} = 0, \tag{7}$$

$$C \cdot S = 0, \tag{8}$$

$$S_p{}^r C_{rikj} = S_k{}^r C_{rjpi}. \tag{9}$$

Proof. Applying to the Patterson's identity the operation $R\cdot$ and using (1), in view of $L \neq 0$, we get

$$\begin{aligned} & g_{hm} Q(S,C)_{lijkpt} + g_{lm} Q(S,C)_{ihjkpt} + g_{im} Q(S,C)_{hljkpt} \\ & \quad + g_{hj} Q(S,C)_{likmpt} + g_{lj} Q(S,C)_{ihkmpt} + g_{ij} Q(S,C)_{hlkmpt} \\ & \quad + g_{hk} Q(S,C)_{limjpt} + g_{lk} Q(S,C)_{ihmjpt} + g_{ik} Q(S,C)_{hlmjpt} = 0. \end{aligned} \tag{10}$$

Using the definition of the tensor $Q(S, C)$, by a standard calculation, we obtain

$$\begin{aligned} & Q(S,C)_{lijkpm} + Q(S,C)_{likmpj} + Q(S,C)_{limjpk} \\ & \quad = 2(S_{pm} C_{likj} + S_{pj} C_{limk} + S_{pk} C_{lijm}) \\ & \quad - (S_{ml} C_{pijk} + S_{im} C_{lpjk} + S_{jl} C_{pikm} + S_{ij} C_{lpkm} + S_{kl} C_{pimj} + S_{ik} C_{lpmj}), \end{aligned}$$

$$g^{rs}Q(S,C)_{rljkps} = -\kappa C_{pljk} + S_p{}^r C_{rljk} - S_l{}^r C_{rpjk} - S_j{}^r C_{rlpk} - S_k{}^r C_{rljp}.$$

Contracting (10) with g^{ht} and using the above relations, we have

$$\begin{aligned}
&2(S_{pm}C_{likj} + S_{pj}C_{limk} + S_{pk}C_{lijm}) - (S_{lm} - \kappa g_{lm})C_{pijk}\\
&\quad - (S_{lj} - \kappa g_{lj})C_{pikm} - (S_{lk} - \kappa g_{lk})C_{pimj} - (S_{im} - \kappa g_{im})C_{lpjk}\\
&\quad - (S_{ij} - \kappa g_{ij})C_{lpkm} - (S_{ik} - \kappa g_{ik})C_{lpmj}\\
&\quad + S_i{}^r(g_{lm}C_{rpjk} + g_{lj}C_{rpkm} + g_{lk}C_{rpmj})\\
&\quad + S_l{}^r(g_{im}C_{rpkj} + g_{ij}C_{rpmk} + g_{ik}C_{rpjm})\\
&\quad + g_{im}(S_p{}^r C_{rljk} - S_j{}^r C_{rlpk} - S_k{}^r C_{rljp})\\
&\quad + g_{ij}(S_p{}^r C_{rlkm} - S_k{}^r C_{rlpm} - S_m{}^r C_{rlkp})\\
&\quad + g_{ik}(S_p{}^r C_{rlmj} - S_m{}^r C_{rlpj} - S_j{}^r C_{rlmp})\\
&\quad + g_{lm}(S_j{}^r C_{ripk} + S_k{}^r C_{rijp} - S_p{}^r C_{rijk})\\
&\quad + g_{lj}(S_k{}^r C_{ripm} + S_m{}^r C_{rikp} - S_p{}^r C_{rikm})\\
&\quad + g_{lk}(S_m{}^r C_{ripj} + S_j{}^r C_{rimp} - S_p{}^r C_{rimj}) = 0.
\end{aligned} \tag{11}$$

Hence, by contraction with g^{lp}, we obtain (7). In local coordinates the relation (1) takes the form

$$\begin{aligned}
&C_{rijk}R^r{}_{hlm} + C_{hrjk}R^r{}_{ilm} + C_{hirk}R^r{}_{jlm} + C_{hijr}R^r{}_{klm}\\
&\quad = L(S_{hl}C_{mijk} - S_{hm}C_{lijk} + S_{il}C_{hmjk} - S_{im}C_{hljk} + S_{jl}C_{himk}\\
&\qquad - S_{jm}C_{hilk} + S_{kl}C_{hijm} - S_{km}C_{hijl}).
\end{aligned}$$

Contracting this equality with g^{hk}, in virtue of (7) and the assumption $L \neq 0$, we get $S_i{}^r C_{rjlm} + S_j{}^r C_{rilm} = 0$, i.e., (8). Applying (7) to (11) we find

$$\begin{aligned}
&2(S_{mp}C_{likj} + S_{jp}C_{limk} + S_{kp}C_{lijm}) - (S_{lm} - \kappa g_{lm})C_{pijk}\\
&\quad - (S_{lj} - \kappa g_{lj})C_{pikm} - (S_{lk} - \kappa g_{lk})C_{pimj} - (S_{im} - \kappa g_{im})C_{lpjk}\\
&\quad - (S_{ij} - \kappa g_{ij})C_{lpkm} - (S_{ik} - \kappa g_{ik})C_{lpmj}\\
&\quad + S_i{}^r(g_{lm}C_{rpjk} + g_{lj}C_{rpkm} + g_{lk}C_{rpmj})\\
&\quad + S_l{}^r(g_{im}C_{rpkj} + g_{ij}C_{rpmk} + g_{ik}C_{rpjm}) = 0.
\end{aligned} \tag{12}$$

Contracting (12) with g^{lm}, we obtain $2S_p{}^r C_{rikj} = S_j{}^r C_{rkip} + S_k{}^r C_{rjpi}$ and next, in view of (8), we get (9). This completes the proof. $\square$

Proposition 4.1. *Let (M,g), $\dim M = 4$, be a semi-Riemannian manifold satisfying the condition* (1). *If $L \neq 0$ at x, then the following relations are fulfilled at x:*

$$S(W,\mathcal{C}(X,Y)Z) = \frac{\kappa}{4} C(X,Y,Z,W), \tag{13}$$

$$\begin{aligned} T(U,X)C(W,V,Y,Z) &+ T(U,Y)C(W,V,Z,X) \\ &+ T(U,Z)C(W,V,X,Y) - T(V,X)C(W,U,Y,Z) \\ &- T(V,Y)C(W,U,Z,X) - T(V,Z)C(W,U,X,Y) = 0, \end{aligned} \tag{14}$$

where $T = S - \frac{\kappa}{4} g$ and $X,Y,Z,W,U,V \in T_x(M)$,

$$S^2 = \frac{\kappa}{2} S - \frac{\kappa^2}{16} g. \tag{15}$$

Moreover, (M,g) is Ricci-pseudosymmetric at x and

$$R \cdot S = \frac{\kappa}{12} Q(g,S). \tag{16}$$

Proof. Transvecting the Patterson's identity with S^{hm} and using the equality $S^{rs}C_{rijs} = 0$, which is an obvious consequence of (8), we get

$$\kappa C_{lijk} = S_l{}^r C_{rijk} - S_i{}^r C_{rljk} - S_j{}^r C_{rkil} - S_k{}^r C_{rjli}.$$

This relation, in virtue of (9) takes the form $\kappa C_{lijk} = 2S_l{}^r C_{rijk} - 2S_i{}^r C_{rljk}$ whence, in view of (8), we obtain (13). Using (13) we have

$$\begin{aligned} &S_i{}^r(g_{lm}C_{rpjk} + g_{lj}C_{rpkm} + g_{lk}C_{rpmj}) + S_l{}^r(g_{im}C_{rpkj} + g_{ij}C_{rpmk} + g_{ik}C_{rpjm}) \\ &= \frac{\kappa}{4}(g_{lm}C_{ipjk} + g_{lj}C_{ipkm} + g_{lk}C_{ipmj} + g_{im}C_{lpkj} + g_{ij}C_{lpmk} + g_{ik}C_{lpjm}). \end{aligned}$$

Substituting this equality into (12) we obtain

$$\begin{aligned} 2(S_{mp}C_{likj} + S_{jp}C_{limk} + S_{kp}C_{lijm}) &= \Big(S_{lm} - \frac{3}{4}\kappa g_{lm}\Big) C_{pijk} \\ &+ \Big(S_{lj} - \frac{3}{4}\kappa g_{lj}\Big) C_{pikm} + \Big(S_{lk} - \frac{3}{4}\kappa g_{lk}\Big) C_{pimj} \\ &+ \Big(S_{im} - \frac{3}{4}\kappa g_{im}\Big) C_{lpjk} + \Big(S_{ij} - \frac{3}{4}\kappa g_{ij}\Big) C_{lpkm} \\ &+ \Big(S_{ik} - \frac{3}{4}\kappa g_{ik}\Big) C_{lpmj}. \end{aligned} \tag{17}$$

On the other hand, transvecting the Patterson's identity with ${S_p}^h$ and using (13) we get

$$S_{pm}C_{lijk} + S_{pj}C_{likm} + S_{pk}C_{limj} + \frac{\kappa}{4}(g_{im}C_{pljk} + g_{ij}C_{plkm} + g_{ik}C_{plmj} - g_{lm}C_{pijk} - g_{lj}C_{pikm} - g_{lk}C_{pimj}) = 0.$$

Substituting this equality into (17) we have (14).

Transvecting now (14) with ${T_h}^m$ and using (13) we obtain $T^2_{hl}C_{pijk} = T^2_{hi}C_{pljk}$.

Applying now Lemma 2.1 for $A_{lpis_1s_2s_3} = T^2_{s_1l}C_{pis_2s_3}$, in virtue of $C \neq 0$, we get $T^2 = 0$. Thus we have (15). Finally, taking into account the equality (2), we have

$$\begin{aligned} {S_l}^r C_{rijk} &= {S_l}^r R_{rijk} - \frac{1}{2}(S_{lk}S_{ij} - S_{lj}S_{ik}) \\ &\quad - \frac{1}{2}\left(g_{ij}S^2_{lk} - g_{ik}S^2_{lj}\right) + \frac{\kappa}{6}(S_{lk}g_{ij} - S_{lj}g_{ik}). \end{aligned}$$

This equality, in virtue of (15) takes the form

$$\begin{aligned} {S_l}^r C_{rijk} &= {S_l}^r R_{rijk} - \frac{1}{2}(S_{lk}S_{ij} - S_{lj}S_{ik}) \\ &\quad - \frac{\kappa}{12}(g_{ij}S_{lk} - g_{ik}S_{lj}) + \frac{\kappa^2}{32}(g_{ij}g_{lk} - g_{ik}g_{lj}). \end{aligned}$$

Symmetrizing this relation in i, l and using (8) we have

$${S_l}^r R_{rijk} + {S_i}^r R_{rljk} = \frac{\kappa}{12}(g_{lj}S_{ik} - g_{lk}S_{ij} + g_{ij}S_{lk} - g_{ik}S_{lj}),$$

i.e., equality (16). This completes the proof. □

Theorem 4.1. *Let (M, g), $\dim M = 4$, be a semi-Riemannian manifold satisfying the condition (1). If $L \neq 0$ and $S \neq \frac{\kappa}{4}g$ at x, then the following relations hold at x:*

(i) $\kappa \neq 0$,

(ii) $\mathfrak{S}_{X,Y,Z}\, a(X)\mathcal{C}(Y, Z) = 0$ *for a certain non-zero covector a at x,*

(iii) $L = \frac{1}{3}$,

(iv) $Q(T, C) = 0$.

Consequently, (M,g) *is pseudosymmetric at* x *and* $L_R = \frac{\kappa}{12}$.

Proof. Since $T \neq 0$ at x, we may choose a vector V at x (with local components V^r), such that the scalar $\rho = a(V)$ is non-zero, where a is the covector defined by $a(X) = T(V,X)$. We also put $F_{ij} = V^r V^s C_{rijs}$, $E_{ijk} = V^r C_{rijk}$. Transvecting (14) with $V^l V^m$ we get

$$\rho C_{pijk} + a_j E_{kip} + a_k E_{jpi} + a_i E_{pjk} + T_{ij} F_{pk} - T_{ik} F_{pj} = 0, \tag{18}$$

which by transvection with V^p, leads to

$$E_{ijk} = \frac{1}{\rho}(a_k F_{ij} - a_j F_{ik}). \tag{19}$$

Symmetrizing (18) with respect to p, i we obtain

$$a_i E_{pjk} + a_p E_{ijk} + T_{ij} F_{pk} - T_{ik} F_{pj} + T_{pj} F_{ik} - T_{pk} F_{ij} = 0,$$

which, in view of (19), leads to $Q(F, T - \frac{1}{\rho} a \otimes a) = 0$. Applying now Lemma 2.2, in view of $F \neq 0$, we have $T - \frac{1}{\rho} a \otimes a = \omega F$, $\omega \in \mathbb{R}$.

We consider two cases: (I) $\omega = 0$ and (II) $\omega \neq 0$.

(I) $\omega = 0$. Then we have $T_{ij} = \frac{1}{\rho} a_i a_j$ and substituting this into (18) and using (19) we have

$$\rho^2 C_{pijk} = a_p a_k F_{ij} - a_p a_j F_{ik} + a_i a_j F_{pk} - a_i a_k F_{pj}. \tag{20}$$

This implies

$$a_l C_{pijk} + a_p C_{iljk} + a_i C_{lpjk} = 0 \tag{21}$$

and, in view of Lemma 2.3 also $C \cdot C = 0$. Taking into account the relation (3) and using (13) and (1) we have ($n = 4$)

$$(1 - 2L) Q(S, C) = \frac{\kappa}{12} Q(g, C). \tag{22}$$

This implies $\kappa = 0$ if and only if $L = \frac{1}{2}$. We assert that $\kappa \neq 0$ at x. Suppose that $\kappa = 0$ at x. Then $T = S = \frac{1}{\rho} a \otimes a$ and using (20) we easily obtain $Q(S,C) = 0$, a contradiction. Now (22) implies $Q(S,C) = \frac{\kappa}{12(1-2L)} Q(g,C)$ whence, in virtue of (1), we have $R \cdot C = \frac{\kappa L}{12(1-2L)} Q(g,C)$. But (M,g) is Ricci-pseudosymmetric with $L_S = \frac{\kappa}{12}$. So we get $\frac{\kappa L}{12(1-2L)} = \frac{\kappa}{12}$ and we have (iii). It is easy to see that for $L = \frac{1}{3}$ the equality (22) takes the form (iv).

Now we consider the second case:

(II) $\omega \neq 0$. Then

$$\begin{aligned} F_{ij} &= \frac{1}{\omega}\Big(T_{ij} - \frac{1}{\rho}a_i a_j\Big), \\ E_{ijk} &= \frac{1}{\rho\omega}(a_k T_{ij} - a_j T_{ik}). \end{aligned} \tag{23}$$

Substituting these relations into (18) we get

$$\begin{aligned} \rho^2\omega C_{lijk} = {} & \rho(T_{ik}T_{lj} - T_{ij}T_{lk}) \\ & + 2(a_l a_k T_{ij} - a_l a_j T_{ik} + a_i a_j T_{lk} - a_i a_k T_{lj}). \end{aligned} \tag{24}$$

Contracting (23) with g^{ij} we have $a_r a^r = 0$. Using this equality and (15), after contraction of (24) with g^{ij}, we get $a_i a^r T_{rj} + a_j a^r T_{ri} = 0$ which immediately implies $a^r T_{rj} = 0$. Transvecting now (24) with a^l and using the above equality we obtain $a^r C_{rijk} = 0$. This implies, by transvection of the Patterson's identity with a^h, $a_m C_{lijk} + a_j C_{likm} + a_k C_{limj} = 0$. Substituting (24) to this equality we have

$$a_m(T_{lk}T_{ij} - T_{lj}T_{ik}) + a_j(T_{lm}T_{ik} - T_{lk}T_{im}) + a_k(T_{lj}T_{im} - T_{lm}T_{ij}) = 0. \tag{25}$$

We assert that $\kappa \neq 0$ at x. Suppose that $\kappa = 0$ at x. Then $T = S$ and

$$\rho\omega C_{lijk} = (S_{ik}S_{lj} - S_{ij}S_{lk}) + \frac{2}{\rho}(a_l a_k S_{ij} - a_l a_j S_{ik} + a_i a_j S_{lk} - a_i a_k S_{lj}).$$

Using this relation and (25), after standard but somewhat lenghty calculations we obtain $Q(S, C) = 0$, a contradiction. The proof of (iii) and (iv) is the same as in the case (I). This completes the proof. □

The existence of manifolds satisfying all relations of the above theorem can be established (see [10], Example 5.1) as follows

Example 4.2. Let $(N, \tilde{g})$, $\dim N = 3$, be a semi-Riemannian manifold such that its Ricci tensor $\tilde{S}$ is of rank one and its scalar curvature $\tilde{\kappa}$ vanishes identically on N. An example of such manifold is presented in [10] (Example 5.1 (ii)). Furthermore, let F, defined by $F(x^1) = a\exp(bx^1)$,

$a = \text{const.} \neq 0$, $b = \text{const.} \neq 0$, be a function on a 1-dimensional manifold $(\overline{M}, g_1)$. It is shown in [10] that warped product $\overline{M} \times_F N$ satisfies (1) with $L = \frac{1}{3}$ and is a pseudosymmetric manifold with $L_R = \frac{\kappa}{12}$, where κ is the scalar curvature of $\overline{M} \times_F N$. Moreover, it is easy to verify that $\kappa = -3b^2 \neq 0$ and one can see that the condition (ii) of the last theorem is satisfied at every point x of $\overline{M} \times_F N$ by arbitrary covector a at x such that the only non-zero component of a is a_2.

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