

**On the mean value formula for the non-symmetric
form of the approximate functional equation
of $\zeta^2(s)$ in the critical strip**

By ISAO KIUCHI (Yamaguchi)

Abstract. The object of this paper is to derive the mean value formula of the error term $R^*\left(s; \frac{l}{k}\right)$ of the non-symmetric form in the approximate functional equation for $\zeta^2(s)$ in the critical strip $0 \leq \sigma \leq 1$.

1. Introduction

Let $s = \sigma + it$ ($0 \leq \sigma \leq 1$, $t \geq 1$) be a complex variable, $\zeta(s)$ the Riemann zeta-function, $d(n)$ the number of positive divisors of n , γ the Euler constant, k and l co-prime integers with $1 \leq l \leq k$. The error term $R^*(s; l/k)$ in the approximate functional equation for $\zeta^2(s)$ is defined by

$$\zeta^2(s) = \sum'_{n \leq \frac{lt}{2\pi k}} \frac{d(n)}{n^s} + \chi^2(s) \sum'_{n \leq \frac{kt}{2\pi l}} \frac{d(n)}{n^{1-s}} + R^*\left(s; \frac{l}{k}\right)$$

where

$$(1.1) \quad \chi(s) = 2^s \pi^{s-1} \sin\left(\frac{1}{2}\pi s\right) \Gamma(1-s),$$

and $\sum'_{n \leq y}$ indicates that the last term is to be halved if y is an integer. For $k \neq l$, this is called the “non-symmetric form” of the approximate functional equation for $\zeta^2(s)$. By using the method of MEURMAN [6] and

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the Motohashi formula (2.2) given below, the mean value formula of the function $|R^*(1/2 + it; l/k)|$ was first studied by KIUCHI [4], who obtained, for $kl \leq T(\log T)^{-20}$, the asymptotic formula

$$(1.2) \quad \int_1^T \left| R^* \left(\frac{1}{2} + it; \frac{l}{k} \right) \right|^2 dt = \sqrt{2\pi} C_{k,l} T^{1/2} + K_{k,l}(T)$$

with

$$(1.3) \quad K_{k,l}(T) = O \left((kl)^{3/4} T^{1/4} \log^3 T \right)$$

where

$$C_{k,l} = \sum_{n=1}^{\infty} \frac{d^2(n) H_{k,l}^2(n)}{\sqrt{n}}$$

and

$$(1.4) \quad H_{k,l}(n) = (kl)^{-1/4} \sqrt{\frac{2}{\pi}} \int_0^{\infty} \left(y + \frac{n\pi}{kl} \right)^{-1/2} \\ \times \cos \left(y + \left(\frac{\bar{k}}{l} + \frac{\bar{l}}{k} \right) n\pi + \frac{\pi}{4} \right) dy^{\dagger} \ll \frac{(kl)^{1/4}}{\sqrt{n}}.$$

Here the residue classes $\bar{k} \pmod{l}$ and $\bar{l} \pmod{k}$ are defined by $k\bar{k} \equiv 1 \pmod{l}$ and $l\bar{l} \equiv 1 \pmod{k}$, respectively. The purpose of this paper is to derive the mean value formula of the function $|R^*(s; l/k)|$ in the critical strip $0 \leq \sigma \leq 1$, and the basic tool is the non-symmetric form of the Motohashi formula (2.2) given below. The principle of the proof is the same as in KIUCHI [5], and the main result is

Theorem. For $1 \leq l \leq k$, $(k, l) = 1$, $kl \leq T(\log T)^{-20}$ and $T \geq 1$, we have

$$(1.5-6-7) \quad \int_1^T \left| R^* \left(s; \frac{l}{k} \right) \right|^2 dt \\ = \begin{cases} A_{k,l}(\sigma) T^{3/2-2\sigma} + O \left(\left(\frac{l}{k} \right)^{1-2\sigma} (kl)^{3/4} T^{5/4-2\sigma} \log^3 T \right) & \text{if } 0 \leq \sigma \leq 5/8, \\ A_{k,l}(\sigma) T^{3/2-2\sigma} + B_{k,l}(\sigma) + O \left(\left(\frac{l}{k} \right)^{1-2\sigma} (kl)^{3/4} T^{5/4-2\sigma} \log^3 T \right) & \text{if } 5/8 < \sigma \leq 1, \\ \pi \sqrt{\frac{k}{l}} C_{k,l} \log T + B_{k,l} \left(\frac{3}{4} \right) + O(k^{5/4} l^{1/4} T^{-1/4} \log^3 T) & \text{if } \sigma = 3/4, \end{cases}$$

[†]The author corrects a misprint of the function $H_{k,l}(n)$ in [4].

where

$$A_{k,l}(\sigma) = \frac{(2\pi)^{2\sigma-1/2}}{3-4\sigma} \left(\frac{l}{k}\right)^{1-2\sigma} C_{k,l}$$

and $B_{k,l}(\sigma)$ is a certain constant.

Corollary. For $1 \leq l \leq k$, $(k, l) = 1$, $kl \leq t(\log t)^{-20}$ and $t \geq 2$, this theorem includes the fact that

$$\left| R^* \left(s; \frac{l}{k} \right) \right| = \begin{cases} \Omega \left(\left(\frac{l}{k} \right)^{1/2-\sigma} C_{k,l}^{1/2} t^{1/4-\sigma} \right) & \text{if } 0 \leq \sigma < 3/4, \\ \Omega \left(\left(\frac{k}{l} \right)^{1/4} C_{k,l}^{1/2} t^{-1/2} \sqrt{\log t} \right) & \text{if } \sigma = 3/4. \end{cases}$$

Comparing (1.6) and (1.7), we observe that the line $\sigma = 3/4$ has a kind of critical property. This is a situation similar to the case of the error term $R(s; t/(2\pi))$ in the “symmetric form” of the approximate functional equation for $\zeta^2(s)$, which is defined by

$$\zeta^2(s) = \sum'_{n \leq \frac{t}{2\pi}} \frac{d(n)}{n^s} + \chi^2(s) \sum'_{n \leq \frac{t}{2\pi}} \frac{d(n)}{n^{1-s}} + R \left(s; \frac{t}{2\pi} \right)$$

for a fixed number σ ($0 \leq \sigma \leq 1$). KIUCHI and MATSUMOTO [3] first showed that

$$(1.8) \quad \int_1^T \left| R \left(\frac{1}{2} + it; \frac{t}{2\pi} \right) \right|^2 dt = \sqrt{2\pi} C T^{1/2} + K(T)$$

with $K(T) = O(T^{1/4} \log T)$, and in [5], the improvement $K(T) = O(\log^4 T)$ has recently proved by KIUCHI, where

$$C = \sum_{n=1}^{\infty} \frac{d^2(n) h^2(n)}{\sqrt{n}}$$

and

$$h(n) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} (y + n\pi)^{-1/2} \cos \left(y + \frac{\pi}{4} \right) dy.$$

Further, a simple argument to deduce sharp results on the mean square of $|R(s; t/(2\pi))|$ was discovered by KIUCHI [5], who proved, for $0 \leq \sigma \leq 1$, the asymptotic formula

$$(1.9) \quad \int_1^T \left| R\left(s; \frac{t}{2\pi}\right) \right|^2 dt = \begin{cases} A_1(\sigma)T^{3/2-2\sigma} + O(T^{1-2\sigma} \log^4 T) & \text{if } 0 \leq \sigma \leq 1/2, \\ A_1(\sigma)T^{3/2-2\sigma} + A_2(\sigma) + O(T^{1-2\sigma} \log^4 T) & \text{if } 1/2 < \sigma \leq 1, \sigma \neq 3/4, \\ \pi C \log T + A_2\left(\frac{3}{4}\right) + O(T^{-1/2} \log^4 T) & \text{if } \sigma = 3/4, \end{cases}$$

with a certain constant $A_2(\sigma)$, where

$$A_1(\sigma) = \frac{(2\pi)^{2\sigma-1/2}}{3-4\sigma} C.$$

From (1.9), KIUCHI has observed, as already pointed out in [5], that the line $\sigma = 3/4$ is a kind of “critical line” in the theory of the Riemann zeta-function, or at least for the function $R(s; t/(2\pi))$. Our theorem indicates that the similar critical property on the line $\sigma = 3/4$ appears in the mean value formulas of more generalized quantity $R^*(s; l/k)$. Comparing (1.5)–(1.7) and (1.9), one may formulate the following

Conjecture. For $0 \leq \sigma \leq 1$, $1 \leq l \leq k$, $(k, l) = 1$, and $kl \leq T(\log T)^{-20}$, the error term $O\left(\left(\frac{l}{k}\right)^{1-2\sigma} (kl)^{3/4} T^{5/4-2\sigma} \log^3 T\right)$ in Theorem can be replaced by

$$O\left(\left(\frac{l}{k}\right)^{1-2\sigma} (kl)^{3/4} T^{1-2\sigma} \log^4 T\right).$$

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2. Application of the Motohashi formula

Let a and b be integers with $a \geq 1$ and $(a, b) = 1$. For $x \geq 1$, we put

$$(2.1) \quad \Delta\left(x; \frac{b}{a}\right) = \sum'_{n \leq x} d(n) e\left(\frac{b}{a}n\right) - \frac{x}{a} \left(\log \frac{x}{a^2} + 2\gamma - 1\right) - E\left(0; \frac{b}{a}\right)$$

where $e(\alpha) = \exp(2\pi i\alpha)$, and $E(0; b/a)$ is the value at $s = 0$ of the analytic continuation of

$$E\left(s; \frac{b}{a}\right) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} e\left(\frac{b}{a}n\right)$$

which is first defined for $\operatorname{Re} s > 1$. Our starting point is the following “non-symmetric form” of the Riemann–Siegel formula for $\zeta^2(s)$, which was proved by MOTOHASHI [8; Theorem 7] (see also [7]):

For $t \geq 2$ and $0 \leq \sigma \leq 1$, we have, uniformly for $kl \leq t(\log t)^{-20}$,

$$(2.2) \quad \chi(1-s)R^*\left(s; \frac{l}{k}\right) = M\left(s; \frac{l}{k}\right) + \overline{M\left(1-\bar{s}; \frac{k}{l}\right)} \\ + O\left(\left(\frac{l}{k}\right)^{1/2-\sigma} \left(\frac{kl}{t}\right)^{1/2} \log^3 t\right),$$

where

$$(2.3) \quad M\left(s; \frac{l}{k}\right) = -e\left(-\frac{1}{8}\right) \left(\frac{t}{2\pi}\right)^{-1/2} \left(\frac{l}{k}\right)^{-s} \Delta\left(\frac{lt}{2\pi k}; -\frac{k}{l}\right) \\ + \frac{1}{2} e\left(-\frac{1}{8}\right) \left(\frac{kl}{2\pi t}\right)^{1/4} \left(\frac{l}{k}\right)^{1/2-s} \sum_{n=1}^{\infty} \frac{d(n)}{n^{1/4}} e\left(\frac{\bar{k}}{l}n\right) \\ \times \sin\left(2\sqrt{\frac{2\pi tn}{kl}} + \frac{\pi}{4}\right) \int_0^{\infty} (\xi + n\pi)^{-3/2} \exp\left(\frac{i\xi}{kl}\right) d\xi.$$

JUTILA [2] (see also (2.6.7) of [8]) proved the following formula, which is an analogue of the Voronoï formula for (2.1):

$$\Delta\left(x; \frac{b}{a}\right) = \frac{a^{1/2}x^{1/4}}{\pi\sqrt{2}} \sum_{n=1}^{\infty} \frac{d(n)}{n^{3/4}} e\left(-\frac{\bar{b}}{a}n\right) \cos\left(4\pi\frac{\sqrt{nx}}{a} - \frac{\pi}{4}\right) + O(a^{3/2}x^{-1/4}),$$

where $x \geq a^2(\log 2a)^3$, and the residue class $\bar{b} \pmod{a}$ is defined by $b\bar{b} \equiv 1 \pmod{a}$. Applying this formula to (2.3) and using integration by parts, we have

$$(2.4) \quad M\left(s; \frac{l}{k}\right) = \frac{i}{\sqrt{2\pi k}} \left(\frac{l}{k}\right)^{1/4-s} \left(\frac{t}{2\pi}\right)^{-1/4} \sum_{n=1}^{\infty} \frac{d(n)}{n^{1/4}} e\left(\frac{\bar{k}}{l}n\right) \sin\left(2\sqrt{\frac{2\pi tn}{kl}} + \frac{\pi}{4}\right) \times \int_0^{\infty} \left(\xi + \frac{n\pi}{kl}\right)^{-1/2} \exp\left(i\left(\xi - \frac{\pi}{4}\right)\right) d\xi + O(k^{1/4+\sigma}l^{5/4-\sigma}t^{-3/4}),$$

and

$$(2.5) \quad \overline{M\left(1-s; \frac{k}{l}\right)} = \frac{-i}{\sqrt{2\pi l}} \left(\frac{l}{k}\right)^{3/4-s} \left(\frac{t}{2\pi}\right)^{-1/4} \sum_{n=1}^{\infty} \frac{d(n)}{n^{1/4}} e\left(-\frac{\bar{l}}{k}n\right) \times \sin\left(2\sqrt{\frac{2\pi tn}{kl}} + \frac{\pi}{4}\right) \int_0^{\infty} \left(\xi + \frac{n\pi}{kl}\right)^{-1/2} \exp\left(-i\left(\xi - \frac{\pi}{4}\right)\right) d\xi + O(k^{1/4+\sigma}l^{5/4-\sigma}t^{-3/4})$$

for $t \geq 2\pi kl(\log 2k)^3$. Substituting (2.4) and (2.5) into (2.2), we obtain, for $kl \leq t(\log t)^{-20}$ and $t \geq 2$,

$$\chi(1-s)R^*\left(s; \frac{l}{k}\right) = \left(\frac{l}{k}\right)^{1/2-\sigma} \left\{ \left(\frac{k}{l}\right)^{it} \left(\frac{t}{2\pi}\right)^{-1/4} \sum_{n=1}^{\infty} \frac{d(n)}{n^{1/4}} e\left(\frac{1}{2}\left(\frac{\bar{k}}{l} - \frac{\bar{l}}{k}\right)n\right) \times \sin\left(2\sqrt{\frac{2\pi tn}{kl}} + \frac{\pi}{4}\right) H_{k,l}(n) + O\left(\left(\frac{kl}{t}\right)^{1/2} \log^3 t\right) \right\}$$

where

$$H_{k,l}(n) = (kl)^{-1/4} \sqrt{\frac{2}{\pi}} \int_0^\infty \left(y + \frac{n\pi}{kl}\right)^{-1/2} \cos\left(y + \left(\frac{\bar{k}}{l} + \frac{\bar{l}}{k}\right)\pi n + \frac{\pi}{4}\right) dy.$$

Put $\sigma = 1/2$ in this formula, and compare it with the above. Then, using the relation $\chi(1-s)\chi(s) = 1$, we have

$$R^*\left(s; \frac{l}{k}\right) = \left(\frac{l}{k}\right)^{1/2-\sigma} \chi(s) \left\{ \chi\left(\frac{1}{2} - it\right) R^*\left(\frac{1}{2} + it; \frac{l}{k}\right) + E_{k,l}(t) \right\}$$

with

$$(2.6) \quad E_{k,l}(t) = O\left(\left(\frac{kl}{t}\right)^{1/2} \log^3 t\right).$$

Therefore, we obtain the following

Lemma. For $0 \leq \sigma \leq 1$, $1 \leq l \leq k$, $(k, l) = 1$ and $kl \leq t(\log t)^{-20}$, we have

$$(2.7) \quad \left| R^*\left(s; \frac{l}{k}\right) \right|^2 = \left(\frac{l}{k}\right)^{1-2\sigma} |\chi(s)|^2 \left\{ \left| R^*\left(\frac{1}{2} + it; \frac{l}{k}\right) \right|^2 + F_{k,l}(t) \right\}$$

where

$$(2.8) \quad F_{k,l}(t) \ll \left| R^*\left(\frac{1}{2} + it; \frac{l}{k}\right) \right| |E_{k,l}(t)| + |E_{k,l}(t)|^2.$$

3. Proof of the Theorem

It follows from the asymptotic formula (see (1.25) of Ivić [1]) of (1.1) that

$$(3.1) \quad |\chi(s)|^2 = \left(\frac{t}{2\pi}\right)^{1-2\sigma} + G_\sigma(t) \quad (t \geq t_0 > 0)$$

with

$$(3.2) \quad G_\sigma(t) = O(t^{-2\sigma}).$$

From (2.7), we have

$$(3.3) \quad \int_1^T \left| R^*\left(s; \frac{l}{k}\right) \right|^2 dt = \left(\frac{l}{k}\right)^{1-2\sigma} \left\{ I_1(1, T) + I_2(1, T) \right\},$$

where

$$I_1(T_1, T_2) = \int_{T_1}^{T_2} |\chi(s)|^2 \left| R^* \left(\frac{1}{2} + it; \frac{l}{k} \right) \right|^2 dt,$$

and

$$I_2(T_1, T_2) = \int_{T_1}^{T_2} |\chi(s)|^2 F_{k,l}(t) dt.$$

Hereafter we assume that $T_1 < T_2 \leq 2T_1$. Applying (1.2), (3.1) and integrating by parts, we have, for $\sigma \neq 3/4$,

$$\begin{aligned} (3.4) \quad I_1(T_1, T_2) &= \frac{(2\pi)^{2\sigma-1/2}}{3-4\sigma} C_{k,l} t^{3/2-2\sigma} + \left(\frac{t}{2\pi} \right)^{1-2\sigma} K_{k,l}(t) \Big|_{T_1}^{T_2} \\ &\quad + (2\pi)^{2\sigma-1} (2\sigma-1) \int_{T_1}^{T_2} t^{-2\sigma} K_{k,l}(t) dt \\ &\quad + \int_{T_1}^{T_2} G_\sigma(t) \left| R^* \left(\frac{1}{2} + it; \frac{l}{k} \right) \right|^2 dt. \end{aligned}$$

From (1.3), we have

$$\int_{T_1}^{T_2} t^{-2\sigma} K_{k,l}(t) dt = O((kl)^{3/4} T_1^{5/4-2\sigma} \log^3 T_1).$$

From (1.2), (1.3), (1.4), (3.2) and $kl \leq t(\log t)^{-20}$, we obtain

$$\begin{aligned} \int_{T_1}^{T_2} G_\sigma(t) \left| R^* \left(\frac{1}{2} + it; \frac{l}{k} \right) \right|^2 dt &\ll \max_{T_1 \leq t \leq T_2} |G_\sigma(t)| \int_{T_1}^{T_2} \left| R^* \left(\frac{1}{2} + it; \frac{l}{k} \right) \right|^2 dt \\ &\ll (kl)^{1/2} T^{1/2-2\sigma}. \end{aligned}$$

Hence we obtain, for $0 \leq \sigma \leq 5/8$,

$$(3.5) \quad I_1(1, T) = \frac{(2\pi)^{2\sigma-1/2}}{3-4\sigma} C_{k,l} T^{3/2-2\sigma} + O((kl)^{3/4} T^{5/4-2\sigma} \log^3 T),$$

and for $5/8 < \sigma \leq 1$ ($\sigma \neq 3/4$),

$$(3.6) \quad I_1(1, T) = \frac{(2\pi)^{2\sigma-1/2}}{3-4\sigma} C_{k,l} T^{3/2-2\sigma} + (2\pi)^{2\sigma-1} T^{1-2\sigma} K_{k,l}(T) \\ + (2\pi)^{2\sigma-1} (2\sigma-1) \int_1^\infty t^{-2\sigma} K_{k,l}(t) dt \\ + \int_1^\infty G_\sigma(t) \left| R^* \left(\frac{1}{2} + it; \frac{l}{k} \right) \right|^2 dt \\ + c_1(\sigma; k, l) + O((kl)^{3/4} T^{5/4-2\sigma} \log^3 T),$$

where the constant $c_1(\sigma; k, l)$ depends on σ , k and l . Similarly in case $\sigma = 3/4$, we obtain, from (1.2), (3.1) and integration by parts,

$$(3.7) \quad I_1(1, T) = \pi C_{k,l} \log T + \sqrt{2\pi} T^{-1/2} K_{k,l}(T) \\ + \sqrt{\frac{\pi}{2}} \int_1^\infty t^{-3/2} K_{k,l}(t) dt + \int_1^\infty G_{3/4}(t) \left| R^* \left(\frac{1}{2} + it; \frac{l}{k} \right) \right|^2 dt \\ + c_1\left(\frac{3}{4}; k, l\right) + O((kl)^{3/4} T^{-1/4} \log^3 T).$$

From (1.2), (1.3), (1.4), (2.6), (2.8), (3.1), (3.2) and Schwarz's inequality we have

$$I_2(T_1, T_2) \ll \left(\int_{T_1}^{T_2} |\chi(s)|^2 \left| R^* \left(\frac{1}{2} + it; \frac{l}{k} \right) \right|^2 dt \right)^{1/2} \left(\int_{T_1}^{T_2} |\chi(s)|^2 |E_{k,l}(t)|^2 dt \right)^{1/2} \\ + \int_{T_1}^{T_2} |\chi(s)|^2 |E_{k,l}(t)|^2 dt \\ \ll (kl)^{3/4} T_1^{5/4-2\sigma} \log^3 T_1.$$

Hence we have

$$(3.8) \quad I_2(1, T) \\ = \begin{cases} O((kl)^{3/4} T^{5/4-2\sigma} \log^3 T) & \text{if } 0 \leq \sigma \leq 5/8, \\ I_2(1, \infty) + O((kl)^{3/4} T^{5/4-2\sigma} \log^3 T) & \text{if } 5/8 < \sigma \leq 1. \end{cases}$$

Substituting (3.5) and (3.8) into (3.3), we obtain, for $0 \leq \sigma \leq 5/8$,

$$\begin{aligned} \int_1^T \left| R^* \left(s; \frac{l}{k} \right) \right|^2 dt &= \frac{(2\pi)^{2\sigma-1/2}}{3-4\sigma} \left(\frac{l}{k} \right)^{1-2\sigma} C_{k,l} T^{3/2-2\sigma} \\ &\quad + O \left(\left(\frac{l}{k} \right)^{1-2\sigma} (kl)^{3/4} T^{5/4-2\sigma} \log^3 T \right). \end{aligned}$$

Similarly in case $5/8 < \sigma \leq 1$, we obtain

$$\begin{aligned} &\int_1^T \left| R^* \left(s; \frac{l}{k} \right) \right|^2 dt \\ &= \begin{cases} \frac{(2\pi)^{2\sigma-1/2}}{3-4\sigma} \left(\frac{l}{k} \right)^{1-2\sigma} C_{k,l} T^{3/2-2\sigma} + B_{k,l}(\sigma) \\ \quad + O \left(\left(\frac{l}{k} \right)^{1-2\sigma} (kl)^{3/4} T^{5/4-2\sigma} \log^3 T \right) & \text{if } \sigma \neq 3/4, \\ \pi \sqrt{\frac{k}{l}} C_{k,l} \log T + B_{k,l} \left(\frac{3}{4} \right) + O(k^{5/4} l^{1/4} T^{-1/4} \log^3 T) & \text{if } \sigma = 3/4. \end{cases} \end{aligned}$$

with a certain constant $B_{k,l}(\sigma)$. Therefore now we have the assertion of Theorem.

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ISAO KIUCHI
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
YAMAGUCHI UNIVERSITY
YOSHIDA 1677-1, YAMAGUCHI, 753-8512
JAPAN

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