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A Hosszú-like functional equation

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Abstract. The equation $f(x+y)+f(xy)=f(xy+x)+f(y)$ is solved over both the natural numbers and the integers.

1. Introduction

Hosszú's equation [2] is

$$f(x+y-xy)+f(xy)=f(x)+f(y), \tag{1}$$

where the domain of f is understood to be a ring, and the codomain is an abelian group. When the domain in equation (1) is a field with at least 5 elements then Hosszú's equation is equivalent to Cauchy's equation [5], in the sense that any solution of the one is a solution of the other. Of course, Cauchy's equation in its affine form is

$$f(x+y)+f(0)=f(x)+f(y). \tag{2}$$

A few years ago [3] I mentioned that the equation

$$f(x+y)+f(xy)=f(xy+x)+f(y) \tag{3}$$

might be worthy of discussion. In equation (3) the domain is a subset of a ring, the variables enter in a bilinear way as they do in equation (1), and any solution of equation (2) is a solution of equation (3). Hence the last equation is Hosszú-like. I proved (unpublished) that any solution

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of equation (3) where the domain of f is the rational field must satisfy equation (2): again making the equation Hosszú-like. At ISFE 17 (1979) W. Benz proved that every continuous solution of equation (3) (the domain being the real field) is indeed an affine Cauchy function.

Recently Girgensohn and Lajkó ([6], to appear) showed that over a field of characteristic not 2 or 3 any solution of equation (3) is a solution of equation (2).

Indeed Lajkó [8] has shown that if f satisfies equation (3) then $g : x \longmapsto f(-x)$ satisfies equation (1). This implies in particular that when the domain of equation (3) is a field with at least 5 elements all solutions must satisfy equation (2).

In this paper I solve equation (3) in two, I believe, interesting cases; Firstly when the domain of f is $\mathbb{N} = \{1, 2, 3, \dots\}$ the set of natural numbers and secondly when the domain is $\mathbb{Z}$, the ring of (rational) integers. Basically what I show is that every solution is a linear combination (in the codomain) of four functions:

$$\chi_2^0 : x \mapsto \begin{cases} 1 & \text{if } x \text{ is even} \\ 0 & \text{if } x \text{ is odd} \end{cases} \qquad \chi_2^1 : x \mapsto \begin{cases} 0 & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd} \end{cases}$$

$$q_2 : x \mapsto q_2(x) \quad \text{where } x = 2q_2(x) + r_2(x) \text{ and } r_2(x) \in \{0, 1\}$$

$$q_3 : x \mapsto q_3(x) \quad \text{where } x = 3q_3(x) + r_3(x) \text{ and } r_3(x) \in \{0, 1, 2\}\,.$$

See also my paper ([6], to appear).

2. Linearizing the equation

Let $f : S \to H$. Here S is either $\mathbb{N}$ or $\mathbb{Z}$ and H is an abelian group written additively.

Proposition 1. *If $f : S \to H$ satisfies equation* (3) *then for each $x \in S$*

$$f(2x) = f(x+1) + f(x) - f(1) \tag{4}$$

and

$$f(2x+1) = f(x+3) + f(x+2) - f(3) - f(2) + f(1). \tag{5}$$

Proof. Put $y = 1$ in equation (3) to deduce (4). With $y = 2, 3$ in equation (3) we deduce that

$$f(2x) + f(x+2) = f(3x) + f(2) \tag{6}$$

$$f(3x) + f(x+3) = f(4x) + f(3) \tag{7}$$

for all $x \in S$.

Now from (4) we deduce that

$$f(4x) = f(2x+1) + f(2x) - f(1)$$

so substituting this in (7) yields

$$f(3x) + f(x+3) = f(2x+1) + f(2x) - f(1) + f(3).$$

Finally using equation (6) we deduce (5). □

We notice that $f(2x+4)$ and $f(2x+1)$ differ by a constant $(f(3) + f(2) - 2f(1))$: so some 3-periodicity has already surfaced. It is this phenomenon that inspires our next step.

Definition. Suppose $f : S \to H$ satisfies equation (3). For each $x \in S$ we set

$$\hat{f}(x) := f(x+4) + f(x+3) - f(x+1) - f(x), \tag{8}$$

so, $\hat{f} : S \to H$ too.

Proposition 2. *If f satisfies equation* (3) *then, for each $x \in S$*

$$\hat{f}(2x) = \hat{f}(x) \tag{9}$$

and

$$\hat{f}(2x+1) = \hat{f}(x+1). \tag{10}$$

Proof. Immediate. □

Corollary.

(i) *If* $S = \mathbb{N}$ *then, for all* $x \in S$

$$\hat{f}(x) = \hat{f}(1). \tag{11}$$

(ii) *If* $S = \mathbb{Z}$, *then for all* $x \in S$,

$$\hat{f}(x) = \begin{cases} \hat{f}(1) & x \in \mathbb{N}, \\ \hat{f}(0) & x \notin \mathbb{N}. \end{cases} \tag{12}$$

Proof. Use induction $x \to x+1$ when $x \in \mathbb{N}$ and $x \mapsto x-1$ when $x \notin \mathbb{N}$, and the proposition. □

We need the next result to complete our exploration of (some) linear consequences of equation (3).

Proposition 3. *If* $f : \mathbb{Z} \to H$ *satisfies equation* (3) *then*

$$f(5) + f(0) = f(3) + f(2). \tag{13}$$

Proof. Put $x = 6$, $y = -1$ in equation (3):

$$f(-6) + f(5) = f(0) + f(-1). \tag{14}$$

Now

$$\begin{aligned} f(-6) &= f(-2) + f(-3) - f(1) \\ &= f(0) + f(-1) - f(1) + f(1) + f(0) - f(3) \\ &\quad - f(2) + f(1) - f(1), \end{aligned}$$

so

$$f(-6) = 2f(0) + f(-1) - f(3) - f(2) \tag{15}$$

using equations (4) and (5) to expand $f(-2)$, $f(-3)$ respectively. Substituting (15) into (14) and simplifying we deduce equation (13). □

Corollary. *If* $f : \mathbb{Z} \to H$ *satisfies equation* (3) *then*

$$\hat{f}(1) = \hat{f}(0). \tag{16}$$

Proof. $\hat{f}(1) = \hat{f}(0)$ iff $f(5) + f(4) - f(2) - f(1) = f(4) + f(3) - f(1) - f(0)$ iff $f(5) + f(0) = f(3) + f(2)$. □

We can now state and prove the main result of this section.

Theorem 4. *Suppose* $f : S \to H$ *satisfies equation* (3) *where* S *is* $\mathbb{N}$ *or* $\mathbb{Z}$. *Then*

$$\text{(a)} \quad f = 0 \quad \textit{if } f(1) = 0,\ f(2) = 0,\ f(3) = 0, \quad \textit{and } f(5) = 0$$

and, for all $x \in S$

$$\text{(17)} \quad \text{(b)} \quad f(x+6) - f(x) = \hat{f}(1).$$

Proof. (a) Assume $f(1) = 0$, $f(2) = 0$, $f(3) = 0$ and $f(5) = 0$. Then $f(4) = (f(3) + f(2) - f(1)) = 0$ also, and so $\hat{f}(x) = \hat{f}(1) = 0$ for all $x \in \mathbb{N}$. If $S = \mathbb{N}$ this yields $f(x) = 0$ for all $x \in \mathbb{N}$. If $S = \mathbb{Z}$ we see that $\hat{f}(x) = \hat{f}(0) = 0$ (by the Corollary to Proposition 3) for all $x \in \mathbb{Z} \backslash \mathbb{N}$. Again this yields $f(x) = 0$ for all $x \notin \mathbb{N}$.

(b) Since $\hat{f}(1) = \hat{f}(0)$ we deduce that

$$\hat{f}(x) = \hat{f}(1) \quad \text{for all } x \in S.$$

So $\hat{f}(x+1) - \hat{f}(x) = 0$, for all $x \in S$

$$f(x+5) - f(x+3) - f(x+2) + f(x) = 0. \tag{18}$$

Thus, replacing x by $x+1$,

$$f(x+6) - f(x+4) - f(x+3) + f(x+1) = 0.$$

Rewriting this we deduce that

$$f(x+6) - [f(x+4) + f(x+3) - f(x+1) - f(x)] - f(x) = 0.$$

So $f(x+6) - \hat{f}(1) - f(x) = 0$, as claimed. □

From part (b) of the theorem we see that 2-periodicity and 3-periodicity are consequences of equation (3).

3. Solving the equation

Let $g : S \to H$, define $G : S^2 \to H$

$$G(a,b) := g(ab) + g(a+b) - g(ab+a) - g(b). \tag{19}$$

Definition. $(a,b) \in S^2$ is *admissible* (for equation (3)) iff $G(a,b) = 0$.

Thus g satisfies equation (3) if, and only if S^2 is admissible.

Proposition 5. *Let $p \in \mathbb{N}$, $\alpha \in H$. Suppose*

$$g : S \to H \quad \textit{satisfies}$$

$$g(x+p) = g(x) + \alpha \tag{20}$$

for all $x \in S$. Then g satisfies equation (3) *if and only if*

$$G(a,b) = 0, \qquad 1 \le a \le p \quad \textit{and} \quad 1 \le b \le p. \tag{21}$$

Proof. The necessity of condition (21) is clear. For the sufficiency, first we note that for all $t \in S$

$$g(x+tp) = g(x) + t\alpha. \tag{22}$$

Next we note that given $x, y \in S$ there are $a, b \in S$ with $1 \le a \le p$ and $1 \le b \le p$ such that $x = a + sp$, $y = b + tp$ where $s, t \in S \cup \{0\}$. Now

$$\begin{aligned} G(x,y) &= g(xy) + g(x+y) - g(xy+x) - g(y) \\ &= g(ab + (at + bs + stp)p) + g(a + b + (s+t)p) \\ &\quad - g(ab + a + (at + bs + stp + s)p) - g(b + tp) \\ &= g(ab) + (at + bs + stp)\alpha + g(a+b) + (s+t)\alpha \\ &\quad - g(ab+a) - (at + bs + stp + s)\alpha - g(b) - t\alpha = G(a,b). \end{aligned}$$

So

$$G(x,y) = 0 \qquad \forall (x,y) \in S^2,$$

if, and ony if

$$G(a,b) = 0 \qquad \forall (a,b) \in [1,p]^2 . \qquad \square$$

We now search for solutions when $p = 2$ and $p = 3$, and $H = \mathbb{Z}$.

Proposition 6.

(a) χ_2^0, χ_2^1 *and* q_2 *all satisfy equation* (3).

(b) q_3 *satisfies equation* (3).

Proof. (a) Suppose $g(x+2) = g(x) + \alpha$. Then $G(1,1) = 0$, $G(1,2) = 0$, $G(2,1) = g(2) + g(3) - g(4) - g(1) = g(2) + g(1) + \alpha - g(2) - \alpha - g(1) = 0$, and

$G(2,2) = g(4)+g(4)-g(6)-g(2) = g(2)+\alpha+g(2)+\alpha-g(2)-2\alpha-g(2) = 0$. So every function that is 2-periodic on S to $\mathbb{Z}$ satisfies equation (3). But these functions are of the form

$$g(x) = \chi_2^0(x)\,[g(1)+g(2)-g(3)] + \chi_2^1(x)\,[g(1)] + q_2(x)\,[g(3)-g(1)]$$

and $\alpha = g(3)-g(1)$, as it must.

(b) Let $g : S \to \mathbb{Z}$ satisfy, for all $x \in S$

$$g(x+3) = g(x) + \alpha.$$

Then $G(1,b) = 0$ for all b, and

$$\begin{aligned} G(3,b) &= g(3b) + g(3+b) - g(3b+3) - g(b) \\ &= g(3) + (b-1)\alpha + g(b) + \alpha - g(3) - b\alpha - g(b) = 0. \end{aligned}$$

Now

$$\begin{aligned} G(2,1) &= g(2) + g(3) - g(4) - g(1) \\ &= g(2) + g(3) - g(1) - \alpha - g(1) = g(3) + g(2) - 2g(1) - \alpha, \\ G(2,2) &= g(4) + g(4) - g(6) - g(2) \\ &= 2g(1) + 2\alpha - g(3) - \alpha - g(2) = -\left[g(3) + g(2) - 2g(1) - \alpha\right] \end{aligned}$$

and $G(2,3) = 0$. So g satisfies equation (3) if, and only if $\alpha = g(3)+g(2)-2g(1)$; that is

$$g(x+3) = g(x) + g(3) + g(2) - 2g(1).$$

Now $q_3(x+3) = q_3(x) + 1$ and $1 = q_3(3) + q_3(2) - 2q_3(1)$. Thus q_3 is a solution, as claimed. □

Putting the above proposition and Theorem 4 together we have:

Theorem 7. *Let S be $\mathbb{N}$ or $\mathbb{Z}$, and H an abelian group written additively. Then $f : S \to H$ satisfies equation* (3) *if and only if there are elements h_1, h_2, h_3 and h_4 in H such that*

$$f(x) = \chi_2^0(x)h_1 + \chi_2^1(x)h_2 + q_2(x)h_3 + q_3(x)h_4$$

for all $x \in S$.

Proof. Clearly, by Proposition 6 each of

$$x \mapsto \chi_2^0(x)h_1, \quad x \mapsto \chi_2^1(x)h_2, \quad x \mapsto q_2(x)h_3, \quad x \mapsto q_3(x)h_4$$

satisfies equation (3). So does their sum hence the 'if' part is proved.

Now suppose $f : S \to H$ satisfies equation (3). Define

$$h_1 := f(2) + f(3) - f(5) \qquad h_2 := f(1)$$
$$h_3 := f(5) - f(3) \qquad h_4 := 2f(3) - f(1) - f(5)$$

and consider

$$\tilde{f}(x) := f(x) - \chi_2^0(x)h_1 - \chi_2^1(x)h_2 - q_2(x)h_3 - q_3(x)h_4.$$

Then $\tilde{f}$ satisfies equation (3). Moreover

$$\tilde{f}(1) = f(1) - h_2 = 0$$
$$\tilde{f}(2) = f(2) - h_1 - h_3 = f(2) - f(2) - f(3) + f(5) - f(5) + f(3) = 0$$
$$\begin{aligned}\tilde{f}(3) &= f(3) - h_2 - h_3 - h_4 \\ &= f(3) - f(1) - f(5) + f(3) - 2f(3) + f(1) + f(5) = 0\end{aligned}$$
$$\begin{aligned}\tilde{f}(5) &= f(5) - h_2 - 2h_3 - h_4 \\ &= f(5) - f(1) - 2f(5) + 2f(3) - 2f(3) + f(1) + f(5) = 0.\end{aligned}$$

So $\tilde{f}$ is the zero function by Theorem 4 part (a). Hence f has the form claimed. □

4. Concluding remarks

What is interesting about the linearization of equation (3) is that it leads to "coupled" equations (4) and (5) whose solutions are very different on $\mathbb{N}$ and $\mathbb{Z}$. A non-zero $\mathbb{Z}$-solution of equations (4) and (5) can be identically zero on $\mathbb{N}$. This phenomenon is worthy of further investigation.

Finally the Pexiderization of equation (3) over $\mathbb{N}$ and $\mathbb{Z}$ is also not without interest.

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