

On some properties of simplices in spaces of constant curvature

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Abstract. Let $k \geq 2$ and $d \geq k + 2$. MARTINI [MH93] proved for $k = 2$ that a d -dimensional simplex S_d in spaces of constant curvature is regular if the 2-faces of S_d are congruent. We shall prove analogous theorems for $k = 3$ and $k = 4$ and show that a similar statement for $d = k + 1$ is false.

1. Introduction

1.1. A tetrahedron S_3 is called isosceles if all 2-faces of S_3 are congruent. Isosceles tetrahedra have interesting properties in Euclidean space R^3 . The following analogous equivalent statements for a tetrahedron S_3 in 3-dimensional spherical and hyperbolic space was proved by the author [HJ69, HJNL96].

- 1.1.1. The tetrahedron S_3 is an isosceles tetrahedron.
- 1.1.2. The 2-faces of S_3 have equal areas.
- 1.1.3. The stereoangles at the vertices of S_3 are congruent.
- 1.1.4. The measures of the above stereoangles are equal.
- 1.1.5. The circumcircles of the 2-faces of S_3 have the same radius.

Let O , K and L denote the midpoint of the circumscribed ball, the midpoint of the inscribed ball and the centroid of S_3 .

- 1.1.6. At least two of the points O , K and L coincide.
- 1.1.7. The perimeters of the faces of S_3 are equal.
- 1.1.8. The perimeters of the vertex figures of S_3 are equal.

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BUI VAN DUNG [BVD84] proved the majority of the above properties for tetrahedra in hyperbolic 3-space whose vertices are ideal.

1.2. H. MARTINI [MH93] proved that the following four properties of a d -simplex in spaces of constant curvature (R^d, S^d, H^d) for $d \geq 4$ are equivalent.

1.2.1. The simplex S_d is regular.

1.2.2. The 2-faces of S_d have equal areas.

1.2.3. The 2-faces of S_d are congruent.

1.2.4. The measures of the four stereoangles of each 3-face of S_d are equal to each other.

As a consequence of the results in [HJ69] and [HJNL96] it is sufficient to show that the statements 1.2.1 and 1.2.3 are equivalent. These results imply the equivalences of the statements according to 1.1.5–1.1.8.

Analogous results were derived in [FPMH90] for Euclidean simplices.

Let us consider a d -simplex S_d in R^d, S^d, H^d . An edge of length a is called an a -edge. Let $(0, 1, 2, \dots, n)$ denote the simplex with vertices $0, 1, 2, \dots, n$.

2. Results

The 3-faces of S_d are congruent isosceles tetrahedra in cases 1.2.2–1.2.4. The result of MARTINI can be formulated as follows. If the 3-faces of S_d are congruent isosceles tetrahedra for $d \geq 4$, then S_d is regular.

Theorem 1. *If the 3-faces of a d -simplex S_d in d -dimensional spaces of constant curvature for $d \geq 5$ are congruent, then S_d is regular.*

PROOF. It is sufficient to prove the statement for $d = 5$. The number of the edges of S_5 is 15.

Let $01 = a$. There exists an a -edge in (2345) too. Let $23 = a$. Then (0123) has two skew a -edges. Hence (1245) has also two skew a -edges. There are two possibilities.

- $\alpha)$ $12 = 45 = a$. It follows from this that the simplex (0123) has 3 joining a -edges and the number of the a -edges of S_5 is at least 4. The simplex (0345) contains also at least 3 a -edges. Hence S_5 has at least 6 a -edges.

β) $25 = 14 = a$ ($15 = 24 = a$). The simplex S_5 contains at least 4 a -edges. There are at least 2 a -edges in (0345) , thus that S_5 has at least 6 a -edges.

If S_5 is not regular, then there exists a b -edge with $a \neq b$. By similar arguments the number of the b -edges is at least 6.

The role of a and b can be interchanged hence the number of the a -edges and the b -edges is equal. Keeping in mind that the number of the edges of S_5 is 15, the number of the a -edges is at most 5. But this is a contradiction to α) and β) and the theorem is proved. $\square$

Theorem 2. *If the 4-faces of a d -simplex S_d in d -dimensional spaces of constant curvature for $d \geq 6$ are congruent, then S_d is regular.*

PROOF. It is sufficient to prove the statement for $d = 6$. The number of the edges of S_6 is 21. We prove that the number of the a -edges is at least 8.

Let $01 = a$. The simplex (23456) has also an a -edge. Let $23 = a$. Then (01234) has two skew a -edges. Hence (03456) has also two skew a -edges. In this case there is an a -edge among the edges with endpoints 4 or 5 or 6 and 0 or 3. Let $04 = a$. Then (01234) has at least 3 a -edges, two of them intersect (01 and 04) and the third (23) is in a screw position in regard to the intersecting a -edges. It follows that the simplex (12456) has also at least 3 a -edges of preceding types. Then there is an a -edge among the edges with endpoints 1, 2 or 4. The simplex (01234) has also a fourth a -edge and these a -edges joint to one another. For example let $12 = a$. The simplex (12456) contains also at least 4 a -edges, 12 and at least three other a -edges (e.g. $16 = 65 = 54 = a$). It follows that S_6 has at least 7 a -edges. But in this case there are at least 5 a -edges in (01456) , and the same holds in (01234) , too. The fifth a -edge is different from the preceding a -edges. Hence the number of the a -edges of S_6 is at least 8.

It can be proved analogously to the proof of Theorem 1 that S_6 contains at most 7 a -edges. But this is a contradiction to the number 8 and the theorem is proved. $\square$

Conjecture. Let $k \geq 5$ and $d \geq k + 2$. If the k -faces of a d -simplex S_d in d -dimensional spaces of constant curvature are congruent, then S_d is regular.

Remark. The conjecture is false for $d = k + 1$. Let the graph of the simplex S_{k+1} be a regular $(k+2)$ -gon. The edges of S_{k+1} which correspond to the equal sides or diagonals of the $(d+1)$ -gon, are equal. If we omit a vertex, then we get the graph of a k -face. It is clear that the k -faces of S_{k+1} are congruent but the simplex S_{k+1} is not regular.

There exists such a simplex S_{k+1} of the above type in R^{k+1} whose congruent examples are the faces of a tessellation in R^{k+1} .

References

- [BVD84] BUI VAN DUNG, Some properties of equilateral tetrahedra with ideal vertices in the hyperbolic space, *Mat. Lapok* **32** (1–3) (1984), 127–135. (in *Hungarian*)
- [FPMH90] P. FRANK and H. MACHARA, Simplices with given 2-face areas, *European J. Combin.* **11** (1990), 241–247.
- [HJ69] J. HORVÁTH, A property of equilateral tetrahedra in spaces of constant curvature, *Mat. Lapok* **20** (1–3) (1969), 357–364. (in *Hungarian*)
- [HJNL96] J. HORVÁTH and L. NÉMETH, Ultraideal simplices in the hyperbolic space, *BDTF Tud. Közl. X. Természettud.* **5** (1996), 3–25. (in *Hungarian*)
- [MH93] H. MARTINI, Regular simplices in spaces of constant curvature, *Amer. Math. Monthly* **100** (1993), 169–171.

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