

On generalized quasi Einstein manifolds

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Abstract. The notions of a generalized quasi Einstein manifold and a manifold of generalized quasi constant curvature are introduced and some properties of such manifolds are obtained.

Introduction

The notion of a quasi Einstein manifold $(QE)_n$ was introduced in a recent paper [1] by the author and R. K. MAITY. According to them a non-flat Riemannian manifold (M^n, g) ($n \geq 3$) is called *quasi Einstein* if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$(1) \quad S(X, Y) = ag(X, Y) + bA(X)A(Y)$$

where a, b are scalars of which $b \neq 0$ and A is a non-zero 1-form such that

$$(2) \quad g(X, U) = A(X) \quad \forall X, \text{ and } U \text{ is a unit vector field.}$$

This paper deals with *generalized quasi Einstein manifolds* which are defined as follows:

A non-flat Riemannian manifold (M^n, g) ($n \geq 3$) is called a generalized quasi Einstein manifold $G(QE)_n$ if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$(3) \quad S(X, Y) = ag(X, Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)]$$

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where a, b, c are scalars of which $b \neq 0$, A, B are non-zero 1-forms such that

$$(4) \quad g(X, U) = A(X), \quad g(X, V) = B(X) \quad \forall X$$

and U, V are two unit vector fields perpendicular to each other.

In such a case a, b, c are called the associated scalars, A, B are called the associated 1-forms and U, V are called *generators* of the manifold. Such an n -dimensional manifold shall be denoted by the symbol $G(QE)_n$.

If $c = 0$, then (3) takes the form (1) and the manifold becomes a quasi Einstein manifold introduced by Chaki and Maity. This justifies the name generalized quasi Einstein manifold for the manifold defined by (3) and the use of the symbol $G(QE)_n$ for an n -dimensional manifold of this kind.

In 1972 CHEN and YANO [2] introduced the notion of a manifold of quasi-constant curvature. According to them a Riemannian *manifold* (M^n, g) ($n > 3$) is said to be of *quasi-constant curvature* $(QC)_n$ (see also [3]) if its curvature tensor $'R$ of type $(0, 4)$ satisfies the condition

$$(5) \quad \begin{aligned} 'R(X, Y, Z, W) = & a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ & + b[g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W) \\ & + g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z)] \end{aligned}$$

where a, b are scalars of which $b \neq 0$ and A is a non-zero 1-form such that

$$(6) \quad g(X, U) = A(X), \quad \forall X,$$

U being a unit vector field. In such a case a and b are called the associated scalars, A is called the associated 1-form and U is called the *generator* of the manifold. Such an n -dimensional manifold was denoted by the symbol $(QC)_n$ [3].

A generalization of a manifold of quasi-constant curvature, called a manifold of *generalized quasi-constant curvature* is needed for the study of a $G(QE)_n$. Such a manifold is defined as follows: A non-flat Riemannian manifold (M^n, g) ($n \geq 3$) shall be called a manifold of generalized quasi-constant curvature if its curvature tensor $'R$ of type $(0, 4)$ satisfies the

condition

$$\begin{aligned}
 (7) \quad 'R(X, Y, Z, W) = & a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\
 & + b[g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W) \\
 & + g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z)] \\
 & + c[g(Y, Z)\{A(X)B(W) + A(W)B(X)\} \\
 & - g(X, Z)\{A(Y)B(W) + A(W)B(Y)\} \\
 & + g(X, W)\{A(Y)B(Z) + A(Z)B(Y)\} \\
 & - g(Y, W)\{A(X)B(Z) + A(Z)B(X)\}]
 \end{aligned}$$

where a, b, c are scalars of which $b \neq 0$, A, B are two non-zero 1-forms such that

$$(8) \quad g(X, U) = A(X), \quad g(X, V) = B(X) \quad \forall X,$$

where U, V are two mutually perpendicular unit vector fields. Such an n -dimensional manifold shall be denoted by the symbol $G(QC)_n$. In such a case a, b, c are called the associated scalars, A, B are called the associated 1-forms and U, V are called the generators of the manifold. If $c = 0$, then (7) takes the form (5) and the manifold becomes a manifold of quasi constant curvature. This justifies the name manifold of generalized quasi constant curvature and the use of the symbol $G(QC)_n$ for an n -dimensional manifold of this kind.

In this paper it is shown that in a $G(QE)_n$ the scalars a and $a + b$ are the Ricci curvatures in the directions of the vector fields V and U respectively and the scalar c is $< \frac{1}{\sqrt{2}}\ell$, where ℓ is the length of the Ricci tensor S . Other results established in this paper are as follows:

- i) Every $G(QE)_3$ is a $G(QC)_3$, but a $G(QE)_n$ ($n > 3$) is not, in general a $G(QC)_n$.
- ii) A $G(QE)_n$ ($n > 3$) is a $G(QC)_n$ if it is conformally flat.
- iii) A $G(QC)_n$ ($n > 3$) is a conformally flat $G(QE)_n$.
- iv) If $U^\perp$ denotes the $(n - 1)$ -dimensional distribution of a $G(QE)_n$ ($n > 3$) orthogonal to the generator U , then the sectional curvature of the plane determined by the vectors X, Y is $\frac{(3n-2)a+b}{(n-1)(n-2)}$ when $X, Y \in U^\perp$, while the sectional curvature of the plane determined by the vectors X, U is $\frac{(3n-2)a+nb}{(n-1)(n-2)}$ when $X \in U^\perp$.

**1. Scalar curvature and
the associated scalars of a $G(QE)_n$ ($n \geq 3$)**

In this section we consider a $G(QE)_n$ with associated scalars a, b, c associated 1-forms A, B and generators U and V corresponding to A and B , respectively. Then (3) and (4) will hold. Since U and V are mutually perpendicular unit vector fields, we have

$$(1.1) \quad g(U, U) = 1, \quad g(V, V) = 1 \quad \text{and} \quad g(U, V) = 0.$$

In virtue of (8) $g(U, V) = 0$ can be expressed as

$$(1.2) \quad A(V) = B(U) = 0.$$

Now contracting (3) over X and Y we get

$$(1.3) \quad r = na + b$$

where r is the scalar curvature. Again from (3) we get

$$(1.4) \quad S(U, U) = a + b$$

$$(1.5) \quad S(V, V) = a$$

$$(1.6) \quad S(U, V) = c.$$

If X is a unit vector field, then $S(X, X)$ is the Ricci curvature in the direction of X . Hence, from (1.4) and (1.5), we can state that $a + b$ and a are the Ricci curvatures in the directions of U and V , respectively.

Let

$$(1.7) \quad g(LX, Y) = S(X, Y)$$

and ℓ^2 denote the square of the length of the Ricci tensor S . Then

$$(1.8) \quad \ell^2 = S(Le_i, e_i),$$

where $\{e_i\}$, $i = 1, 2, \dots, n$ is an orthonormal basis of the tangent space $(T_p M)$ of $G(QE)_n$. Now from (3) we get

$$S(Le_i, e_i) = (n-1)a^2 + (a+b)^2 + 2c^2.$$

Hence

$$\ell^2 - 2c^2 = (n-1)a^2 + (a+b)^2.$$

Therefore

$$(1.9) \quad \ell^2 > 2c^2 \quad [\because \ell^2 - 2c^2 \neq 0].$$

From (1.9) we get (1.10) $c < \frac{1}{\sqrt{2}}\ell$.

Summing up, we can state the following theorem:

Theorem 1. *In a $G(QE)_n$ ($n \geq 3$) the scalars $a+b$ and a are the Ricci curvatures in the directions of the generators U and V , respectively and the associated scalar c is less than $\frac{1}{\sqrt{2}}\ell$, where ℓ is the length of the Ricci tensor S .*

2. Three-dimensional generalized quasi Einstein manifold

In this section we consider a $G(QE)_3$. In this case (2.1) $r = 3a + b$.

It is known [5] that in a 3-dimensional Riemannian manifold (M^3, g) the curvature tensor $'R$ of type $(0, 4)$ has the following form:

$$(2.1) \quad \begin{aligned} 'R(X, Y, Z, W) = & [g(Y, Z)S(X, W) - g(X, Z)S(Y, W) \\ & + g(X, W)S(Y, Z) - g(Y, W)S(X, Z)] \\ & + \frac{r}{2}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \end{aligned}$$

Using (3), we can express (2.1) as follows:

$$(2.2) \quad \begin{aligned} 'R(X, Y, Z, W) = & \left(2a + \frac{r}{2}\right)[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ & + b[g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W) \\ & + g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z)] \\ & + c[g(Y, Z)\{A(X)B(W) + A(W)B(X)\} \\ & - g(X, Z)\{A(Y)B(W) + A(W)B(Y)\} \\ & + g(X, W)\{A(Y)B(Z) + A(Z)B(Y)\} \\ & - g(Y, W)\{A(X)B(Z) + A(Z)B(X)\}]. \end{aligned}$$

In virtue of (7), it follows from (2.2) that a $G(QE)_3$ is a $G(QC)_3$. This leads to the following result:

Theorem 2. *Every $G(QE)_3$ is a $G(QC)_3$.*

3. Conformally flat $G(QE)_n$ ($n > 3$)

A $G(QE)_n$ ($n > 3$) is not, in general, a $G(QC)_n$. In this section we consider a conformally flat $G(QE)_n$ ($n > 3$) and show that such a $G(QE)_n$ is a $G(QC)_n$.

It is known [6] that in a conformally flat Riemannian manifold (M^n, g) ($n > 3$) the curvature tensor $'R$ of type $(0, 4)$ has the following form:

$$(3.1) \quad \begin{aligned} 'R(X, Y, Z, W) = & \frac{1}{n-2} [g(Y, Z)S(X, W) - g(X, Z)S(Y, W) \\ & + g(X, W)S(Y, Z) - g(Y, W)S(X, Z)] \\ & + \frac{r}{(n-1)(n-2)} [g(Y, Z)g(X, W) \\ & - g(X, Z)g(Y, W)]. \end{aligned}$$

Using (3) we can express (3.1) as follows:

$$(3.2) \quad \begin{aligned} 'R(X, Y, Z, W) = & \frac{2a(n-1) + r}{(n-1)(n-2)} \\ & \times [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ & + \frac{b}{n-2} [g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W) \\ & + g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z)] \\ & + \frac{c}{n-2} [g(Y, Z)\{A(X)B(W) + A(W)B(X)\} \\ & - g(X, Z)\{A(Y)B(W) + A(W)B(Y)\} \\ & + g(X, W)\{A(Y)B(Z) + A(Z)B(Y)\} \\ & - g(Y, W)\{A(X)B(Z) + A(Z)B(X)\}]. \end{aligned}$$

In virtue of (7) it follows from (3.2) that the manifold under consideration is a $G(QC)_n$ ($\because b \neq 0$). Therefore we can state as follows:

Theorem 3. Every conformally flat $G(QE)_n$ ($n > 3$) is a $G(QC)_n$.

We now enquire whether every $G(QC)_n$ ($n \geq 3$) is a $G(QE)_n$. Contracting (7) over Y and Z we get

$$(3.3) \quad S(X, W) = [a(n-1) + b]g(X, W) + b(n-2)A(X)A(W) \\ + c(n-2)[A(X)B(W) + A(W)B(X)].$$

In virtue of (3) it follows from (3.3) that a $G(QC)_n$ ($n \geq 3$) is a $G(QE)_n$ ($\because b \neq 0$).

In a Riemannian manifold (M^n, g) ($n > 3$) the conformal curvature tensor $'C$ of type $(0, 4)$ has the following form:

$$(3.4) \quad 'C(X, Y, Z, W) = 'R(X, Y, Z, W) - \frac{1}{n-2}[g(Y, Z)S(X, W) \\ - g(X, Z)S(Y, W) + g(X, W)S(Y, Z) - g(Y, W)S(X, Z)] \\ + \frac{r}{(n-1)(n-2)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)].$$

Using (7) and (3.3) it follows from (3.4) that

$$(3.5) \quad 'C(X, Y, Z, W) = 0$$

i.e. the manifold under consideration is conformally flat. Thus we can state the following theorem:

Theorem 4. *Every $G(QC)_n$ ($n \geq 3$) is a $G(QE)_n$ while every $G(QC)_n$ ($n > 3$) is a conformally flat $G(QE)_n$.*

4. Sectional curvatures at a point of a conformally flat $G(QE)_n$ ($n > 3$)

Let $U^\perp$ denote the $(n-1)$ -dimensional distribution in a conformally flat $G(QE)_n$ ($n > 3$) orthogonal to U . Then for any $X \in U^\perp$, $g(X, U) = 0$ or, $A(X) = 0$. In this section we shall determine sectional curvature K at the plane determined by the vectors $X, Y \in U^\perp$ or by X, U . Putting $Z = Y$ and $W = X$ in (3.1) we get

$$(4.1) \quad 'R(X, Y, Y, X) = \frac{2a(n-1) + r}{(n-1)(n-2)}[g(X, X)g(Y, Y) - \{g(X, Y)\}^2].$$

Then

$$\begin{aligned}
 (4.2) \quad K(X, Y) &= \frac{{}'R(X, Y, Y, X)}{g(X, X)g(Y, Y) - \{g(X, Y)\}^2} \\
 &= \frac{2a(n-1) + r}{(n-1)(n-2)} \quad [\text{by (4.1)}] \\
 &= \frac{2a(n-1) + na + b}{(n-1)(n-2)} = \frac{(3n-2)a + b}{(n-1)(n-2)},
 \end{aligned}$$

and

$$(4.3) \quad K(X, U) = \frac{{}'R(X, U, U, X)}{g(X, X)g(U, U) - \{g(X, U)\}^2}.$$

But

$$(4.4) \quad {}'R(X, U, U, X) = \left[\frac{2a(n-1) + r}{(n-1)(n-2)} + \frac{b}{n-2} \right] g(X, X).$$

Hence, from (4.3), we get

$$\begin{aligned}
 (4.5) \quad K(X, U) &= \frac{2a(n-1) + r}{(n-1)(n-2)} + \frac{b}{n-2} \\
 &= \frac{2a(n-1) + na + b}{(n-1)(n-2)} + \frac{b}{n-2} \\
 &= \frac{(3n-2)a + nb}{(n-1)(n-2)}.
 \end{aligned}$$

Summing up we can state the following theorem.

Theorem 5. *In a conformally flat $G(QE)_n$ ($n > 3$) the sectional curvature of the plane determined by two vectors $X, Y \in U^\perp$ is $\frac{(3n-2)a+b}{(n-1)(n-2)}$, while the sectional curvature of the plane determined by two vectors X, U is $\frac{(3n-2)a+nb}{(n-1)(n-2)}$.*

I conclude by stating as follows:

The importance of a $G(QE)_n$ lies in the fact that such a four-dimensional semi-Riemannian manifold is relevant to the study of a general relativistic fluid space-time admitting heat flux [7], where U is taken as the velocity vector field of the fluid and V is taken as the heat flux vector field.

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