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Group valued q-additive functions on polynomial values

By I. KÁTAI (Budapest) and M. V. SUBBARAO (Edmonton)

Abstract. Let G be an Abelian toplogical group, $P(n) = A_k n^k + \cdots + A_0$, $A_j \in \mathbb{Z}$, $A_k > 0$, $f : \mathbb{N}_0 \to G$ be a q-additive function. It is proved that $\lim f(P(n)) = \alpha$ implies that f is a very special function.

1. Introduction

Let $q \geq 2$, $q \in \mathbb{N}$, $\mathbb{A} = \{0, 1, \ldots, q-1\}$, G be an additively written Abelian group. Let $\varepsilon_j(n)$ be the sequence of the digits in the q-ary expansion of n, i.e.

$$n = \varepsilon_0(n) + \varepsilon_1(n)q + \ldots, \quad \varepsilon_j(n) \in \mathbb{A}. \tag{1.1}$$

Let $\mathbb{N}_0$ be the set of non-negative integers. (a, b) denotes the greatest common divisor of a and b.

We say that $f : \mathbb{N}_0 \to G$ is a q-additive (G-valued) arithmetical function, if $f(0) = 0$ and

$$f(n) = \sum_{j=0}^{\infty} f(\varepsilon_j(n)q^j) \tag{1.2}$$

is satisfied for every $n \in \mathbb{N}_0$.

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Since $\varepsilon_j(n) = 0$ for all but finitely many j, thus the right hand side of (1.2) is considered as a finite sum.

Let $\mathcal{A}_q(G)$ be the class of q-additive functions.

In a recent paper [1] INDLEKOFER and KÁTAI proved that there exists an absolute constant c_1 with the following property: if $f(p) =$ constant on the set of primes with the possible exception of a subset of relative density 0, then there is an integer $k \in [1, c_1]$ such that $kf(nq) = 0$ for every $n \in \mathbb{N}_0$.

The conjecture of Gelfond that for the sum of digit function $\alpha(n) = \sum \varepsilon_j(n)$ the sequence $\alpha(p)$ on the set of primes contains infinitely many odd, and even values remains open. The above result is an affirmative assertion in this direction.

The problem is interesting for another subsets of the integers, e.g. for polynomial values.

In this paper we shall prove

Theorem 1. *Let $P(n) = A_k n^k + \cdots + A_1 n + A_0$, where $A_j \in \mathbb{N}_0$ $(j = 0, \dots, k)$ $A_k > 0$. Let G be an Abelian topological group, $f \in \mathcal{A}_q(G)$ with the property that*

$$\lim_{n\to\infty} f(P(n)) = \alpha,$$

with some $\alpha \in G$.

Then $f(P(n)) = f(P(0))$ $(n \in \mathbb{N}_0)$ and there exists an integer $v \in \mathbb{N}_0$ such that $f(nq^v) = nf(q^v)$ for $n \in \mathbb{N}_0$.

Let Δ be the largest common divisor of the values $\{A_k m^k + \dots$ $\cdots + A_1 m \mid m \in \mathbb{N}_0\}$. Let Δ_1 be the largest divisor of Δ which is coprime to q. Let D_1 be the largest divisor of $k!A_k$ which is coprime to q.

If (D_1, Δ_1)=1 then there exists an integer $M(\geq 0)$ such that $f(bq^u)$=0 for every $b \in \mathbb{A}$ and $u \geq M$.

Remark. In the special case $P(n) = n^k$ we have that $f(bq^u) = 0$ for every large u, and $b \in \mathbb{A}$.

In what follows let $\mathcal{B} \subseteq \mathbb{N}_0$, G be a compact Abelian group, and $f \in \mathcal{A}_q(G)$ be such that the set $\mathcal{E}$ of limit points of the sequence $\{f(d) \mid d \in \mathcal{B}\}$ contains only finitely many elements.

Assume that $\mathcal{B} = \{P(n) \mid n \in \mathbb{N}_0\}$, $P(x) = A_k x^k + \cdots + A_1 x + A_0$, $A_\nu \in \mathbb{N}_0$, $A_k > 0$. We expect that under the above assumptions there exist suitable $D \in \mathbb{N}$, $u \in \mathbb{N}_0$ such that $Df(nq^u) = 0$ $(n \in \mathbb{N}_0)$. Here we prove it for $P(n) = n^2$, $q =$ odd.

Theorem 2. *Let q be odd, $q \geq 2$. Let $f \in \mathcal{A}_q(G)$, and assume that $\{f(n^2) \mid n \in \mathbb{N}_0\}$ is a finite set. Then there exist integers $D \geq 1$, $R \geq 0$ such that $Df(nq^R) = 0$ for every $n \in \mathbb{N}_0$.*

Theorem 3. *Let q be odd, $q \geq 3$. Let $f \in \mathcal{A}_q(G)$, and assume that the set $\mathcal{E}$ of the limit points of $\{f(n^2) \mid n \in \mathbb{N}_0\}$ is a finite set. Then $Df(nq^R) = 0$ with some integers $D \geq 1$, $R \geq 0$.*

2. Lemmata

Lemma 1. *Let $q \geq 2$, $D, Y \in \mathbb{N}$, $(D, q) = 1$, $Y > 6D$, $Y_1 = \left[\frac{Y}{3}\right]$. Assume that $f \in \mathcal{A}_q(G)$, and that*

$$f(nD) = 0 \quad \text{for } n \in [0, q^Y - 1]. \tag{2.1}$$

Then $f(h) = hf(1)$ whenever $h < q^{Y_1}$. Furthermore, $Df(1) = 0$.

Proof. Let $T \in \mathbb{N}$ be defined such that $Dq^T < q^Y$.

Assume that $n_1, n_2 < q^T$, $n_1, n_2 \in \mathbb{N}_0$ and that $n_1 \equiv n_2 \pmod{D}$. Then we can find some $u \in [0, D-1]$ by which $m_j = n_j + u \cdot q^T$ $(j = 1, 2)$ are multiples of D. Since $m_j < Dq^T < q^T$, from the assumption (2.1) we obtain that $0 = f(m_j) = f(n_j) + f(u \cdot q^T)$, whence the assertion

$$f(n_1) = f(n_2) \quad \text{if} \quad n_1 \equiv n_2 \pmod{D},\ n_1, n_2 < q^T \tag{2.2}$$

immediately follows.

Let $T = 2Y_1 + \varphi(D) + 2$. Then $Y_1 < T - Y_1 - \varphi(D)$, and there is some integer k for which $Y_1 \leq k\varphi(D) < T - Y_1$. Furthermore, $Dq^T < q^Y$. Let $u, v \in [0, q^{Y_1 - 1} - 1]$. By the Euler–Fermat theorem $q^{k\varphi(D)} \equiv 1 \pmod{D}$, therefore by (2.2) we get

$$f(u + v) = f(u) + f(v \cdot q^{k\varphi(D)}) = f(u) + f(v). \tag{2.3}$$

The assertion of Lemma 1 immediately follows. □

Lemma 2. *Let $q \ge 2$, $D = D_1 \cdot D_2$, where $(D_1, q) = 1$ and all the prime factors of D_2 divide q. Let $u \in \mathbb{N}_0$ be defined so that $D_2 \mid q^u$. Let $Z > 6D_1$, $Z_1 = \left[\frac{Z}{3}\right]$, $Y = Z + u - \left[\frac{\log D_2}{\log q}\right]$.*

Assume that $f \in \mathcal{A}_q(G)$ and that

$$f(nD) = 0 \quad \text{for } n \in [0, q^Y - 1]. \tag{2.4}$$

Then $f(nq^u) = nf(q^u)$ holds for every $n < q^{Z_1}$. Furthermore, $D_1 f(q^u) = 0$.

Proof. Let $f_h(n) := f(nq^h)$ $(h \in \mathbb{N}_0)$. Then $f_h \in \mathcal{A}_q(G)$ for every h. From (2.4) we have that

$$0 = f\left(\nu \cdot \frac{q^u}{D_2} D\right) = f_u(D_1\nu), \quad \text{if } \nu < q^Z.$$

Thus Lemma 2 is a direct consequence of Lemma 1. □

3. Proof of Theorem 1

Assume that the conditions are satisfied.

Let $\ell \in \mathbb{N}_0$ be fixed. We have

$$P(x + \ell) = B_\ell^{(0)} + B_\ell^{(1)} x + \cdots + B_\ell^{(k)} x^k, \tag{3.1}$$

where

$$B_\ell^{(0)} = P(\ell),\ B_\ell^{(1)} = \frac{P'(\ell)}{1!}, \ldots,\ B_\ell^{(k)} = \frac{P^{(k)}(\ell)}{k!}, \quad B_\ell^{(k)} = A_k.$$

Let ε be a small positive number, $R_k, \ldots, R_0$ be such a subsequence of positive integers for which $R_k < \varepsilon R_{k-1} < \cdots < \varepsilon^k R_0$ holds. Let m_k be run over the set of integers up to $q^{\varepsilon R_k}$. We define

$$m_{k-1} = 1 + m_k \cdot q^{R_k},\ m_{k-2} = 1 + m_{k-1} q^{R_{k-1}}, \ldots,\ m_0 = 1 + m_1 q^{R_1}.$$

Let $\beta := f(B_\ell^{(0)}) - f(B_0^{(0)}) = f(P(\ell)) - f(P(0))$.

If ε is small and R_k is large, then

$$B_j^{(j_0)} m_0^{j_0} < q^{R_0} \quad (j_0 = 0, \ldots, k-1), \tag{3.2}$$

and from

$$P\big(\ell + m_0 q^{R_0}\big) = \sum_{j_0=0}^{k} B_\ell^{(j)} m_0^{j_0} q^{j_0 R_0},$$

we have

$$f\big(P(\ell + m_0 q^{R_0})\big) = \sum_{j_0=0}^{k} f\big(B_\ell^{(j_0)} m_0^{j_0} q^{j_0 R_0}\big).$$

Writing this formula with $\ell = 0$ as well, observing that $B_\ell^{(k)} = B_0^{(k)}$, subtracting, we deduce that

$$\begin{aligned} &f(P(\ell + m_0 q^{R_0})) - f(P(m_0 q^{R_0})) \\ &= \sum_{j_0=0}^{k} \big(f(B_\ell^{(j_0)} m_0^{j_0} q^{j_0 R_0}\big) - f\big(B_0^{(j_0)} m_0^{j_0} q^{j_0 R_0})\big) = \sum_{j_0=1}^{k-1} + \beta. \end{aligned} \tag{3.3}$$

The left hand side tends to zero as $m_0 q^{R_0} \to \infty$. Let $A_0(m_0 \mid R_0)$ be the sum $\sum_{j_0=1}^{k-1}$ on the right hand side of (3.3).

If $k = 1$, then this sum is empty, $\beta = 0$.

We may assume that $k > 1$. We have that $A_0(m_0 \mid R_0) \to -\beta$ as $R_k \to \infty$.

Since

$$m_0^{j_0} = \sum_{j_1=0}^{j_0} \binom{j_0}{j_1} m_1^{j_1} q^{j_1 R_1},$$

$$\begin{aligned} A_0\big(1 + m_1 q^{j_1 R_1} \mid R_0\big) &= A_0(1 \mid R_0) + A_0(m_1 q^{j_1 R_1} \mid R_0\big) \\ &\quad + A_1(m_1 \mid R_0, R_1), \end{aligned}$$

where

$$\begin{aligned} A_1(m_1 \mid R_0, R_1) = \sum_{j_0=2}^{k-1} \sum_{j_1=1}^{j_0-1} \Bigg(& f\left(B_\ell^{(j_0)} \binom{j_0}{j-1} m_1^{j_1} q^{j_1 R_1 + j_0 R_0}\right) \\ & - f\left(B_0^{j_0} \binom{j_0}{j_1} m_1^{j_1} q^{j_1 R_1 + j_0 R_0}\right)\Bigg). \end{aligned}$$

For $R_k \to \infty$ we have $A_1(m_1 \mid R_0, R_1) \to \beta$. We can continue this process. We obtain that

$$A_1(m_1 \mid R_0, R_1) = A_1(1 \mid R_0, R_1) + A_1(m_2 q^{R_2} \mid R_0, R_1) + A_2(m_2 \mid R_0, R_1, R_2),$$

where

$$A_2(m_2 \mid R_0, R_1, R_2) = \sum_{j_0=3}^{k-1} \sum_{j_1=2}^{j_0-1} \sum_{j_2=1}^{j_1-1} \big(f(\xi_\ell(j_0, j_1, j_2)) - f(\xi_0(j_0, j_1, j_2))\big),$$

and

$$\xi_1(j_0, j_1, j_2) = B_\ell^{(j)} \binom{j_0}{j_1} \binom{j_1}{j_2} m_2^{j_2} q^{j_2 R_2 + j_1 R_1 + j_0 R_0} \quad (\ell := \ell, 0).$$

Furthermore, $A_2(m_2 \mid R_0, R_1, R_2) \to -\beta$ as $R_k \to \infty$.

Defining $A_h(m_h \mid R_0, \dots, R_h)$ similarly, it tends to $-(-1)^h\beta$. The largest exponent of m_h occurring in the definition of $A_h(\ )$ is $k-(h+1)$. We have $A_{k-1}(m_{k-1} \mid R_0, \dots, R_{k-1}) = f\Big(B_\ell^{(k)} \binom{k}{k-1} \dots \binom{2}{1} m_{k-1} q^{\sigma_k}\Big) - f\Big(B_0^{(k)} \binom{k}{R-1} \dots \binom{2}{1} m_{k-1} q^{\sigma_k}\Big)$, where $\sigma_k = R_{k-1} + 2R_{k-2} + \dots + kR_0$.

The right hand side is clearly 0, otherhand it tends to $-(-1)^{k-1}\beta$, i.e. $\beta = 0$.

Thus we proved that $f(P(\ell)) = f(P(0))$ for every $\ell \in \mathbb{N}_0$. Thus

$$\begin{aligned} f(A_0) = f(P(0)) &= f\big(P(m_0 q^{R_0})\big) \\ &= f(A_0) + f(A_1 m_0 q^{R_0}) + \dots + f(A_R m_0^R q^{kR_0}), \end{aligned} \tag{3.4}$$

and so

$$\sum_{j_0=1}^{k} f\big(A_{j_0} m_0^{j_0} q^{j_0 R_0}\big) = 0.$$

Repeating the above argument we deduce that $f(A_k k! m q^{\sigma R}) = 0$ as $m < q^{\varepsilon R_k}$.

We shall apply now Lemma 2 with $D = k!A_k$ and with f_{σ_k} instead of f, where $f_h(m) := f(mq^h)$. For each integer $T > 0$ there exists a bound c_0 such that if $R_k > c_0$, then for every $n < q^T$, $f_{\sigma_k}(nq^u) = nf_{\sigma_k}(q^u) = nf(q^{u+\sigma_k})$ for every $n \in [0, q^T - 1]$.

Here u is the integer defined in Lemma 2. Since σ_k takes on each large integer values when $R_k, \dots, R_0$ run over all appropriate sequences, we obtain that there is a fixed v such that

$$f(nq^v) = nf(q^v) \quad (n \in \mathbb{N}_0). \tag{3.5}$$

Furthermore $D_1 f(q^v) = 0$.

Let us write m, R instead of m_0, R_0 into (3.4), and assume that R is large. By using (3.5), we deduce that

$$0 = \sum_{j=1}^{k} f(A_j m^j q^{jR}) = \left(\sum_{j=1}^{k} A_j m^j q^{jR-v} \right) f(q^v)$$

i.e.

$$mq^{R-v}(A_1 + A_2 m q^R + \cdots + A_k m^{k-1} q^{(k-1)R}) f(q^v) = 0. \tag{3.6}$$

Let δ be the order of $f(q^v)$. Then $\delta \mid D_1$, $(\delta, q) = 1$. Let R be a multiple of $\varphi(\delta)$, $R = j\varphi(\delta)$. Then $q^{\ell R} \equiv 1 \pmod{\delta}$, thus $\delta \mid A_1 m + A2m^2 + \cdots + A_k m^k$ for every $m \in \mathbb{N}_0$. Thus $\delta \mid \Delta_1$, and from $(\Delta, D_1) = 1$ we infer that $\delta = 1$.

The proof of the theorem is finished.

4. Proof of Theorem 2

Let $\mathcal{B} = \{\beta_1, \dots, \beta_s\} = \{f(n^2) \mid n \in \mathbb{N}\}$. Let $\mathcal{B}_0 \subseteq \mathcal{B}$ be the subset of those β_ν which have finite order, and let D_1 be the least common multiple of their orders. Then, by $f_1(n) := D_1 f(n)$, $f_1 \in \mathcal{A}_q(G)$, and

$$\mathcal{D} := \{f_1(n^2) \mid n \in \mathbb{N}_0\}$$

is a finite set, it contains zero, all the other elements are of infinite order. It is enough to prove the assertion for f_1 instead of f.

Let $0 \le m < q^s/2$, $s \in \mathbb{N}$, $n = 1 + m \cdot q^s$. Then $f_1(n^2)$, $f_1(1)$, $f_1(m^2 q^{2s}) \in \mathcal{D}$, thus $f_1(n^2) = f_1(1) + f_1(m^2 q^{2s}) + f_1(2mq^s)$, consequently $f_1(2mq^s)$ belongs to the finite set

$$\mathcal{E} = (\mathcal{D} + \mathcal{D}) - \mathcal{D}.$$

Arguing, as in the proofs of Lemma 1 and 2, we obtain that there exists a finite set $\mathcal{F}$ such that

$$f_1(mq^S) \in \mathcal{F}, \tag{4.1}$$

if $S > S_0 = a$ constant, and $(1 \le)m < \frac{q^{s-1}}{3} < q^{s-3}$.

Let $\mathcal{F}_1 \subseteq \mathcal{F}$ be the set of elements having finite order, and D_2 be the least common multiple of their orders. Let $f_2(n) = D_2 f_1(n)$. Let $\mathcal{F}_2 := \{0\} \cup (\mathcal{F} \setminus \mathcal{F}_1)$.

To finish the proof it is enough to show that either $\mathcal{F}_2 = \{0\}$, or for every $0 \ne \beta \in \mathcal{F}_2$ the set $\{bq^\ell \mid b \in A,\ \ell \in \mathbb{N}_0,\ f_2(bq^\ell) = \beta\}$ is finite.

Assume indirectly that there exists some $0 \ne \beta \in \mathcal{F}_2$ for which $f_2(b_\nu q^{\ell_\nu}) = \beta$ $(\nu = 1, 2, \dots)$ $\ell_1 < \ell_2 < \dots$. From (4.1) we obtain that for every $S > S_0$ the interval $[S, 2S-4]$ does not contain more than $\mathrm{card}(\mathcal{F}_2)$ elements.

Indeed, if

$$S \le \ell_u < \ell_{u+1} < \dots < \ell_{u+T} < 2S - 4,$$

then applying (4.1) with

$$m_r = b_u q^{\ell_u - S} + \dots + b_{u+r} q^{\ell_u + r - S} \quad (r = 0, \dots, T)$$

$f_2(m_r q^S) = r\beta (\in \mathcal{F})$. Since the order of β is infinite, therefore $r_1\beta \ne r_2\beta$ if $r_1 \ne r_2$, thus $T + 1 \le \mathrm{card}(\mathcal{F})$.

Let $M \ge 2$ be an integer, $1 \le \nu < q^{M-1}$, $\mu = 1 + \nu q$. It is easy to show that for every ν there exists such an $n_\nu < q^M$ for which $n_\nu^2 \equiv \mu \pmod{q^M}$. Let t_ν be defined from the equation $n_\nu^2 = \mu + q^M t_\nu$.

Let $\ell_{u+1} < M \le \ell_{u+2}$,

$$\nu_j = b_2 q^{\ell_2 - 1} + \dots + b_{j+2} q^{l_{j+2} - 1} \qquad (j = 0, \dots, u-1)$$

$\mu_j = 1 + \nu_j q$.

Then $n_{\nu_j}^2 = \mu_j + q^M t_{\nu_j}$,

$$f_2(n_{\nu_j}^2) = (j+1)\beta + f_2(q^M t_\nu) \qquad (j = 0, \dots, u-1). \tag{4.2}$$

Observe that $f_2(q^M t_\nu)$ belongs to the set $\mathcal{F} + \mathcal{F}$ for large M. Indeed, if we write $t_\nu = t_\nu^{(0)} + q^H t_\nu^{(1)}$, $H = \left[\frac{M}{2}\right]$, $0 \le t_\nu^{(0)} < q^H$, $0 \le q^{M-H}$, then $f_2(q^M t_\nu) = f_2(q^M t_\nu^{(0)}) + f_2(q^{M+H} t_\nu^{(1)})$ and both summands belongs to $\mathcal{F}$.

From (4.2) we deduce that $(j+1)\beta$ $(j = 0, \dots, u-1)$ belongs to a finite set. Thus for $M \to \infty$ the variable u should be bounded.

The proof is completed.

5. Proof of Theorem 3

Theorem 3 is an easy consequence of Theorem 2. We shall use the following notation: For some fixed ℓ consider the sequences

$$\xi_{m,R} = 1 + \ell m q^R, \quad \eta_{m,R} = \ell + m q^R$$

for such values of $m,\ R > 0$ for which $2\ell m < q^R$, $\ell^2 < q^R$ hold true.

Then

$$f(\xi_{m,R}^2) = f(1) + f(2\ell m q^R) + f(\ell^2 m^2 q^{2R})$$

$$f(\eta_{m,R}^2) = f(\ell^2) + f(2\ell m q^R) + f(m^2 q^{2R}),$$

whence

$$f(\ell^2) - f(1) = f(\eta_{m,R}^2) - f(\xi_{m,R}^2) - f(m^2 q^{2R}) + f(\ell^2 m^2 q^{2R}).$$

Each summand tend to $\mathcal{E}$ as $mq^R \to \infty$, thus

$$f(\ell^2) - f(1) \in \mathcal{E} - \mathcal{E} - \mathcal{E} + \mathcal{E}.$$

Consequently $f(\ell^2)$ belongs to the finite set $f(1) + \mathcal{E} + \mathcal{E} - (\mathcal{E} + \mathcal{E})$, Theorem 2 can be applied.

References

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I. KÁTAI
DEPARTMENT OF COMPUTER ALGEBRA
EÖTVÖS LORAND UNIVERSITY
PÁZMÁNY PÉTER SÉTÁNY I/D
H–1117 BUDAPEST
HUNGARY

E-mail: katai@compalg.inf.elte.hu

M. V. SUBBARAO
UNIVERSITY OF ALBERTA
EDMONTON AB, T6G 2G1
CANADA

E-mail: subbarao@ualberta.ca

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