

## On submanifolds of Lorentzian almost paracontact manifolds

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**Abstract.** Lorentzian almost paracontact structures are constructed from Riemannian almost paracontact structures on a differentiable manifold. Some properties of submanifolds of a Lorentzian  $s$ -paracontact manifold are presented. Non-existence of any anti-invariant distribution  $\mathcal{A}$  on a submanifold tangent to the structure vector field of an  $LP$ -Sasakian manifold such that the structure vector field is orthogonal to  $\mathcal{A}$  is proved. This non-existence implies that an  $LP$ -Sasakian or  $LSP$ -Sasakian manifold does not admit any proper  $CR$ , generalized  $CR$ , semi-invariant or almost semi-invariant submanifold. It is also proved that a Lorentzian  $s$ -paracontact manifold can not admit any proper mixed foliated semi-invariant submanifold.

### 1. Introduction

K. MATSUMOTO introduced [9] the notion of a Lorentzian almost paracontact manifold. Later on several authors studied Lorentzian almost paracontact manifolds including those of [5], [10]–[13], [16]. Different types of submanifolds of Lorentzian almost paracontact manifold have been studied in [3], [4], [6], [7], [15], [17]–[19], [21], [23], [24].

In this paper, we study submanifolds of Lorentzian almost paracontact manifolds. The paper is organized as follows. Section 2 is devoted to preliminaries. In Section 3, we prove a theorem which interrelates the Riemannian and Lorentzian almost paracontact structures on a differentiable manifold. In this way, we find a general method for constructing

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Lorentzian almost paracontact structures from Riemannian almost paracontact structures on a differentiable manifold. In Section 4, some properties of submanifolds of a Lorentzian  $s$ -paracontact manifold are presented. In Section 5, we mainly prove the non-existence of any anti-invariant distribution  $\mathcal{A}$  on a submanifold tangent to the structure vector field of an  $LP$ -Sasakian manifold such that the structure vector field is orthogonal to  $\mathcal{A}$ . Therefore, we are able to state that a  $LP$ -Sasakian or  $LSP$ -Sasakian manifold can not admit any proper  $CR$ , generalized  $CR$ , semi-invariant or almost semi-invariant submanifold. In the last section, we prove that a Lorentzian  $s$ -paracontact manifold can not admit any proper mixed foliated semi-invariant submanifold.

## 2. Preliminaries

A differentiable manifold  $\overline{M}$  is said to admit an *almost paracontact Riemannian structure*  $(\phi, \xi, \eta, g)$ , where  $\phi$  is a  $(1, 1)$  tensor field,  $\xi$  is a vector field,  $\eta$  is a 1-form and  $g$  is a Riemannian metric on  $\overline{M}$  such that

$$(1) \quad \phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1,$$

$$(2) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vector fields  $X$  and  $Y$  on  $\overline{M}$  (see [20]).

On the other hand,  $\overline{M}$  is said to admit a *Lorentzian almost paracontact structure*  $(\phi, \xi, \eta, g)$ , if  $\phi$  is a  $(1, 1)$  tensor field,  $\xi$  is a vector field,  $\eta$  is a 1-form and  $g$  is a Lorentzian metric on  $\overline{M}$ , which makes  $\xi$  a timelike unit vector field, such that

$$(3) \quad \phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1,$$

$$(4) \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for all vector fields  $X$  and  $Y$  on  $\overline{M}$  (see [9], [10]).

For both the structures mentioned above, it follows that

$$(5) \quad \eta \circ \phi = 0, \quad \phi\xi = 0,$$

$$(6) \quad g(\xi, X) = \eta(X),$$

$$(7) \quad g(\phi X, Y) = g(X, \phi Y).$$

A Lorentzian almost paracontact manifold is called

1. *Lorentzian para Sasakian* (in brief, *LP-Sasakian*) manifold [9] if

$$(8) \quad (\bar{\nabla}_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X,$$

where  $\bar{\nabla}$  is the covariant differentiation with respect to  $g$ ,

2. *Lorentzian special para Sasakian* (in brief, *LSP-Sasakian*) manifold [9] if

$$(9) \quad \Phi(X, Y) = \varepsilon g(\phi X, \phi Y), \quad \varepsilon^2 = 1.$$

An *LSP-Sasakian* manifold is always *LP-Sasakian* [9].

Let  $M$  be a submanifold of a Lorentzian almost paracontact manifold  $\bar{M}$  with Lorentzian almost paracontact structure  $(\phi, \xi, \eta, g)$ . Let the induced metric on  $M$  also be denoted by  $g$ . Then Gauss and Weingarten formulae are given respectively by

$$(10) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad X, Y \in TM,$$

$$(11) \quad \bar{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad N \in T^\perp M,$$

where  $\nabla$  is the induced connection on  $M$ ,  $h$  is the second fundamental form of the immersion, and  $-A_N X$  and  $\nabla_X^\perp N$  are the tangential and normal parts of  $\bar{\nabla}_X N$ . From (10) and (11) one gets

$$(12) \quad g(h(X, Y), N) = g(A_N X, Y).$$

Moreover, we have

$$(13) \quad (\bar{\nabla}_X \phi)Y = ((\nabla_X P)Y - A_{FY}X - th(X, Y)) \\ + ((\nabla_X F)Y + h(X, PY) - fh(X, Y)),$$

where

$$(14) \quad \phi X \equiv PX + FX, \quad PX \in TM, \quad FX \in T^\perp M,$$

$$(15) \quad \phi N \equiv tN + fN, \quad tN \in TM, \quad fN \in T^\perp M,$$

$$(16) \quad (\nabla_X P)Y \equiv \nabla_X PY - P\nabla_X Y, \quad (\nabla_X F)Y \equiv \nabla_X^\perp FY - F\nabla_X Y.$$

Let  $\xi \in TM$ . We write  $TM = \{\xi\} \oplus \{\xi\}^\perp$ , where  $\{\xi\}$  is the distribution spanned by  $\xi$  and  $\{\xi\}^\perp$  is the complementary orthogonal distribution of  $\{\xi\}$  in  $M$ . Then we get

$$(17) \quad P\xi = 0 = F\xi, \quad \eta \circ P = 0 = \eta \circ F,$$

$$(18) \quad P^2 + tF = I + \eta \otimes \xi, \quad FP + fF = 0,$$

$$(19) \quad f^2 + Ft = I, \quad tf + Pt = 0.$$

A submanifold  $M$  of a Lorentzian almost paracontact manifold  $\overline{M}$  with  $\xi \in TM$  is an *almost semi-invariant submanifold* of  $\overline{M}$  if  $TM$  can be decomposed as a direct sum of mutually orthogonal differentiable distributions

$$TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{D} \oplus \{\xi\},$$

where  $\mathcal{D}^1 = TM \cap \phi(TM)$ ,  $\mathcal{D}^0 = TM \cap \phi(T^\perp M)$  (see [7]). A submanifold  $M$  of a Lorentzian almost paracontact manifold  $\overline{M}$  with  $\xi \in TM$  is a *generalized CR-submanifold* [21] of  $\overline{M}$  if  $TM$  can be decomposed as a direct sum of mutually orthogonal differentiable distributions  $TM = \mathcal{D}^0 \oplus \mathcal{D} \oplus \{\xi\}$ , where  $\mathcal{D}^0 = TM \cap \phi(T^\perp M)$ .

A submanifold  $M$  of a Lorentzian almost paracontact manifold  $\overline{M}$  is an *invariant* (resp. *anti-invariant*) *submanifold* of  $\overline{M}$  if  $\phi(TM) \subset TM$  (resp.  $\phi(TM) \subset T^\perp M$ ). An almost semi-invariant submanifold of a Lorentzian almost paracontact manifold is a *semi-invariant submanifold* if  $\mathcal{D} = \{0\}$ . A semi-invariant submanifold of a Lorentzian almost paracontact manifold becomes an invariant submanifold (resp. anti-invariant submanifold) if  $\mathcal{D}^0 = \{0\}$  (resp.  $\mathcal{D}^1 = \{0\}$ ). An almost semi-invariant submanifold is *proper* if none of the distributions  $\mathcal{D}^1$ ,  $\mathcal{D}^0$  and  $\mathcal{D}$  is zero. A semi-invariant submanifold is *proper* if  $\mathcal{D}^0 \neq \{0\} \neq \mathcal{D}^1$ .

### 3. Riemannian and Lorentzian almost paracontact structures

An example of a 5-dimensional *LP*-Sasakian manifold is given as follows.

*Example 3.1* (K. MATSUMOTO, I. MIHAI and R. ROSCA [11]). Let  $\mathbb{R}^5$  be the 5-dimensional real number space with a coordinate system  $(x, y, z, t, s)$ . Defining

$$\begin{aligned}\eta &= ds - ydx - tdz, & \xi &= \frac{\partial}{\partial s}, \\ g &= \eta \otimes \eta - (dx)^2 - (dy)^2 - (dz)^2 - (dt)^2, \\ \phi\left(\frac{\partial}{\partial x}\right) &= -\frac{\partial}{\partial x} - y\frac{\partial}{\partial s}, & \phi\left(\frac{\partial}{\partial y}\right) &= -\frac{\partial}{\partial y}, \\ \phi\left(\frac{\partial}{\partial z}\right) &= -\frac{\partial}{\partial z} - t\frac{\partial}{\partial s}, & \phi\left(\frac{\partial}{\partial t}\right) &= -\frac{\partial}{\partial t}, & \phi\left(\frac{\partial}{\partial s}\right) &= 0,\end{aligned}$$

the structure  $(\phi, \xi, \eta, g)$  becomes an *LP*-Sasakian structure in  $\mathbb{R}^5$ .

Now, we prove the following theorem which interrelates the Riemannian and Lorentzian almost paracontact structures on a differentiable manifold.

**Theorem 3.2.** *A differentiable manifold  $\overline{M}$  admits an almost paracontact Riemannian structure if and only if it admits a Lorentzian almost paracontact structure.*

PROOF. Let  $\overline{M}$  admit an almost paracontact Riemannian structure  $(\phi, \xi, \eta, g)$ . We define a 1-form  $\beta$  by

$$(20) \quad \beta(X) \equiv -\eta(X)$$

and a  $(0, 2)$  tensor field  $\gamma$  (see page 148, B. O'NEILL [14]) by

$$(21) \quad \gamma(X, Y) \equiv g(X, Y) - 2\eta(X)\eta(Y).$$

From (20), it is clear that

$$\beta(X)\beta(Y) = \eta(X)\eta(Y)$$

and hence (21) becomes equivalent to

$$(22) \quad g(X, Y) = \gamma(X, Y) + 2\beta(X)\beta(Y).$$

In view of (20), (1) transforms to

$$\phi^2 = I + \beta \otimes \xi, \quad \beta(\xi) = -1.$$

From (21), it is clear that  $\gamma$  is symmetric. Moreover,

$$\begin{aligned} \gamma(\phi X, \phi Y) &= g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \\ &= \gamma(X, Y) + \eta(X)\eta(Y) = \gamma(X, Y) + \beta(X)\beta(Y), \end{aligned}$$

where (21), (5), (2) and (20) have been used. Consequently,

$$\gamma(\xi, X) = \beta(X) = -\eta(X) = -g(\xi, X).$$

Thus  $X$  is orthogonal to  $\xi$  with respect to  $g$  if and only if  $X$  is orthogonal to  $\xi$  with respect to  $\gamma$ . From the last equation we see that  $\gamma(\xi, \xi) = -1$ , that is,  $\gamma$  makes  $\xi$  a timelike unit vector field. If  $X$  and  $Y$  are orthogonal to  $\xi$  with respect to  $\gamma$ , then  $\gamma(X, Y) = g(X, Y)$ , that is,  $X$  and  $Y$  are spacelike. Thus the metric  $\gamma$  is a Lorentzian metric associated with the structure  $(\phi, \xi, \beta)$ . This proves the necessary part.

Conversely, let  $(\phi, \xi, \beta, \gamma)$  be a Lorentzian almost paracontact structure on  $\overline{M}$ . Then it is easy to check that  $(\phi, \xi, \eta, g)$  is an almost paracontact Riemannian structure on  $\overline{M}$ , where  $\eta$  and  $g$  are defined by (20) and (22) respectively.  $\square$

*Remark 3.3.* In view of the preceding theorem, it is now easy to construct a Lorentzian almost paracontact structure by an almost paracontact Riemannian structure and vice-versa.

#### 4. Submanifolds of Lorentzian $s$ -paracontact manifolds

We begin this section with the following definition, which is analogous to the definition of special paracontact Riemannian manifolds.

*Definition 4.1.* We call a Lorentzian almost paracontact manifold as a *Lorentzian  $s$ -paracontact manifold* if

$$(23) \quad \phi X = \overline{\nabla}_X \xi.$$

It can be verified that an  $LP$ -Sasakian manifold is always a Lorentzian  $s$ -paracontact manifold.

*Definition 4.2.* The distribution  $\{\xi\}^\perp$  in a Lorentzian almost paracontact manifold will be called the *paracontact distribution*.

First, we prove

**Theorem 4.3.** *On a Lorentzian  $s$ -paracontact manifold  $\overline{M}$  the paracontact distribution  $\{\xi\}^\perp$  is integrable.*

PROOF. Let  $X, Y \in \{\xi\}^\perp$ . Then  $\eta(X) = 0 = \eta(Y)$  and consequently, in view of (23) and (7), it follows that  $\eta[X, Y] = 0$ ,  $X, Y \in \{\xi\}^\perp$ .  $\square$

In view of Definition 4.2 and Theorem 4.3, we can state the following theorem.

**Theorem 4.4.** *Let  $M$  be a submanifold of a Lorentzian  $s$ -paracontact manifold such that  $\xi$  is tangential to  $M$ . Then the paracontact distribution  $\{\xi\}^\perp$  on  $M$  is integrable.*

In view of the above theorem we have the following corollary.

**Corollary 4.5.** *Let  $M$  be either a semi-invariant or an almost semi-invariant or a generalized CR-submanifold of a Lorentzian  $s$ -paracontact manifold. Then the paracontact distribution  $\{\xi\}^\perp$  on  $M$  is integrable.*

Now, we prove a lemma.

**Lemma 4.6.** *For a submanifold  $M$  of a Lorentzian  $s$ -paracontact manifold, we have*

$$(24) \quad \phi X = \nabla_X \xi + h(X, \xi), \quad \xi \in TM,$$

$$(25) \quad \phi X = -A_\xi X + \nabla_X^\perp \xi, \quad \xi \in T^\perp M,$$

$$(26) \quad \eta(A_N X) = 0, \quad \xi \in T^\perp M,$$

$$(27) \quad \eta(A_N X) = g(\phi X, N), \quad \xi \in TM$$

for all  $X \in TM$  and  $N \in T^\perp M$ .

PROOF. From (23) and Gauss formula, we get (24) and (25). In view of (6), we get (26). In last, for  $\xi \in TM$ , we have

$$\eta(A_N X) = g(\xi, A_N X) = -g(\xi, \overline{\nabla}_X N) = g(\overline{\nabla}_X \xi, N) = g(\phi X, N),$$

where (6), (11) and (23) have been used.  $\square$

In view of (24), we can state the following

**Theorem 4.7.** *Let  $M$  be a submanifold of a Lorentzian  $s$ -paracontact manifold such that  $\xi$  is tangential to  $M$ . Then  $M$  is an invariant submanifold if and only if  $h(X, \xi) = 0$ , and  $M$  is an anti-invariant submanifold if and only if  $\nabla_X \xi = 0$ .*

Now, we prove the following

**Theorem 4.8.** *If  $M$  is a totally umbilical submanifold of a Lorentzian  $s$ -paracontact manifold such that  $\xi$  is tangential to  $M$ , then*

- (a)  *$M$  is necessarily minimal and consequently totally geodesic, and*
- (b)  *$M$  is an invariant submanifold and  $\nabla_X \xi \neq 0$  for any non-zero  $X$  that is not in  $\{\xi\}$ .*

PROOF. Let  $M$  be totally umbilical. Using (24),  $\phi\xi = 0$  and (6) we get

$$0 = h(\xi, \xi) = g(\xi, \xi)H = H, \quad H \equiv \text{trace}(h) / \dim(M),$$

hence we have (a). The second part is obvious from Theorem 4.7 and Theorem 4.8 (a).  $\square$

Next, we prove the following

**Theorem 4.9.** *A submanifold  $M$  of a Lorentzian  $s$ -paracontact manifold such that  $\xi$  is normal to  $M$  is an anti-invariant submanifold if and only if  $A_\xi X = 0$ . Consequently, if  $M$  is totally geodesic then it is anti-invariant.*

PROOF. Since,  $\xi$  is normal to  $M$ , therefore in view of (25) and (12), we obtain

$$g(\phi X, Y) = -g(A_\xi X, Y), \quad X, Y \in TM,$$

which proves the theorem.  $\square$

We complete this section by the following

*Remark 4.10.* Since  $LP$ -Sasakian and  $LSP$ -Sasakian manifolds are Lorentzian  $s$ -paracontact manifolds, therefore the results of this section are valid for the submanifolds of  $LP$ -Sasakian and  $LSP$ -Sasakian manifolds.

### 5. Nonexistence of an anti-invariant distribution

Let  $M$  be a submanifold of a Lorentzian  $s$ -paracontact manifold  $\overline{M}$  with  $\xi \in TM$ . Then in view of (27) and (14) we get

$$(28) \quad \eta(A_N X) = g(FX, N), \quad X \in TM, \quad N \in T^\perp M.$$

Moreover, if  $\overline{M}$  is  $LP$ -Sasakian, then in view of (8) and (13) we get

$$(29) \quad (\nabla_X P)Y - A_{FY}X - th(X, Y) = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X.$$

Now, we prove the following

**Theorem 5.1.** *Let  $M$  be a submanifold of a  $LP$ -Sasakian manifold  $\overline{M}$  with  $\xi \in TM$ . Then there does not exist any anti-invariant distribution  $\mathcal{A}$  such that  $\mathcal{A} \perp \{\xi\}$ .*

PROOF. We shall prove that  $\mathcal{A} = \{0\}$ . Let  $X \in \mathcal{A}$  and  $Y \in TM$ . We get

$$\begin{aligned} g(A_{FX}X, Y) &= g(h(Y, X), FX) = g(th(Y, X), X) \\ &= g(\nabla_Y PX - P\nabla_Y X - A_{FX}Y - g(\phi Y, \phi X)\xi - \eta(X)\phi^2 Y, X) \\ &= -g(\nabla_Y X, PX) - g(A_{FX}Y, X) = -g(A_{FX}X, Y), \end{aligned}$$

which implies that

$$A_{FX}X = 0, \quad X \in \mathcal{A}$$

and consequently

$$0 = \eta(A_{FX}X) = g(FX, FX) = g(\phi X, \phi X) = g(X, X),$$

that is,  $\mathcal{A} = \{0\}$ . □

Since an  $LSP$ -Sasakian manifold is  $LP$ -Sasakian, therefore we can state the following

**Corollary 5.2.** *Let  $M$  be a submanifold of a  $LSP$ -Sasakian manifold  $\overline{M}$  with  $\xi \in TM$ . Then there does not exist any anti-invariant distribution  $\mathcal{A}$  such that  $\mathcal{A} \perp \{\xi\}$ .*

In view of the definitions of  $CR$  [3], [17], generalized  $CR$  [21], semi-invariant [6] and almost semi-invariant [7] submanifolds of Lorentzian almost paracontact manifolds and in view of Theorem 5.1 and Corollary 5.2 we have the following theorem.

**Theorem 5.3.** *An  $LP$ -Sasakian or  $LSP$ -Sasakian manifold does not admit any proper  $CR$ , generalized  $CR$ , semi-invariant or almost semi-invariant submanifold. In fact, in these cases the anti-invariant distribution  $\mathcal{D}^0$  becomes  $\{0\}$ .*

*Remark 5.4.* The geometry of  $CR$ -submanifolds of Kaehler manifolds was initiated by A. BEJANCU in 1978 (see A. BEJANCU [2] and K. YANO & M. KON [25]). The definition of  $CR$ -submanifolds of Lorentzian almost paracontact manifolds resembles with the definition of semi-invariant submanifolds. However, the name  $CR$ -submanifold does not seem to be appropriate as possibility of getting a  $CR$ -structure on the so called  $CR$ -submanifold of a  $LAP$ -manifold is very far from reality. In [4], [23] it is proved that a Lorentzian para-Sasakian manifold does not admit proper semi-invariant submanifold. In [6], [18] it is claimed that the distributions  $\mathcal{D}^0$  and  $\mathcal{D}^0 \oplus \{\xi\}$  are never integrable on semi-invariant submanifolds of  $LP$ -Sasakian manifolds. The same result is claimed in [7] for almost semi-invariant submanifolds of  $LP$ -Sasakian manifolds. Contrary to these claims, the authors of [21] claim that the distribution  $\mathcal{D}^0$  is always integrable on generalized  $CR$ -submanifolds of  $LP$ -Sasakian manifolds. However, in view of the above theorem, in these cases the anti-invariant distribution  $\mathcal{D}^0$  becomes  $\{0\}$ , which makes a number of results of [3], [6], [7], [17], [18], [21] redundant.

## 6. Nonexistence of proper mixed foliated semi-invariant submanifolds

In [8], a semi-invariant submanifold is said to be *mixed foliated* if  $\mathcal{D}^1 \oplus \{\xi\}$  is integrable and  $h(Z + \xi, X) = 0$  for all  $Z \in \mathcal{D}^1$  and  $X \in \mathcal{D}^0$ . In [22], the first author of this paper proved that a Sasakian manifold does not admit any proper mixed foliated semi-invariant submanifold.

In similar manner, we prove the following

**Theorem 6.1.** *A Lorentzian  $s$ -paracontact manifold can not admit any proper mixed foliated semi-invariant submanifold.*

PROOF. For a submanifold  $M$  of a Lorentzian  $s$ -paracontact manifold, it follows that

$$\phi X = \bar{\nabla}_X \xi = \nabla_X \xi + h(X, \xi), \quad \xi, X \in TM.$$

If  $M$  is semi-invariant, then for  $X \in \mathcal{D}^0$  we get  $\nabla_X \xi = 0$  and  $\phi X = h(X, \xi)$ . Moreover, if  $M$  is assumed to be mixed foliated, then for  $X \in \mathcal{D}^0$  we get  $\phi X = 0$ , that is,  $\mathcal{D}^0 = \{0\}$ .  $\square$

*Remark 6.2.* The authors of [4] prove that  $LP$ -Sasakian manifolds do not admit proper mixed foliated semi-invariant submanifolds. But in view of Theorem 5.3,  $LP$ -Sasakian manifolds do not admit even proper semi-invariant submanifolds. Therefore, that result of [4] is redundant.

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