

On ϕ -skew symmetric conformal vector fields

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Abstract. The notion of the J -skew symmetric vector field was introduced in [MNR]. In the present paper, we deal with ϕ -skew symmetric conformal vector fields on a Kenmotsu manifold $M(\phi, \Omega, \eta, \xi, g)$. A necessary and sufficient condition for M to admit such a vector field C is given. In this case, C defines an infinitesimal relative conformal transformation of Ω and ϕC is a relatively integral invariant of Ω .

0. Introduction

Let $M(\phi, \Omega, \eta, \xi, g)$ be a $(2m + 1)$ -dimensional almost contact Riemannian manifold, where the structure tensors ϕ , η and ξ are a $(1, 1)$ -tensor field, a closed 1-form and the Reeb vector field, respectively, satisfying

$$\phi^2 = -Id + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

M is said to be a *Kenmotsu manifold* if the following conditions

$$(0.1) \quad (\nabla_Z \phi)Z' = -\eta(Z')\phi Z - g(Z, \phi Z')\xi,$$

$$(0.2) \quad \nabla_Z \xi = Z - \eta(Z)\xi, \quad Z, Z' \in \Gamma TM$$

hold good.

In the present paper, we assume that M carries a ϕ -skew symmetric conformal (abbr. SSC) vector field C (in the sense of [R1]), that is

$$(0.3) \quad \nabla C = fdp + C \wedge \phi C,$$

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where dp denotes the canonical vector valued 1-form and $\wedge$ the wedge product of vector fields on M .

It is known that (0.3) implies

$$\mathcal{L}_C g = \rho g, \quad \rho = 2f \in \Lambda^0 M.$$

We prove that the dual 1-form $C^\flat$ of C with respect to g is an exterior recurrent 1-form having $(\phi C)^\flat$ as the recurrent form and that $C^\flat$, $(\phi C)^\flat$ and η belong to the same ideal I . In consequence, M is foliated by 3 surfaces tangent to the distributions spanned by $\{\xi, C\}$, $\{\xi, \phi C\}$ and $\{C, \phi C\}$ respectively, and if Z is any vector field orthogonal to $\{\xi, \phi C\}$, then $\mathcal{L}_C \mathcal{R}(C, Z)$ defines an infinitesimal conformal transformation for $g(C, Z)$, where $\mathcal{R}$ denotes the Ricci tensor field of M .

Finally, by using the Lie algebra induced by $C^\flat$ and $(\phi C)^\flat$, it is shown that C defines an infinitesimal relative conformal transformation of Ω , i.e.

$$d(\mathcal{L}_C \Omega) = (d\rho + 2\rho\eta) \wedge \Omega$$

and that Ω is a relatively integral invariant [AM] of Ω , i.e.

$$d(\mathcal{L}_{\phi C} \Omega) = 0.$$

1. Preliminaries

Let (M, g) be an n -dimensional connected manifold and let ∇ be the covariant differential operator defined by the metric tensor g (we assume that M is oriented and ∇ is the Levi-Civita connection).

Let ΓTM be the set of sections of the tangent bundle and $\flat : TM \rightarrow T^*M$ and $\sharp : T^*M \rightarrow TM$ the classical isomorphisms defined by g (i.e., $\flat$ is the index lowering operator and $\sharp$ is the index raising operator).

We denote by

$$A^q(M, TM) = \Gamma \text{Hom}(\Lambda^q TM, TM)$$

the set of vector valued q -forms ($q \leq \dim M$) and following [P] we write for the covariant derivative with respect to ∇

$$d^\nabla : A^q(M, TM) \rightarrow A^{q+1}(M, TM)$$

(it should be noticed that in general $d^{\nabla^2} = d^{\nabla} \circ d^{\nabla} \neq 0$, unlike $d^2 = d \circ d = 0$).

If $p \in M$, then $dp \in A^1(M, TM)$ is the canonical vector valued 1-form and is called *the soldering form* of M . Since ∇ is symmetric, one has $d^{\nabla}(dp) = 0$.

The cohomology operator $[GL]$ is defined by

$$(1.1) \quad d^{\omega} = d + e(\omega)$$

and is acting on ΛM , where $e(\omega)$ denotes the exterior product by the closed 1-form ω . One has $d^{\omega} \circ d^{\omega} = 0$ and a form $\omega \in \Lambda M$ with $d^{\omega} \omega = 0$ is said to be d^{ω} -closed.

Let $\mathcal{O} = \{e_A \mid A = 1, \dots, n\}$ be a local field of orthonormal frames over M and let $\mathcal{O}^* = \{\omega^A\}$ be its associated coframe. The E. CARTAN's structure equation [C] written in the index-less manner are

$$(1.2) \quad \nabla e = \theta \otimes e,$$

$$(1.3) \quad d\omega = -\theta \wedge \omega,$$

$$(1.4) \quad d\theta = -\theta \wedge \theta + \Theta.$$

In the above equations θ (resp. Θ) are the local connections forms in the tangent bundle TM (resp. the curvature forms on M).

2. ϕ -skew symmetric conformal vector fields

Let $M(\phi, \Omega, \eta, \xi, g)$ be a $(2m+1)$ -dimensional Kenmotsu manifold [K], [MRV].

As is known, the quintuple of the structure tensor fields $(\phi, \Omega, \eta, \xi, g)$ satisfies the following equations:

$$(2.1) \quad \begin{cases} \phi^2 = -Id + \eta \otimes \xi, & \phi\xi = 0, & \eta(\xi) = 1, \\ g(Z, Z') = g(\phi Z, \phi Z') + \eta(Z)\eta(Z'), & \eta(Z) = g(\xi, Z), \\ (\nabla\phi)Z = -\eta(Z)\phi dp - (\phi Z)^{\flat} \otimes \xi, \\ \nabla\xi = dp - \eta \otimes \xi, \\ \Omega(Z, Z') = g(\phi Z, Z'), \end{cases}$$

for any vector fields $Z, Z' \in \Gamma TM$, and moreover we have

$$(2.2) \quad d\eta = 0, \quad d\Omega = 2\eta \wedge \Omega.$$

It should be noted that the equations (2.2) show that the pairing (η, Ω) defines a *conformal cosymplectic structure* $1 \times CS(m, \mathbf{R})$ [R1], [BR]. We also recall [R2] that the structure vector field ξ is (as in the case of a Sasakian manifold) always exterior concurrent (abbr. EC), that is

$$(2.3) \quad d^\nabla(\nabla\xi) = \nabla^2\xi = \xi \wedge dp.$$

In the present paper, we assume that M carries a vector field C such that its covariant differential satisfies

$$(2.4) \quad \nabla C = fdp + C \wedge \phi C, \quad f \in \Lambda^0 M.$$

As an extension of the concept of J -skew symmetric vector field [MNR], we agree to define C as a *ϕ -skew symmetric conformal vector field*.

Let Z be any vector field on M . If we denote by Z^A ($A \in \{0, \dots, 2m\}$) its components with respect to an orthonormal frame $\mathcal{O} = \{e_0 = \xi, e_1, \dots, e_m, e_{m+1} = \phi e_1, \dots, e_{2m} = \phi e_m\}$, then, on behalf of the 4-th equation of (2.1), its covariant derivative is expressed by

$$(2.5) \quad \nabla Z = (dZ^A + Z^B \theta_B^A + Z^0 \omega^A) \otimes e_A + (dZ^0 + Z^b) \otimes \xi, \quad Z^0 = \eta(Z).$$

With respect to $\mathcal{O}^*$, we have

$$(2.6) \quad \Omega = \sum_{a=1}^m \omega^a \wedge \omega^{a^*}, \quad a^* = a + m.$$

We come back to the case under discussion. Since (2.4) is expressed as

$$(2.7) \quad \nabla C = fdp + (\phi C)^b \otimes C - C^b \otimes \phi C,$$

one quickly finds

$$g(\nabla_Z C, Z') + g(\nabla_{Z'} C, Z) = 2fg(Z, Z'),$$

which is equivalent to $\mathcal{L}_C g = 2fg$.

This, as is known, shows that C is a conformal vector field having $\rho = 2f$ as the conformal scalar.

Further by (2.5) and (2.7) one may write

$$(2.8) \quad \begin{cases} dC^a + C^A \theta_A^a = (f - C^a) \omega^a + C^a (\phi C)^b + C^{a*} C^b, \\ dC^{a*} + C^A \theta_A^{a*} = \theta - C^0 \omega^{a*} + C^{a*} (\phi C)^b - C^A C^b, \\ dC^0 = (f - 1) \eta + C^0 (\phi C)^b + C^b. \end{cases}$$

Since $C^b = C^0 \eta + \sum_{A=1}^{2m} C^A \omega^A$, then by E. Cartan's structure equations one infers from (2.8)

$$(2.9) \quad dC^b = 2(\phi C)^b \wedge C^b.$$

This proves that C^b is a recurrent form [D] having $2(\phi C)^b$ as the recurrence form, and so one refinds ROSCA's lemma induced by the concept of skew symmetric vector fields [R1], [R2].

Next, since

$$(2.10) \quad (\phi C)^b = \sum_{a=1}^m (C^a \omega^{a*} - C^{a*} \omega^a),$$

one infers by (2.8) and E. Cartan's structure equations

$$(2.11) \quad d(\phi C)^b = 2(f - C^0) \Omega + \eta \wedge (C^b + (\phi C)^b).$$

Now by the exterior differentiation of (2.11), one derives on behalf of (2.2)

$$(2.12) \quad f = C^0$$

and

$$(2.13) \quad \eta \wedge C^b \wedge (\phi C)^b = 0.$$

Hence by (2.13) one may say that the forms η , C^b and $(\phi C)^b$ belong to the same ideal I .

Conversely, by straightforward computations, one may prove that if a vector field C on M satisfies (2.9) and (2.13), then it implies (2.4).

By (2.12) one has

$$(2.14) \quad d(\phi C)^b = \eta \wedge (C^b + (\phi C)^b) = 0.$$

Then by (2.9) and (2.13) it is seen that the three 2-forms $(\phi C)^b \wedge C^b$, $\eta \wedge C^b$ and $\eta \wedge (\phi C)^b$ are closed. Therefore, if the Kenmotsu manifold M

under consideration carries a ϕ -SSC vector field C , then it is foliated by 3 surfaces tangent to the distributions spanned by $\{\xi, C\}$, $\{\xi, \phi C\}$ and $\{C, \phi C\}$, respectively.

Next by (2.12) and the third equation of (2.8) one may write

$$(2.15) \quad \text{grad } \rho = (\rho - 2)\xi + \rho\phi C + 2C.$$

On the other hand, by the third equation of (2.1) one derives

$$(2.16) \quad \nabla\phi C = (\phi C)^\flat \otimes \phi C + C^\flat C - \left((\phi C)^\flat + \frac{\rho}{2}C^\flat\right) \otimes \xi$$

and by a standard calculation one gets

$$(2.17) \quad \text{div } \phi C = \|C\|^2 - \frac{\rho^2}{4}.$$

Since C is a conformal vector field on M , one has as is known $\text{div } C = \frac{2m+1}{2}\rho$ and also finds

$$(2.18) \quad \langle dp, \phi C \rangle = \rho \left(2f - \frac{\rho^2}{4} \right), \quad \langle dp, \xi \rangle = 2(\rho - 1).$$

Hence by (2.15) and (2.16) one gets

$$\Delta\rho = -\text{div}(\text{grad } \rho) = 2(1 + 2m) - (3 + 4m + 4l)\rho + \frac{\rho}{2},$$

where $2l = \|C\|^2$.

Now by Yano's formula [B], that is

$$\mathcal{L}_C K = 2m\Delta\rho - K\rho,$$

one may write

$$\mathcal{L}_C K = 2(1 + 2m)2m - [2m(3 + 4m + 2l) + K]\rho - \frac{\rho^3}{2},$$

where K denotes the scalar curvature of M .

Next, by the general formula for conformal vector fields (see [B]), since we know that

$$2\mathcal{L}_C \mathcal{R}(Z, Z') = (\Delta\rho)g(Z, Z') - (2m - 1)\text{Hess}_\rho(Z, Z'),$$

where $\mathcal{R}$ means the Ricci tensor of M , one finds, after some calculation, that

$$\begin{aligned} 2\mathcal{L}_C\mathcal{R}(C, Z) &= [\Delta\rho - 2\rho(1+l)(2m-1) + (2m-1)]g(C, Z) \\ &\quad - (2m-1)\frac{\rho}{2}(\rho-2)g(\phi C, Z) + 4l\left(1 - \frac{\rho^2}{4}\right)\eta(Z). \end{aligned}$$

Hence for any Z orthogonal to the surface S tangent to the distribution spanned by $\{\xi, \phi C\}$, $\mathcal{L}_C\mathcal{R}(C, Z)$ defines an infinitesimal transformation for $g(C, Z)$.

On the other hand, by (2.1) one has

$$(2.19) \quad \mathcal{L}_C\Omega = \rho\Omega + \eta \wedge (C^b - (\phi C)^b).$$

From (2.19) and (2.13) one derives

$$(2.20) \quad d(\mathcal{L}_C\Omega) = (d\rho + 2\rho\eta) \wedge \Omega$$

and so, according to the definition, it follows that C is an *infinitesimal relative conformal transformation* of Ω .

It should be noticed that since η is closed, then making use of the cohomological transformation operator d^η (see Section 1), one may also write

$$(2.21) \quad d^\eta(\mathcal{L}_C\Omega) = \eta \wedge \mathcal{L}_C\Omega$$

and say that C defines an *infinitesimal conformal cohomological transformation* of Ω .

Next, by a short calculation, one gets from (2.2)

$$\mathcal{L}_{\phi C}\Omega = \frac{1}{2}d\rho \wedge \eta - d(\phi C)^b,$$

and consequently

$$d(\mathcal{L}_{\phi C}\Omega) = 0.$$

Hence, following the definition (see also [AM]), the above equation says that ϕC is a *relatively integral invariant* of Ω .

Summarizing, up these computations, we have the following

Theorem. *Let $M(\phi, \Omega, \eta, \xi, g)$ be a $(2m + 1)$ -dimensional Kenmotsu manifold. Then the necessary and sufficient condition in order that M carries a ϕ -skew symmetric conformal vector field C , that is*

$$\nabla_Z C = fZ + g(Z, \phi C)C - g(Z, C)\phi C, \quad Z \in \Gamma TM,$$

is that

$$dC^\flat = 2(\phi C)^\flat \wedge C^\flat, \quad C^\flat \wedge (\phi C)^\flat \wedge \eta = 0$$

(i.e. $C^\flat$ is exterior recurrent with $(\phi C)^\flat$ as the recurrence form and $C^\flat$, $(\phi C)^\flat$ and η belong to the same ideal).

Any such a Kenmotsu manifold is foliated by 3 surfaces tangent to the distributions spanned by $\{\xi, C\}$, $\{\xi, \phi C\}$ and $\{C, \phi C\}$ respectively.

If Z it is any vector field orthogonal to the surface S tangent to the distribution spanned by $\{\xi, \phi C\}$, then $\mathcal{L}_C \mathcal{R}(C, Z)$ defines an infinitesimal conformal transformation for $g(Z, C)$.

In addition, C defines an infinitesimal relative conformal transformation of Ω , i.e.,

$$d(\mathcal{L}_C \Omega) = (d\rho + 2\rho\eta) \wedge \Omega, \quad \rho = 2f$$

and ϕC is a relatively integral invariant of Ω , i.e.

$$d(\mathcal{L}_{\phi C} \Omega) = 0.$$

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References

- [AM] R. ABRAHAM and J. MARSDEN, Foundations of Mechanics, *Benjamin, New York*, 1972.
- [B] T. BRANSON, Conformally covariant equations on differential forms, *Comm. Part. Diff. Eq.* **11** (1982), 393–431.
- [BR] K. BUCHNER and R. ROSCA, Cosymplectic quasi-Sasakian manifold with a Φ -structure vector field ξ , *An. Ştiinţ. Univ. Al. I. Cuza Iasi* **37** (1991), 215–223.
- [C] E. CARTAN, Systèmes Différentiels Extérieurs. Applications Géométriques, *Hermann, Paris*, 1945.
- [D] D. K. DATTA, Exterior recurrent forms on a manifold, *Tensor NS* **36** (1982), 115–120.

- [GL] F. GUEDIRA and L. LICHNEROWICZ, Géométrie des algèbres de Lie locales de Kirilov, *J. Math. Pures Appl.* **63** (1984), 407–484.
- [K] K. KENMOTSU, A class of almost contact Riemannian manifolds, *Tohoku Math. J.* **24** (1972), 93–103.
- [MMS] K. MATSUMOTO, I. MIHAI and M. SHAHID, Certain submanifolds of a Kenmotsu manifold, Proc. 3-rd Pacific Rim Geom. Conf., *International Press, Cambridge MA*, 1998, 183–193.
- [MNR] I. MIHAI, L. NICOLESCU and R. ROSCA, On para-Kaehlerian manifolds $M(J, g)$ and on skew-symmetric Killing vector fields carried by M , *Portugal. Math.* **54** (1997), 215–228.
- [MRV] I. MIHAI, R. ROSCA and L. VERSTRAELEN, Some Aspects of the Differential Geometry of Vector Fields, PADGE, vol. 2, *K.U. Leuven, K.U. Brussel*, 1996.
- [P] W. A. POOR, Differential Geometric Structures, *McGraw Hill, New York*, 1981.
- [R1] R. ROSCA, On conformal cosymplectic quasi Sasakian manifold, *Giornate di Geometria, Messina*, 1988.
- [R2] R. ROSCA, On exterior concurrent skew symmetric Killing vector field, *Rend. Sem. Mat. Messina* **2** (1993), 137–145.

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