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Normal subgroups of the group of units in group rings of torsion groups

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Dedicated to Professor Kálmán Győry on his 60th birthday

Abstract. Let $\mathbb{F}G$ be the group algebra of a finite p-group G over a field of characteristic p, and $V(\mathbb{F}G)$ the subgroup of units of augmentation 1. We show that $V(\mathbb{F}G)$ and all of its normal subgroups are such that all factors of their upper central series, with the possible exception of the centre, are elementary abelian p-groups. We give an extension of these results for torsion groups G with some restriction on the group ring RG.

1. Introduction

Let RG be the group ring of a group G over an integral domain R. We denote by $U(RG)$ the group of units of the group ring RG and by $V(RG)$ the subgroup of its normalized units; i.e., the set of all units of augmentation 1.

Group algebras have been characterized under different group-theoretical assumption on its group of units, such as being nilpotent, solvable or n-Engel. One of the hardest and most important problems for modular group algebras consists in describing the structure of its group of units.

First Coleman and Passman [6] proved that, if $\mathbb{F}_pG$ is a group algebra of a finite nonabelian p-group G over the field $\mathbb{F}_p$ of p elements, then

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$V(\mathbb{F}_pG)$ is not a p-regular group and has exponent at least p^2. Shalev [13] and Kurdics [10] pointed out that $V(\mathbb{F}_pG)$ is not metabelian for $p > 3$ and Shalev and Mann [11] showed that the wreath product $C_p \wr A$, for $|A| = p^2$, is a section of $V(\mathbb{F}_pG)$ under weak hypotheses on G.

Let $\mathbb{F}$ be a field of characteristic p. Note that $V(\mathbb{F}G) = 1 + A(\mathbb{F}G)$ for a finite p-group G, where $A(\mathbb{F}G)$ is the augmentation ideal of $\mathbb{F}G$ and the adjoint group $(A(\mathbb{F}G), \circ)$ is isomorphic to $V = V(\mathbb{F}G)$. Thus V is a circle group and by Du's Theorem [7], the k-th term $\zeta_k(V)$ of the upper central series of V coincides with $1+C_k$, where C_k is both a subring and a Lie ideal of $\mathbb{F}G$ and, at the same time, the k-th term of the upper central series of the associated Lie algebra of the augmentation ideal $A(\mathbb{F}G)$. As a consequence, we can obtain the following assertion: the factors $\zeta_{i+1}(V)/\zeta_i(V)$ of the upper central series, with the possible exception of the centre of V, are elementary abelian p-groups.

We extend this result for all normal subgroups of the nilpotent group $V(\mathbb{F}G)$. Moreover, this is true for all nilpotent normal subgroups in $V(\mathbb{F}G)$ with weaker assumptions about the group G, namely, for any torsion group G with some restrictions on $\mathbb{F}G$. This generalizes the results proved by Bovdi and Khripta [1], [2] for abelian and solvable normal subgroups.

2. Normal subgroups of the group of units

Let m be the order of an element $g \in G$ and assume that $1 - \alpha^m \in U(R)$ for some $\alpha \in R$. Then, it is easy to see that $g - \alpha \in U(RG)$ and $(g-\alpha)^{-1} = (1-\alpha^m)^{-1} \sum_{i=0}^{m-1} \alpha^{m-1-i} g^i$.

Definition. We say that a torsion group G is R-favourable, if, for each element $g \in G$, $g \neq 1$, there exists $0 \neq \alpha \in R$ such that $g - \alpha \in U(RG)$.

We note that a torsion group G is R-favourable in the following cases:

1. R is an infinite local domain;
2. G is a p-group and R is an integral domain of characteristic p with a nontrivial group of units;
3. R is a field, $R \neq \mathbb{F}_2$ and, if G has 2-torsion, then $|R| \neq 1 + 2^n$.

We will denote by $\Delta(G) = \{h \in G \mid [G : C_G(h)] < \infty\}$ the *FC-centre* of G and by $\Delta^+(G)$ the maximal torsion subgroup of $\Delta(G)$.

Lemma 1. *Let R be a commutative integral domain of characteristic $p \geq 0$ and let G be a torsion R-favourable group. If A is a normal subgroup in $V(RG)$, such that any two elements of A which are conjugate in $V(RG)$ commute, then A has one of the following properties:*

(i) *If either $p = 0$ or $p > 2$ and p does not divide the order of any element of the subgroup $\Delta(G)$, then A is a central subgroup in $V(RG)$.*

(ii) *If $p = 2$ then A^2 is a central subgroup in $V(RG)$.*

(iii) *If $p > 2$ and A is a noncentral subgroup in $V(RG)$, then A^p and $A \cap G$ are central subgroups in $V(RG)$, p divides the order of some element of $\Delta(G)$.*

(iv) *If A is a noncentral subgroup, then any conjugate of $h \in A$ is of the form ha where a is an element of order p.*

Proof. Choose elements $g \in G$ and $h \in A$ such that the commutator of g, h in G is not 1, i.e. $(g, h) \neq 1$. Since G is R-favourable, we have that $g - c$ is a unit for some $0 \neq c \in R$ and thus $1 - c \in U(R)$. Then $(1-c)^{-1}(g - c) \in V(RG)$ and there exist $h_0, h_c \in A$ such that

$$ghg^{-1} = h_0, \qquad (g - c)h = h_c(g - c). \tag{1}$$

Clearly, $(h_0 - h_c)g = c(h - h_c)$. Since h, h_0 and h_c are elements of A, which are conjugate, they commute. Thus the elements $(h_0 - h_c)g$ and h also commute, so we obtain the equality $(h_0 - h_c)hg = (h_0 - h_c)gh$, which implies

$$(h_0 - h)(h_0 - h_c) = 0. \tag{2}$$

From (1) and (2) we can see that $0 = (h - h_0)(h_0 - h_c)g = c(h - h_0)(h - h_c)$. It follows that $(h - h_0)(h - h_c) = 0$ and by (2) we obtain

$$(ghg^{-1} - h)^2 = 0. \tag{3}$$

Consequently, if $p = 2$, then A^2 is a central subgroup in $V(RG)$.

Suppose that $p \neq 2$. By (3) we have that

$$h^2g - 2hgh + gh^2 = 0. \tag{4}$$

Note that, if g commutes with h, the above relation is trivially true, so this identity holds for $g \in G$.

We consider in RG the inner derivation given by $d(x) = hx - xh$, for all $x \in RG$. By (4) we have that $d^2(x) = 0$ for all $x \in RG$. Since

$$d^2(xy) = d^2(x)y + 2d(x)d(y) + xd^2(y),$$

we get that $d(x)d(y) = 0$. If we take $y = zx$, then

$$d(x)(d(z)x + zd(x)) = d(x)d(zx) = 0$$

and we can see that $d(x)zd(x) = 0$ for all $x, z \in RG$ and $RGd(x)RG$ is a nilpotent ideal. By Passman's theorem ([12], Theorem 4.2.13), if the characteristic of R does not divide the order of any element of $\Delta(G)$, then the ring RG is semiprime and, in this case, we have that $RGd(x)RG = 0$. This implies that, if the characteristic of R is either 0 or $p > 2$, then $d(g) = 0$ for all $g \in G$ and A is central in $V(RG)$, which is a contradiction. Hence, (i) follows.

Now assume that p divides the order of some element of $\Delta(G)$. From (3) we conclude that $(h - g^{-1}hg)^p = 0$. Therefore, $h^p = g^{-1}h^pg$ for any $g \in G$ and $h \in A$, which shows that A^p is a central subgroup of $V(RG)$. If $h \in A \cap G$, then from (3) we obtain that $g^{-1}h^2g - 2g^{-1}hgh + h^2 = 0$. Comparing elements in G we get $gh = hg$.

Let $h \in A$ and $u \in V(RG)$. We claim that the commutator (h, u) is of order p. Indeed, since any two conjugate elements in A commute and A^p is central, we see that $(h^{-1}u^{-1}hu)^p = (h^{-1})^p(u^{-1}hu)^p = (h^{-1})^pu^{-1}h^pu = 1$. Consequently, the conjugate of h is of form ha, where $a \in A$ has order p. □

Lemma 2. *Let $\mathbb{F}_2$ be the field of two elements and let G be a locally finite 2-group. If A is a normal subgroup of $V(\mathbb{F}_2G)$, such that any two elements of A which are conjugates in $V(\mathbb{F}_2G)$ commute, then A^4 is central in $V(\mathbb{F}_2G)$.*

Proof. Suppose that A is noncentral and that $gh \neq hg$ for some $g \in G$ and $h \in A$. Let h_2 be a conjugate of h in $V(\mathbb{F}_2G)$. Since G is a locally finite group, the element $g + h_2 + 1$ is a unit. Set $h_0 = ghg^{-1}$ and $h_1 = (g + h_2 + 1)h(g + h_2 + 1)^{-1}$. Then $h_0, h_1 \in A$ and

$$(h_0 + h_1)g = (h_2 + 1)(h + h_1). \tag{5}$$

According to our assumption, the conjugates h, h_1 and h_2 commute, so it follows that the elements $(h_0 + h_1)g$ and h also commute. Thus

$(h_0+h)(h_0+h_1)=0$ and from (5) we have $(h_0+h)(h_2+1)(h+h_1)=0$. From these we can conclude that

$$\begin{aligned}(h_0+h)^2(h_2+1) &= (h+h_0)(h_0+h_1+h_1+h)(h_2+1)\\ &= (h+h_0)(h+h_1)(h_2+1)=0.\end{aligned}$$

If we put $h_2=h$ and $h_2=h_0$, it follows that

$$(h_0+h)^2(h_0+1)=0 \quad \text{and} \quad (h_0+h)^2(h+1)=0.$$

As a consequence we have that $(h_0+h)^3=0$ and $h_0^4=h^4$. Therefore, h^4 is a central element in $V(\mathbb{F}_2G)$. □

Theorem. *Let R be a commutative integral domain of characteristic $p\geq 0$ with a nontrivial group of units and let G be a torsion R-favourable group. Then*

(i) *if $p=0$ or $p>2$ and p does not divide the order of any element of $\Delta(G)$, then every solvable normal subgroup N of $V(RG)$ is central;*

(ii) *if $p>0$ and N is a nilpotent normal subgroup of class c, then all factors of the upper central series of N, with the possible exception of its centre, are elementary abelian p-groups and $N^{p^{c-1}}$ is a central subgroup in $V(RG)$.*

Proof. Suppose that a solvable normal subgroup N of $V(RG)$ is noncentral. Then there exists a noncentral solvable normal subgroup H in $V(RG)$ with abelian commutator subgroup H'. By Lemma 1, H' is central in $V(RG)$. Thus any two elements from H which are conjugate in $V(RG)$ commute. So, we can also apply Lemma 1 to H and conclude that it is a central subgroup of $V(RG)$, which is a contradiction.

Let N be a nilpotent normal subgroup of $V(RG)$ and let

$$1\subset\zeta_1(N)\subset\zeta_2(N)\subset\cdots\subset\zeta_c(N)=N$$

be the upper central series of N. Clearly, the subgroups $\zeta_i(N)$ are normal in $V(RG)$ and $\zeta_2(N)$ is a nilpotent group of class two with the property that conjugate elements commute. By Lemma 1, the factor group $\zeta_2(N)/\zeta_1(N)$ has exponent p. It follows from [8, Theorem 4.2.13] that any factor of the upper central series also has exponent p. Therefore, the exponent of $N/\zeta_1(N)$ is a divisor of p^{c-1}. □

Remark 1. If G is not R-favourable then the statement of Theorem does not always hold. For example, let L be a normal subgroup of odd order in a finite group G such that G/L is isomorphic to the dihedral group D_6 of order 6. Then $V(\mathbb{F}_2G)$ has a noncentral solvable normal subgroup which is isomorphic to D_6 (see [1]).

Corollary 1. *Let R be the ring of p-adic integers and let G be a p-group. Then every solvable normal subgroup of $V(RG)$ is central.*

Suppose that $\mathbb{F}G$ is infinite. If $\Delta(V)$ is the FC-centre of $V(\mathbb{F}G)$ and $\Delta^+(V)$ is the subgroup of $\Delta(V)$ consisting of all elements of finite order in $\Delta(V)$ then, by [4], all elements of the commutator subgroup of $\Delta^+(V)$ are unipotent and central in $\Delta(V)$. By Theorem we can conlude the following.

Corollary 2. *Let $\mathbb{F}G$ be an infinite group algebra of characteristic p such that G is $\mathbb{F}$-favourable. Then $\Delta^+(V)^p$ is a central subgroup.*

As it is well-known, by Khripta's theorem [9], the group of units of a modular group algebra KG of characteristic p is nilpotent if and only if G is nilpotent and the commutator subgroup G' is a finite p-group. Thus, we obtain the following:

Corollary 3. *Let G be a nilpotent p-group with finite commutator subgroup and let $\mathbb{F}$ be a field of characteristic p. Then, all normal subgroups of $V(\mathbb{F}G)$ are such that all factors of their upper central series, with the possible exception of the centre, are elementary abelian p-groups.*

Remark 2. Let G be a nilpotent p-group with finite commutator subgroup and let $\mathbb{F}_p$ be the field with p elements. Then by Coleman's theorem [5] the intersection of the commutator subgroup V' of $V(\mathbb{F}_pG)$ with G is the commutator subgroup G' of G. Therefore, G/G' is isomorphic to a subgroup of V/V' and the factors of the lower central series of V do not need to be elementary abelian subgroups.

Definition. Let G be a finite p-group. The *depth* of the conjugacy class C_g is the nonnegative integer d such that $C_G(g) = C_G(g^{p^d})$ and that $C_G(g^{p^d})$ is a proper subgroup of $C_G(g^{p^{d+1}})$.

Let G be a finite p-group and C_g be the conjugacy class of G containing $g \in G$. An element of the form $\widehat{C_g} = \sum_{g\in C_g} g$ of $\mathbb{F}G$ is called a *class sum.* We recall that the set of all class sums is a basis of the centre of $\mathbb{F}G$ over $\mathbb{F}$.

Let p^k be the exponent of the centre of a finite p-group G and let r be the greatest depth of the conjugacy classes of G. According to the result in [3], if the depth of the conjugacy class C_g is d, then $\widehat{C_g}^{p^i} = \widehat{C_{g^{p^i}}}$ for $i = 1, \dots, d$ and $\widehat{C_g}^{p^{d+1}} = 0$. From these we can conclude that the exponent p^t of the centre of $V(\mathbb{F}G)$ is equal to $\max\{p^k, p^{r+1}\}$.

Let $L(A)$ be the associated Lie algebra of the augmentation ideal $A(\mathbb{F}G)$ and denote by C_k the k-th term of the upper central series of $L(A)$. Then

$$C_k = \{a \in A(\mathbb{F}G) \mid [a, x] \in C_{k-1} \text{ for all } x \in A(\mathbb{F}G)\}.$$

Du's theorem [7] described the k-th term $\zeta_k(V)$ of the upper central series of $V(\mathbb{F}_pG)$ as

$$\zeta_k(V) = 1 + C_k.$$

It is well-known that $L(A)$ is a restricted Lie algebra, thus it satisfies the identity

$$[x, y, p] = [x, y, y, \dots, y] = [x, y^p]. \tag{1}$$

Thus if $u = 1 + a \in \zeta_k(V)$ and $v = 1 + b \in V$ then $a \in C_k$ and $b \in A(\mathbb{F}_pG)$. As a consequence (1), we obtain the assertion: $[a^p, b] \in C_{k-p}$ for all $b \in A(\mathbb{F}_pG)$ and $a^p \in C_{k-p+1}$. By Du's theorem $v^p = 1 + a^p \in \zeta_{k-p+1}(V)$ and the factor group $\zeta_k(V)/\zeta_{k-p+1}(V)$ has exponent p.

Applying Theorem we obtain the following.

Corollary 4. *Let G be a finite p-group and $\mathbb{F}$ be a field of characteristic p. Denote by c and p^n the nilpotency class and the exponent of the group $V(\mathbb{F}G)$, respectively. Then $p^n \leq p^{[\frac{c-1}{p}]+t}$, where $[\frac{c-1}{p}]$ is the upper integral part of $\frac{c-1}{p}$.*

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