

A rigidity theorem for hypersurfaces in a sphere

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Abstract. Let M^n be a closed hypersurface in the unit sphere S^{n+1} . Denote by $|h|^2$ and H the square of the length of its second fundamental form and the mean curvature, respectively, suppose that $|\nabla h|^2 \geq n^2 |\nabla H|^2$. If $|h|^2 < 2\sqrt{n-1}$, M^n is a small hypersphere in S^{n+1} . We also characterize all M^n with $|h|^2 = 2\sqrt{n-1}$. When M^n has constant mean curvature, it is just the result of HOU [3].

1. Introduction

Let S^{n+1} be an $(n+1)$ -dimensional unit sphere with constant sectional curvature 1, let M^n be an n -dimensional closed hypersurface in S^{n+1} , and $e_1, \dots, e_n$ a local orthonormal frame field on M^n , $\omega_1, \dots, \omega_n$ its dual coframe field. Then the second fundamental form of M^n is

$$h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j.$$

Denote by H the mean curvature and $|h|^2$ the square of the length of the second fundamental form. As it is well known, there are many rigidity results for minimal hypersurfaces or hypersurfaces with constant mean curvature H in S^{n+1} by use of J. SIMONS' method, for example, see [1], [4], [6], [8] etc. In [3], HOU proved the following theorem:

Mathematics Subject Classification: 53C42, 53A10.

Key words and phrases: hypersurface, sphere, second fundamental form, mean curvature.

This work is supported in part by the National Natural Science Foundation of China.

Theorem A. *Let M^n be a closed hypersurface of constant mean curvature in S^{n+1} . Then*

- (1) *If $|h|^2 < 2\sqrt{n-1}$, M^n is a small hypersphere $S^n(r)$ of radius $r = \sqrt{\frac{n}{n+|h|^2}}$.*
- (2) *If $|h|^2 = 2\sqrt{n-1}$, M^n is either a small hypersphere $S^n(r_0)$ or $H(r)$ -torus $S^1(r) \times S^{n-1}(s)$, where $r_0^2 = \frac{n}{n+2\sqrt{n-1}}$, $r^2 = \frac{1}{\sqrt{n-1}+1}$ and $s^2 = \frac{\sqrt{n-1}}{\sqrt{n-1}+1}$.*

In the present paper, we would like to use Cheng–Yau’s self-adjoint operator $\square$ to prove the following general rigidity theorem.

Theorem B. *Let M^n be an n -dimensional closed hypersurface in S^{n+1} with $|\nabla h|^2 \geq n^2|\nabla H|^2$. Then*

- (1) *If $|h|^2 < 2\sqrt{n-1}$, M^n is a small hypersphere $S^n(r)$ of radius $r = \sqrt{\frac{n}{n+|h|^2}}$.*
- (2) *If $|h|^2 = 2\sqrt{n-1}$, M^n is either a small hypersphere $S^n(r_0)$ or $H(r)$ -torus $S^1(r) \times S^{n-1}(s)$, where $r_0^2 = \frac{n}{n+2\sqrt{n-1}}$, $r^2 = \frac{1}{\sqrt{n-1}+1}$ and $s^2 = \frac{\sqrt{n-1}}{\sqrt{n-1}+1}$.*

Obviously, when M^n has constant mean curvature, Theorem B becomes Theorem A, so Theorem B is an extension of Theorem A.

2. Preliminaries

Let M^n be an n -dimensional closed hypersurface in S^{n+1} . We choose a local orthonormal frame $e_1, \dots, e_{n+1}$ in S^{n+1} such that at each point of M^n , $e_1, \dots, e_n$ span the tangent space of M^n and form an orthonormal frame there. Let $\omega_1, \dots, \omega_{n+1}$ be its dual coframe. In this paper, we use the following convention on the range of indices:

$$1 \leq A, B, C, \dots \leq n+1; \quad 1 \leq i, j, k, \dots \leq n.$$

Then the structure equations of S^{n+1} are given by

$$(1) \quad d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$(2) \quad d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D,$$

$$(3) \quad K_{ABCD} = (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}).$$

Restrict these forms to M^n , we have

$$(4) \quad \omega_{n+1} = 0.$$

From Cartan's lemma we can write

$$(5) \quad \omega_{n+1i} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}.$$

From these formulas, we obtain the structure equations of M^n :

$$(6) \quad d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$(7) \quad d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,$$

$$(8) \quad R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}),$$

where R_{ijkl} are the components of the curvature tensor of M^n and

$$(9) \quad h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j$$

is the second fundamental form of M^n . We also have

$$(10) \quad R_{ij} = (n-1)\delta_{ij} + nHh_{ij} - \sum_k h_{ik}h_{kj},$$

$$(11) \quad n(n-1)(R-1) = n^2H^2 - |h|^2,$$

where R is the normalized scalar curvature, and H the mean curvature.

Define the first and the second covariant derivatives of h_{ij} , say h_{ijk} and h_{ijkl} by

$$(12) \quad \sum_k h_{ijk} \omega_k = dh_{ij} + \sum_k h_{kj} \omega_{ki} + \sum_k h_{ik} \omega_{kj},$$

$$(13) \quad \sum_l h_{ijkl} \omega_l = dh_{ijk} + \sum_m h_{mjk} \omega_{mi} + \sum_m h_{imk} \omega_{mj} + \sum_m h_{ijm} \omega_{mk}.$$

Then we have the Codazzi equation

$$(14) \quad h_{ijk} = h_{ikj},$$

and the Ricci's identity

$$(15) \quad h_{ijkl} - h_{ijlk} = \sum_m h_{mj} R_{mikl} + \sum_m h_{im} R_{mjkl}.$$

For a C^2 -function f defined on M^n , we define its gradient and Hessian f_{ij} by the following formulas

$$(16) \quad df = \sum_i f_i \omega_i, \quad \sum_j f_{ij} \omega_j = df_i + \sum_j f_j \omega_{ji}.$$

The Laplacian of f is defined by $\Delta f = \sum_i f_{ii}$.

Let $\phi = \sum_{ij} \phi_{ij} \omega_i \otimes \omega_j$ be a symmetric tensor defined on M^n , where

$$(17) \quad \phi_{ij} = nH\delta_{ij} - h_{ij}.$$

Following CHENG-YAU [2], we introduce an operator $\square$ associated to ϕ acting on any C^2 -function f by

$$(18) \quad \square f = \sum_{i,j} \phi_{ij} f_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}) f_{ij}.$$

Since ϕ_{ij} is divergence-free, it follows [2] that the operator $\square$ is self-adjoint relative to the L^2 inner product of M^n , i.e.,

$$(19) \quad \int_M f \square g = \int_M g \square f.$$

We can choose a local frame field $e_1, \dots, e_n$ at any point $p \in M$, such that $h_{ij} = \lambda_i \delta_{ij}$, by use of (18) and (11), we have

$$(20) \quad \begin{aligned} \square(nH) &= nH\Delta(nH) - \sum_i \lambda_i (nH)_{ii} \\ &= \frac{1}{2} \Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i \lambda_i (nH)_{ii} \\ &= \frac{1}{2} n(n-1) \Delta R + \frac{1}{2} \Delta |h|^2 - n^2 |\nabla H|^2 - \sum_i \lambda_i (nH)_{ii}. \end{aligned}$$

On the other hand, through a standard calculation by use of (14) and (15), we get

$$(21) \quad \frac{1}{2}\Delta|h|^2 = \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + \frac{1}{2} \sum_{i,j} R_{ijij} (\lambda_i - \lambda_j)^2.$$

Putting (21) into (20), we have

$$(22) \quad \square(nH) = \frac{1}{2}n(n-1)\Delta R + |\nabla h|^2 - n^2|\nabla H|^2 + \frac{1}{2} \sum_{i,j} R_{ijij} (\lambda_i - \lambda_j)^2.$$

Because M^n is closed, we obtain the following formula by integrating (22) and by noting $\int_{M^n} \Delta R = 0$ and $\int_{M^n} \square(nH) = 0$

$$(23) \quad \int_{M^n} \left[|\nabla h|^2 - n^2|\nabla H|^2 + \frac{1}{2} \sum_{i,j} R_{ijij} (\lambda_i - \lambda_j)^2 \right] = 0.$$

From (8), we have $R_{ijij} = 1 - \lambda_i \lambda_j$, $i \neq j$, and by putting this into (23), we obtain

$$(24) \quad \int_{M^n} \left[|\nabla h|^2 - n^2|\nabla H|^2 + n|h|^2 - n^2H^2 - |h|^4 + nH \sum_i \lambda_i^3 \right] = 0.$$

Let $\mu_i = \lambda_i - H$ and $|Z|^2 = \sum_i \mu_i^2$, we have

$$(25) \quad \sum_i \mu_i = 0, \quad |Z|^2 = |h|^2 - nH^2,$$

$$(26) \quad \sum_i \lambda_i^3 = \sum_i \mu_i^3 + 3H|Z|^2 + nH^3.$$

From (24)–(26), we get

$$(27) \quad \int_{M^n} \left[|\nabla h|^2 - n^2|\nabla H|^2 + |Z|^2(n + nH^2 - |Z|^2) + nH \sum_i \mu_i^3 \right] = 0.$$

We need the following algebraic lemma due to M. OKUMURU [7] (see also [1]).

Lemma 2.1. *Let $\mu_i, i = 1, \dots, n$, be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where $\beta = \text{constant} \geq 0$. then*

$$(28) \quad -\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3,$$

and the equality holds in (28) if and only if at least $(n-1)$ of the μ_i are equal.

By use of Lemma 2.1, we have

$$(29) \quad \int_{M^n} \left[|\nabla h|^2 - n^2 |\nabla H|^2 + |Z|^2 (n + nH^2 - |Z|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} |H||Z|) \right] \leq 0.$$

3. Proof of Thoeorem B

Now consider the quadratic form $Q(u, t) = u^2 - \frac{n-2}{\sqrt{n-1}}ut - t^2$. By the orthogonal transformation

$$\begin{cases} \bar{u} = \frac{1}{\sqrt{2n}} \{ (1 + \sqrt{n-1})u + (1 - \sqrt{n-1})t \} \\ \bar{t} = \frac{1}{\sqrt{2n}} \{ (\sqrt{n-1} - 1)u + (\sqrt{n-1} + 1)t \}. \end{cases}$$

$Q(u, t)$ turns into $Q(u, t) = \frac{n}{2\sqrt{n-1}}(\bar{u}^2 - \bar{t}^2)$, where $\bar{u}^2 + \bar{t}^2 = u^2 + t^2 = |h|^2$.

Take $u = \sqrt{n}|H|$, $t = |Z|$, then

$$\begin{aligned} n + nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} |H||Z| - |Z|^2 &= n + Q(u, t) = nc + \frac{n(\bar{u}^2 - \bar{t}^2)}{2\sqrt{n-1}} \\ (30) \quad &= n + \frac{n(-\bar{u}^2 - \bar{t}^2)}{2\sqrt{n-1}} + \frac{n\bar{u}^2}{\sqrt{n-1}} \geq n - \frac{n}{2\sqrt{n-1}} |h|^2. \end{aligned}$$

From (29) and (30) we have

$$(31) \quad 0 \geq \int_{M^n} \left\{ (|\nabla h|^2 - n^2 |\nabla H|^2) + |Z|^2 \left[n - \frac{n}{2\sqrt{n-1}} |h|^2 \right] \right\}.$$

By the assumption of theorem $|\nabla h|^2 \geq n^2 |\nabla H|^2$, we know if $|h|^2 < 2\sqrt{n-1}$, we have $|Z|^2 \equiv 0$, which means that M^n is totally umbilical and hence is a small hypersphere $S^n(r)$, where $r = \sqrt{\frac{n}{n+|h|^2}}$.

If $|h|^2 = 2\sqrt{n-1}$, then equality holds in Lemma 2.1, and it follows that at least $(n-1)$ of λ_i 's are equal to one another. When $\lambda_1 = \lambda_2 = \dots = \lambda_n$, M^n is totally umbilical and hence a small hypersphere $S^n(r_0)$ where $r_0 = \frac{n}{n+2\sqrt{n-1}}$. When M^n is not totally umbilical, there are exactly $(n-1)$ of λ_i 's that are equal to one another. Then from [3] we know that M^n is $H(r)$ -torus $S^1(r) \times S^{n-1}(s)$, where $r^2 = \frac{1}{\sqrt{n-1}+1}$ and $s^2 = \frac{\sqrt{n-1}}{\sqrt{n-1}+1}$. This completes the proof of Theorem B.

From Theorem B, we have the following corollary immediately.

Corollary 1 [3]. *Let M^n be an n -dimensional closed hypersurface with constant mean curvature in S^{n+1} . Then*

- (1) *If $|h|^2 < 2\sqrt{n-1}$, M^n is a small hypersphere $S^n(r)$ of radius $r = \sqrt{\frac{n}{n+|h|^2}}$.*
- (2) *If $|h|^2 = 2\sqrt{n-1}$, M^n is either a small hypersphere $S^n(r_0)$ or $H(r)$ -torus $S^1(r) \times S^{n-1}(s)$, where $r_0^2 = \frac{n}{n+2\sqrt{n-1}}$, $r^2 = \frac{1}{\sqrt{n-1}+1}$ and $s^2 = \frac{\sqrt{n-1}}{\sqrt{n-1}+1}$.*

Corollary 2. *Let M^n be an n -dimensional closed hypersurface with constant normalized scalar curvature R in S^{n+1} . Suppose $R \geq 1$, then*

- (1) *If $|h|^2 < 2\sqrt{n-1}$, M^n is a small hypersphere $S^n(r)$ of radius $r = \sqrt{\frac{n}{n+|h|^2}}$.*
- (2) *If $|h|^2 = 2\sqrt{n-1}$, M^n is either a small hypersphere $S^n(r_0)$ or $H(r)$ -torus $S^1(r) \times S^{n-1}(s)$, where $r_0^2 = \frac{n}{n+2\sqrt{n-1}}$, $r^2 = \frac{1}{\sqrt{n-1}+1}$ and $s^2 = \frac{\sqrt{n-1}}{\sqrt{n-1}+1}$.*

PROOF. From (11),

$$n^2 H^2 - \sum_{i,j} h_{ij}^2 = n(n-1)(R-1).$$

Taking the covariant derivative of the above expression, and using the fact $R = \text{constant}$, we get

$$n^2 H H_k = \sum_{i,j} h_{ij} h_{ijk}.$$

By Cauchy-Schwarz inequality, we have

$$(32) \quad \sum_k n^4 H^2 (H_k)^2 = \sum_k \left(\sum_{i,j} h_{ij} h_{ijk} \right)^2 \leq \left(\sum_{i,j} h_{ij}^2 \right) \sum_{i,j,k} h_{ijk}^2,$$

that is

$$(33) \quad n^4 H^2 |\nabla H|^2 \leq |h|^2 |\nabla h|^2.$$

On the other hand, from $R \geq 1$, we have $n^2 H^2 - |h|^2 \geq 0$. Thus

$$|\nabla h|^2 \geq n^2 |\nabla H|^2,$$

so Corollary 2 follows from Theorem B. $\square$

Acknowledgement. This note was written during the author's stay at the Max-Planck-Institut für Mathematik in Bonn. The author would like to express his thanks to the institute for its hospitality and support.

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(Received January 31, 2001)