

On functions additive with respect to algorithms

By TIBOR FARKAS (Debrecen)

Abstract. In this paper we prove that for an arbitrary interval filling sequence there exist two algorithms such that the additivity of a function with respect to them implies its linearity. In contrast to some known results cited in Section 2 of this paper, we will prove the linearity of the function without requiring any special properties for the interval filling sequence and any regularity properties for the function.

1. Introduction

Let Λ be the set of the strictly decreasing sequences $\lambda = (\lambda_n)$ of positive real numbers for which $L(\lambda) := \sum_{n=1}^{\infty} \lambda_n < +\infty$. A sequence $(\lambda_n) \in \Lambda$ is called an *interval filling sequence* if, for any $x \in [0, L(\lambda)]$, there exists a sequence (δ_n) such that $\delta_n \in \{0, 1\}$ for all $n \in \mathbb{N}$ (the set of all positive integers) and $x = \sum_{n=1}^{\infty} \delta_n \lambda_n$. This concept has been introduced in [1]. The set of the interval filling sequences will be denoted by IF .

For a number $x \in]0, L(\lambda)[$ there can be more than one sequences $\delta = (\delta_n) \in \{0, 1\}^{\mathbb{N}}$ such that $x = \sum_{n=1}^{\infty} \delta_n \lambda_n$. For example, if $\lambda_n = q^{-n}$ where $1 < q < \frac{1+\sqrt{5}}{2}$ then for every $x \in]0, L(\lambda)[$ the cardinality of the set of such representations of x is continuum [9].

An *algorithm* (with respect to $\lambda = (\lambda_n) \in IF$) is defined in [3] as a sequence of functions $\alpha_n : [0, L(\lambda)] \rightarrow \{0, 1\}$ ($n \in \mathbb{N}$) for which

$$x = \sum_{n=1}^{\infty} \alpha_n(x) \lambda_n \quad (x \in [0, L(\lambda)]).$$

Mathematics Subject Classification: 39B20.

Key words and phrases: interval filling sequence, algorithm, $\mathcal{A}_0$ -additive function.

We denote the set of algorithms (with respect to $\lambda = (\lambda_n) \in IF$) by $\mathcal{A}(\lambda)$. Obviously, $\mathcal{A}(\lambda) \neq \emptyset$ for all $\lambda \in IF$, namely it was proved in [1] and [2] that, if $\lambda = (\lambda_n) \in IF$, $x \in [0, L(\lambda)]$ and

$$\varepsilon_n(x) = \begin{cases} 0 & \text{if } x < \sum_{i=1}^{n-1} \varepsilon_i(x) \lambda_i + \lambda_n, \\ 1 & \text{if } x \geq \sum_{i=1}^{n-1} \varepsilon_i(x) \lambda_i + \lambda_n, \end{cases}$$

or

$$\varepsilon_n^*(x) = \begin{cases} 0 & \text{if } x \leq \sum_{i=1}^{n-1} \varepsilon_i^*(x) \lambda_i + \lambda_n, \\ 1 & \text{if } x > \sum_{i=1}^{n-1} \varepsilon_i^*(x) \lambda_i + \lambda_n, \end{cases}$$

or

$$\varepsilon'_n(x) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n-1} \varepsilon'_i(x) \lambda_i + \sum_{i=n+1}^{\infty} \lambda_i < x, \\ 0 & \text{if } \sum_{i=1}^{n-1} \varepsilon'_i(x) \lambda_i + \sum_{i=n+1}^{\infty} \lambda_i \geq x, \end{cases}$$

then $\varepsilon = (\varepsilon_n)$, $\varepsilon^* = (\varepsilon_n^*)$, $\varepsilon' = (\varepsilon'_n) \in \mathcal{A}(\lambda)$. The algorithms ε , ε^* and ε' are called *regular (or greedy)*, *quasiregular* and *antiregular (or lazy)* algorithms, respectively.

If $\lambda = (\lambda_n) \in IF$, $\mathcal{A}_0 \subset \mathcal{A}(\lambda)$, $\mathcal{A}_0 \neq \emptyset$, $F : [0, L(\lambda)] \rightarrow \mathbb{R}$ and

$$F(x) = \sum_{n=1}^{\infty} \alpha_n(x) F(\lambda_n) \quad (x \in [0, L(\lambda)])$$

for all $(\alpha_n) \in \mathcal{A}_0$ then F will be called an $\mathcal{A}_0$ -*additive* function (with respect to λ) [3]. If

$$F\left(\sum_{n=1}^{\infty} \delta_n \lambda_n\right) = \sum_{n=1}^{\infty} \delta_n F(\lambda_n) \quad (\delta = (\delta_n) \in \{0, 1\}^{\mathbb{N}})$$

i.e. F is additive with respect to any algorithm then f is called *completely additive* [1].

2. Known results

Theorem 2.1 ([4]). *If $\lambda = (\lambda_n) \in IF$ and $F : [0, L(\lambda)] \rightarrow \mathbb{R}$ is a completely additive function (with respect to λ) then there exists $c \in \mathbb{R}$, such that $F(x) = c \cdot x$ for any $x \in [0, L(\lambda)]$ (i.e. briefly: F is linear).*

Theorem 2.2 ([2]). *If $\lambda = (\lambda_n) \in IF$ and $F : [0, L(\lambda)] \rightarrow \mathbb{R}$ is $\{\varepsilon\}$ -additive then F is right continuous.*

Remark 2.3 ([2]). There exist $\lambda = (\lambda_n) \in IF$ and $F : [0, L(\lambda)] \rightarrow \mathbb{R}$ such that F is ε -additive but F is non-continuous at the points of a countably infinite dense set. (At the so-called finite points, i.e. the points x for which $\{n \in \mathbb{N} \mid \varepsilon_n(x) = 1\}$ is a finite set.)

Theorem 2.4 ([2]). *If $\lambda = (\lambda_n) \in IF$ and $F : [0, L(\lambda)] \rightarrow \mathbb{R}$ is $\{\varepsilon, \varepsilon^*\}$ -additive then F is continuous.*

Remark 2.5 ([5]). There exist $\lambda = (\lambda_n) \in IF$ and $F : [0, L(\lambda)] \rightarrow \mathbb{R}$ such that F is $\{\varepsilon, \varepsilon^*\}$ -additive (i.e. continuous) but is non-differentiable at any point $x \in [0, L(\lambda)]$.

Theorem 2.6 ([6]). *Let $F : [0, L(\lambda)] \rightarrow \mathbb{R}$ be a so-called smooth interval filling sequence and let F be $\{\varepsilon, \varepsilon^*\}$ -additive (i.e. continuous). If F is differentiable on a set of positive measure or $F(x) > 0$ for $x > 0$ then F is linear.*

Theorem 2.7 ([7]). *If $\lambda = (\lambda_n) \in IF$ and $F : [0, L(\lambda)] \rightarrow \mathbb{R}$ is $\{\varepsilon, \varepsilon^*\}$ -additive (i.e. continuous) and F is differentiable at a finite point then F is linear.*

Theorem 2.8 ([8]). *Let $\lambda = (\lambda_n) \in IF$ and $\lambda_n \geq \lambda_{n+1} + \lambda_{n+2}$ for $n \in \mathbb{N}$. If $F : [0, L(\lambda)] \rightarrow \mathbb{R}$ is an $\{\varepsilon, \varepsilon^*, \varepsilon'\}$ -additive function then F is linear.*

3. Sufficiency of two algorithms

Definition 3.1. Let $\lambda = (\lambda_n)$ be an interval filling sequence. For $x \in [0, L(\lambda)]$ and $n \in \mathbb{N}$ let

$$\varepsilon_n^M(x) = \begin{cases} \varepsilon_n^*(x) & \text{if } x = \lambda_m \text{ for an } m \in \mathbb{N}, \\ \varepsilon_n(x) & \text{otherwise.} \end{cases}$$

It is obvious that $\varepsilon^M = (\varepsilon_n^M)$ is an algorithm, it will be called the *mixed regular* algorithm.

Theorem 3.2. *If $\lambda = (\lambda_n) \in IF$ and $F : [0, L(\lambda)] \rightarrow \mathbb{R}$ is additive with respect to the mixed regular algorithm then F is continuous.*

PROOF. We will prove that F is $\{\varepsilon, \varepsilon^*\}$ -additive and the continuity will follow from Theorem 2.4. The $\{\varepsilon\}$ -additivity of F is obvious. If $\{n \in \mathbb{N} \mid \varepsilon_n(x) = 1\}$ is an infinite set then $(\varepsilon_n^*(x))$ coincides with $(\varepsilon_n(x))$. The case when $x = \lambda_m$ for an $m \in \mathbb{N}$ is also trivial. Thus we have to deal only with the quasiregular representations of those numbers x for which $\{n \in \mathbb{N} \mid \varepsilon_n(x) = 1\}$ has finitely many, but at least two elements. Let us denote the maximum of this set by k . Then

$$\begin{aligned} F(x) &= F\left(\sum_{n=1}^{\infty} \varepsilon_n^M(x) \lambda_n\right) = \sum_{n=1}^{\infty} \varepsilon_n^M(x) F(\lambda_n) = \sum_{n=1}^{\infty} \varepsilon_n(x) F(\lambda_n) \\ &= \sum_{n=1}^k \varepsilon_n(x) F(\lambda_n) = \sum_{n=1}^{k-1} \varepsilon_n(x) F(\lambda_n) + F(\lambda_k) = \sum_{n=1}^{k-1} \varepsilon_n(x) F(\lambda_n) \\ &\quad + F\left(\sum_{n=1}^{\infty} \varepsilon_n^M(\lambda_k) \lambda_n\right) = \sum_{n=1}^{k-1} \varepsilon_n(x) F(\lambda_n) + \sum_{n=1}^{\infty} \varepsilon_n^M(\lambda_k) F(\lambda_n) \\ &= \sum_{n=1}^{k-1} \varepsilon_n^*(x) F(\lambda_n) + \sum_{n=k+1}^{\infty} \varepsilon_n^*(\lambda_k) F(\lambda_n) = \sum_{n=1}^{\infty} \varepsilon_n^*(x) F(\lambda_n), \end{aligned}$$

so F is $\{\varepsilon^*\}$ -additive and this completes our proof. $\square$

To prove our main result we will need the following two lemmas.

Lemma 3.3. *Let $\lambda = (\lambda_n) \in IF$ and let $F : [0, L(\lambda)] \rightarrow \mathbb{R}$ be a continuous function. If*

$$F\left(\sum_{n=1}^{\infty} \alpha_n \lambda_n\right) = \sum_{n=1}^{\infty} \alpha_n F(\lambda_n)$$

whenever $(\alpha_n) \in \{0, 1\}^{\mathbb{N}}$ and $\{n \in \mathbb{N} \mid \alpha_n = 1\}$ is finite then F is linear.

PROOF. We will prove the complete additivity of F and the linearity will follow from Theorem 2.1. Let $(\delta_n) \in \{0, 1\}^{\mathbb{N}}$. Then

$$\begin{aligned} F\left(\sum_{n=1}^{\infty} \delta_n \lambda_n\right) &= F\left(\lim_{k \rightarrow \infty} \sum_{n=1}^k \delta_n \lambda_n\right) = \lim_{k \rightarrow \infty} F\left(\sum_{n=1}^k \delta_n \lambda_n\right) \\ &= \lim_{k \rightarrow \infty} \left(\sum_{n=1}^k \delta_n F(\lambda_n)\right) = \sum_{n=1}^{\infty} \delta_n F(\lambda_n), \end{aligned}$$

so F is completely additive. $\square$

Lemma 3.4. *If $B^{(i)}$ is a countably infinite set for each $i \in \mathbb{N}$ then there exist pairwise disjoint countably infinite sets $C^{(i)}$ ($i \in \mathbb{N}$) such that*

$$C^{(i)} \subset B^{(i)} \quad (i \in \mathbb{N}).$$

PROOF. Let $\sigma = (\sigma_1, \sigma_2) : \mathbb{N} \rightarrow \mathbb{N}^2$ be a bijection. We define a sequence (c_k) by recursion. Let $c_1 \in B_{\sigma_1(1)}$ and if $k > 1$ then let $c_k \in B_{\sigma_1(k)} \setminus \{c_n \mid n \in \mathbb{N}, n < k\}$. Now

$$C^{(i)} := \{c_k \mid k \in \mathbb{N}, \sigma_1(k) = i\} \quad (i \in \mathbb{N}).$$

These sets $C^{(i)}$ ($i \in \mathbb{N}$) are obviously disjoint and it follows from the definition that $C^{(i)} \subset B^{(i)}$. Since $H_i = \{(i, n) \mid n \in \mathbb{N}\} \subset \mathbb{N}^2$ is an infinite set for every $i \in \mathbb{N}$, $\sigma^{-1}(H_i) \subset \mathbb{N}$ is also infinite. And if $k \in \sigma^{-1}(H_i)$ then $c_k \in C^{(i)}$, so $C^{(i)}$ is infinite. $\square$

Now we are ready to prove our main result:

Theorem 3.5. *Let $\lambda = (\lambda_n)$ be an arbitrary interval filling sequence. There exist two algorithms μ, ν such that if a function $F : [0, L(\lambda)] \rightarrow \mathbb{R}$ is $\{\mu, \nu\}$ -additive then F is linear.*

PROOF. Let $\mu = \varepsilon^M$. By Theorem 3.2, the μ -additivity of F implies its continuity, so, by Lemma 3.3, the proof of the theorem will be complete if we show that there exists an algorithm ν such that if F is $\{\mu, \nu\}$ -additive then

$$F\left(\sum_{n=1}^{\infty} \alpha_n \lambda_n\right) = \sum_{n=1}^{\infty} \alpha_n F(\lambda_n)$$

whenever $(\alpha_n) \in \{0, 1\}^{\mathbb{N}}$ and $\{n \in \mathbb{N} \mid \alpha_n = 1\}$ is finite.

There exist countably many 0, 1-sequences $\alpha = (\alpha_n)$ for which $\{n \in \mathbb{N} \mid \alpha_n = 1\}$ is finite. Hence there exists a sequence $(\alpha^{(i)})$ of *all* these sequences (i.e. $(\alpha_n^{(i)}) \in \{0, 1\}^{\mathbb{N}}$ for every $i \in \mathbb{N}$). Let us denote $\max\{n \in \mathbb{N} \mid \alpha_n^{(i)} = 1\}$ by $m(i)$. We define another sequence $(\beta^{(i)})$ of 0, 1-sequences by the following formula:

$$\beta_n^{(i)} = \begin{cases} \alpha_n^{(i)} & \text{if } n < m(i), \\ 0 & \text{if } n = m(i), \\ \varepsilon_n^*(\lambda_{m(i)}) & \text{if } n > m(i). \end{cases}$$

Then $\sum_{n=1}^{m(i)} \alpha_n^{(i)} \lambda_n = \sum_{n=1}^{\infty} \beta_n^{(i)} \lambda_n$, denote this sum by $x^{(i)}$. Let

$$B^{(i)} = \left\{ x \in \mathbb{R} \mid x = \sum_{n=1}^N \beta_n^{(i)} \lambda_n, N \in \mathbb{N} \right\}.$$

If $x \in B^{(i)}$ then let us denote by $N(x, i)$ the minimal integer N for which $x = \sum_{n=1}^N \beta_n^{(i)} \lambda_n$. The sets $B^{(i)}$ satisfy the conditions of Lemma 3.4, so there exist pairwise disjoint infinite subsets $C^{(i)} \subset B^{(i)}$ ($i \in \mathbb{N}$). At this point we are able to define our second algorithm:

$$\nu_n(x) := \begin{cases} \beta_n^{(i)} & \text{if there is an } i \in \mathbb{N} \text{ such that } x \in C^{(i)} \text{ and } n \leq N(x, i), \\ 0 & \text{if there is an } i \in \mathbb{N} \text{ such that } x \in C^{(i)} \text{ and } n > N(x, i), \\ \mu_n(x) & \text{if } x \notin \bigcup_{k=1}^{\infty} C^{(k)}. \end{cases}$$

The definition of ν is correct because of the disjoint property of sets $C^{(i)}$. Now let i be an arbitrary positive integer and let F be $\{\mu, \nu\}$ -additive. Then

$$\begin{aligned} F\left(\sum_{n=1}^{m(i)} \alpha_n^{(i)} \lambda_n\right) &= F\left(\sum_{n=1}^{\infty} \beta_n^{(i)} \lambda_n\right) = F(x^{(i)}) = \lim_{\substack{x \rightarrow x^{(i)} \\ x \in C^{(i)}}} F(x) \\ &= \lim_{\substack{x \rightarrow x^{(i)} \\ x \in C^{(i)}}} F\left(\sum_{n=1}^{\infty} \nu_n(x) \lambda_n\right) = \lim_{\substack{x \rightarrow x^{(i)} \\ x \in C^{(i)}}} \left(\sum_{n=1}^{\infty} \nu_n(x) F(\lambda_n)\right) \\ &= \lim_{\substack{x \rightarrow x^{(i)} \\ x \in C^{(i)}}} \left(\sum_{n=1}^{N(x, i)} \beta_n^{(i)} F(\lambda_n)\right) = \sum_{n=1}^{\infty} \beta_n^{(i)} F(\lambda_n) \\ &= \sum_{n=1}^{m(i)-1} \alpha_n^{(i)} F(\lambda_n) + \sum_{n=m(i)+1}^{\infty} \varepsilon_n^*(\lambda_{m(i)}) F(\lambda_n) \\ &= \sum_{n=1}^{m(i)-1} \alpha_n^{(i)} F(\lambda_n) + F(\lambda_{m(i)}) = \sum_{n=1}^{m(i)} \alpha_n^{(i)} F(\lambda_n), \end{aligned}$$

which was to be proved. $\square$

Remark 3.6. Note that $\mu(x) = \nu(x)$ for all but countably many points $x \in [0, L(\lambda)]$, so these two algorithms “almost coincide”. Moreover, it is

easy to prove that for $(\lambda_n) = (\frac{1}{2^n})$ the additivity of F with respect to the mixed regular algorithm implies the linearity. It is an open problem to characterize those interval filling sequences $\lambda \in IF$ for which there exists *one* algorithm α such that if a function $F : [0, L(\lambda)] \rightarrow \mathbb{R}$ is $\{\alpha\}$ -additive then F is linear.

References

- [1] Z. DARÓCZY, A. JÁRAI and I. KÁTAI, Intervallfüllende Folgen und volladditive Funktionen, *Acta Sci. Math.* **50** (1986), 337–350.
- [2] Z. DARÓCZY and I. KÁTAI, Interval filling sequences and additive functions, *Acta Sci. Math.* **52** (1988), 337–347.
- [3] Z. DARÓCZY, GY. MAKSA and T. SZABÓ, Some regularity properties of algorithms and additive functions with respect to them, *Aequationes Math.* **41** (1991), 111–118.
- [4] Z. DARÓCZY, I. KÁTAI and T. SZABÓ, On completely additive functions related to interval-filling sequences, *Arch. Math.* **54** (1990), 173–179.
- [5] Z. DARÓCZY and I. KÁTAI, Additive functions, *Anal. Math.* **12** (1986), 85–96.
- [6] Z. DARÓCZY and I. KÁTAI, On functions additive with respect to interval filling sequences, *Acta Math. Hung.* **51** (1988), 185–200.
- [7] Z. DARÓCZY, A. JÁRAI and I. KÁTAI, Some remarks on interval filling sequences and additive functions, Contributions to the Theory of Functional Equations, Proceedings of the Seminar Debrecen–Graz, *Grazer Math. Ber.* **315** (1991), 13–24.
- [8] T. SZABÓ, Triadditive functions, *Ann. Univ. Sci. Budapest, Sect. Comput.* **13** (1992), 25–33.
- [9] P. ERDŐS, I. JOÓ and V. KOMORNIK, Characterization of the unique expansions $1 = \sum_{i=1}^{\infty} q^{-n_i}$ and related problems, *Bull. Soc. Math. France* **118** (1990), 377–390.

TIBOR FARKAS
 INSTITUTE OF MATHEMATICS AND INFORMATICS
 UNIVERSITY OF DEBRECEN
 H-4010 DEBRECEN, P.O. BOX 12
 HUNGARY

(Received June 7, 2001)