

On q -multiplicative functions

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Abstract. The analogon of Delange’s theorem for q -multiplicative functions is investigated for some subsets of integers.

1. Introduction

Let $q \geq 2$ be an integer and $\mathbb{A} = \{0, 1, \dots, q-1\}$. We shall use the standard notations: $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}, \mathbb{C}$, denote the set of positive integers, non-negative integers, integers, real-numbers, complex numbers, respectively. For $x \in \mathbb{R}$ let $\{x\}$ be the fractional part of x , and $\|x\|$ be the distance of x to the closest integer. The q -ary expansion of some $n \in \mathbb{N}_0$ is defined as the unique sequence $\varepsilon_0(n), \varepsilon_1(n), \dots$ for which

$$(1) \quad n = \sum_{j=0}^{\infty} \varepsilon_j(n) q^j, \quad \varepsilon_j(n) \in \mathbb{A}$$

holds. $\varepsilon_0(n), \varepsilon_1(n), \dots$ are called the digits in the q -ary expansion of n .

Let $\mathcal{A}_q$ be the set of real-valued q -additive functions, and $\mathcal{M}_q$ be the set of complex-valued q -multiplicative functions.

A function $f : \mathbb{N}_0 \rightarrow \mathbb{R}$ belongs to $\mathcal{A}_q$, if $f(0) = 0$, and for every $n \in \mathbb{N}_0$,

$$(2) \quad f(n) = \sum_{j=0}^{\infty} f(\varepsilon_j(n) q^j).$$

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A function $g : \mathbb{N}_0 \rightarrow \mathbb{C}$ belongs to $\mathcal{M}_q$, if $g(0) = 1$, and for every $n \in \mathbb{N}_0$,

$$(3) \quad g(n) = \prod_{j=0}^{\infty} g(\varepsilon_j(n)q^j).$$

Since $f(\varepsilon_j(n)q^j) = 0$, $g(\varepsilon_j(n)q^j) = 1$ for all those j for which $q^j > n$, therefore the number of summands on the right hand side of (2), and the number of factors on the right hand side of (3) is finite.

Let $\overline{\mathcal{M}}_q$ be the class of q -multiplicative functions with modulus 1: i.e. $g \in \overline{\mathcal{M}}_q$, if g is q -multiplicative and $|g(n)| = 1$ ($n \in \mathbb{N}_0$). Let $e(\alpha) = e^{2\pi i\alpha}$.

A classical theorem of H. DELANGE [1] asserts that for $g \in \overline{\mathcal{M}}_q$, $N_x = \left\lceil \frac{\log x}{\log q} \right\rceil$,

$$m(x) := \frac{1}{x} \sum_{n < x} g(n) = \prod_{j=0}^{N_x-1} \frac{1}{q} \left(\sum_{b \in \mathbb{A}} g(bq^j) \right) + o_x(1),$$

whence he deduced that $\lim_{x \rightarrow \infty} |m(x)|$ always exists and equals

$$\prod_{j=0}^{\infty} \left| \frac{1}{q} \sum_{b \in \mathbb{A}} g(bq^j) \right|,$$

which is nonzero if and only if

$$(4) \quad \sum_{b \in \mathbb{A}} g(bq^j) \neq 0 \quad (\text{for all } j \in \mathbb{N}_0)$$

and

$$(5) \quad \sum_{j=0}^{\infty} \sum_{b \in \mathbb{A}} \operatorname{Re}(1 - g(bq^j)) < \infty.$$

Furthermore, he proved that $\lim_{x \rightarrow \infty} m(x)$ exists and is nonzero if and only if (4) holds and the series

$$(6) \quad \sum_{j=0}^{\infty} \sum_{b \in \mathbb{A}} (1 - g(bq^j))$$

is convergent.

An interesting problem is to give analogues of DELANGE's theorem [1], if we sum $g(n)$ on some subsets of the integers. Let $\alpha_1, \dots, \alpha_k$ be rationally independent real numbers, i.e. such that $h_1\alpha_1 + \dots + h_k\alpha_k + h_{k+1} \cdot 1 = 0$ has the only solution $h_1 = \dots = h_{k+1} = 0$ in integers $h_1, \dots, h_{k+1}$. Let $I_j = [u_j, v_j) \subset [0, 1)$ be arbitrary proper subintervals of $[0, 1)$, let E be the set of those integers n for which

$$\{\alpha_1 n\} \in I_1, \dots, \{\alpha_k n\} \in I_k$$

simultaneously holds.

Let

$$l(n) = \begin{cases} 1 & \text{if } n \in E, \\ 0 & \text{if } n \in \mathbb{N}_0 \setminus E. \end{cases}$$

Our purpose in this paper is to investigate the sum

$$M(x) := \sum_{n < x} g(n)l(n)$$

for $g \in \overline{\mathcal{M}}_q$.

We shall prove the following

Theorem 1. $\lim_{x \rightarrow \infty} \frac{|M(x)|}{x}$ always exists. It is nonzero if there exist integers $h_1, \dots, h_k$ for which

$$(7) \quad \sum_{l=0}^{\infty} \sum_{b \in \mathbb{A}} \operatorname{Re}(1 - g(bq^l)e((h_1\alpha_1 + \dots + h_k\alpha_k)bq^l)) < \infty$$

and

$$(8) \quad \sum_{b \in \mathbb{A}} g(bq^l)e((h_1\alpha_1 + \dots + h_k\alpha_k)bq^l) \neq 0 \quad l = 0, 1, \dots$$

The relation (7) can be satisfied for at most one choice of $h_1, \dots, h_k \in \mathbb{Z}$. Assume that (7) holds. Then

$$\frac{M(x)}{x} = c_{h_1}^{(1)} \dots c_{h_k}^{(k)} \frac{S_{h_1, \dots, h_k}(x)}{x} + o_x(1)$$

where

$$c_{h_j}^{(j)} = \frac{e(-h_j u_j) - e(-h_j v_j)}{2\pi i h_j} \quad \text{if } h_j \neq 0,$$

and

$$c_0^{(j)} = (v_j - u_j),$$

furthermore,

$$S_{h_1, \dots, h_k}(x) = \sum_{n < x} g(n) e((h_1 \alpha_1 + \dots + h_k \alpha_k) n).$$

$\lim_{x \rightarrow \infty} \frac{M(x)}{x}$ exists if and only if $\lim_{x \rightarrow \infty} \frac{S_{h_1, \dots, h_k}(x)}{x}$ exists. $\lim_{x \rightarrow \infty} \frac{S_{h_1, \dots, h_k}(x)}{x}$ exists if and only if $\sum_{l=0}^{\infty} \sum_{b \in \mathbb{A}} (1 - g(bq^l) e((h_1 \alpha_1 + \dots + h_k \alpha_k) bq^l))$ is convergent.

2. Proof

Let f_j ($j = 1, \dots, k$) be the function defined in $[0, 1)$ by

$$f_j(y) = \begin{cases} 1 & \text{if } y \in I_j, \\ 0 & \text{if } y \in [0, 1) \setminus I_j, \end{cases}$$

and extended periodically mod 1. Then

$$f_j(y) \sim \sum_{m=-\infty}^{\infty} c_m^{(j)} e(my),$$

where $c_m^{(j)} = \frac{e(-mu_j) - e(-mv_j)}{2\pi im}$, if $m \neq 0$ and $c_0^{(j)} = (v_j - u_j) = |I_j|$.

Choosing a small $\Delta > 0$, for

$$f_j^*(u) = \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} f_j(y+u) dy \sim \sum_{m=-\infty}^{\infty} d_m^{(j)}(\Delta) e(mu)$$

we obtain that $d_0^{(j)}(\Delta) = |I_j|$ and $|d_m^{(j)}(\Delta)| \leq \frac{c}{\Delta m^2}$, with an absolute positive constant c . Thus the Fourier series is absolutely convergent and represents $f_j^*(u)$. Let now K be a large integer, $\tilde{f}_j(u) = \sum_{|h| \leq K} d_h^{(j)}(\Delta) e(hu)$. Then $|f_j^*(u) - \tilde{f}_j(u)| \leq \frac{c}{\Delta K}$, and so

$$(9) \quad \left| \prod_{j=1}^k f_j^*(u_j) - \prod_{j=1}^k \tilde{f}_j(u_j) \right| \leq \frac{ck}{\Delta K},$$

since $0 \leq f_j^*(u_j) \leq 1$ holds for $u_j \in \mathbb{R}$. We obviously have $l(n) = \prod_{j=1}^k f_j(n\alpha_j)$. Let $l^*(n) := \prod_{j=1}^k f_j^*(n\alpha_j)$ and $\tilde{l}(n) := \prod_{j=1}^k \tilde{f}_j(n\alpha_j)$. Let us observe that $f_j^*(u) = f_j(u)$ if $u \notin [u_j - \Delta, u_j + \Delta] \cup [v_j - \Delta, v_j + \Delta]$. Therefore $l(n) = l^*(n)$, except when $\{n\alpha_j\} \in [u_j - \Delta, u_j + \Delta] \cup [v_j - \Delta, v_j + \Delta]$ for some j . Furthermore $|l(n) - l^*(n)| \leq 1$ always holds.

Let $S(x) := \sum_{n < x} g(n)\tilde{l}(n)$. We have

$$\begin{aligned} |M(x) - S(x)| &\leq \left| \sum_{n < x} g(n)(l(n) - \tilde{l}(n)) \right| \leq \sum_{n < x} |l(n) - \tilde{l}(n)| \\ &\leq \sum_{n < x} |l(n) - l^*(n)| + \sum_{n < x} |l^*(n) - \tilde{l}(n)| = \sum_1 + \sum_2. \end{aligned}$$

From (9) we have that $\sum_2 \leq \frac{ckx}{\Delta K}$. Furthermore,

$$\sum_1 \leq \sum_{j=1}^k \#\{n \leq x \mid \{\alpha_j n\} \in [u_j - \Delta, u_j + \Delta] \cup [v_j - \Delta, v_j + \Delta]\}$$

and by using that $\alpha_j n$ is uniformly distributed mod 1, we obtain that $\sum_1 \leq c_1 k \Delta x$ with an absolute positive constant c_1 for every large x .

Let us observe furthermore that

$$S(x) = \sum_{h_1, \dots, h_k} d(h_1, \dots, h_k) S_{h_1, \dots, h_k}(x)$$

where $h_1, \dots, h_k$ run over the integers in $[-K, K]$,

$$d(h_1, \dots, h_k) = \prod_{j=1}^k d_{h_j}^{(j)}(\Delta)$$

and

$$S_{h_1, \dots, h_k}(x) = \sum_{n < x} g(n) e((h_1 \alpha_1 + \dots + h_k \alpha_k) n).$$

Lemma 1. Assume that

$$\limsup_{x \rightarrow \infty} \frac{|M(x)|}{x} > 0.$$

Then there are some integers $h_1^*, \dots, h_k^*$ such that

$$(10) \quad \sum_{l=0}^{\infty} \sum_{c \in \mathbb{A}} \operatorname{Re}(1 - g(cq^l) e((h_1^* \alpha_1 + \dots + h_k^* \alpha_k) cq^l))$$

is convergent.

PROOF of Lemma 1. The function $g(n) e((h_1 \alpha_1 + \dots + h_k \alpha_k) n)$ as a function of n belongs to $\overline{\mathcal{M}}_q$. If (10) does not hold, then $\frac{|S_{h_1, \dots, h_k}(x)|}{x} \rightarrow 0$ ($x \rightarrow \infty$) due to DELANGE's theorem [1], and so $\frac{|S(x)|}{x} \rightarrow 0$. Since $\frac{|M(x)|}{x} \leq \frac{|S(x)|}{x} + \frac{|M(x) - S(x)|}{x}$, and the second term is less than $c_1 k \Delta + \frac{ck}{\Delta K}$, therefore

$$(11) \quad \limsup_{x \rightarrow \infty} \frac{|M(x)|}{x} \leq c_1 k \Delta + \frac{ck}{\Delta K}.$$

This inequality holds for each $\Delta > 0$ and each $K > 0$. By letting $K \rightarrow \infty$, then $\Delta \rightarrow 0$, we obtain that

$$\limsup_{x \rightarrow \infty} \frac{|M(x)|}{x} = 0. \quad \square$$

Lemma 2. *The relation*

$$(12) \quad \sum_{j=0}^{\infty} \sum_{b \in \mathbb{A}} \operatorname{Re}(1 - g(bq^j) e((h_1 \alpha_1 + \dots + h_k \alpha_k) bq^j)) < \infty$$

may hold at most for one collection of integers $h_1, \dots, h_k$.

PROOF of Lemma 2. The relation (12) is equivalent to

$$\sum_{j=0}^{\infty} \sum_{b \in \mathbb{A}} \left\| \frac{\arg g(bq^j)}{2\pi} + (h_1 \alpha_1 + \dots + h_k \alpha_k) bq^j \right\|^2 < \infty.$$

Assume that (12) holds with $(h_1, \dots, h_k)$ as well as with $(h_1^*, \dots, h_k^*)$.

Let $\gamma = (h_1 - h_1^*) \alpha_1 + \dots + (h_k - h_k^*) \alpha_k$. If $(h_1, \dots, h_k) \neq (h_1^*, \dots, h_k^*)$, then γ is an irrational number, and

$$(13) \quad \sum_{j=0}^{\infty} \sum_{b \in \mathbb{A}} \|\gamma bq^j\|^2 < \infty.$$

we shall see that (13) is impossible.

From (13) it follows that $\|\gamma q^j\| \rightarrow 0$ ($j \rightarrow \infty$). Let $\gamma q^j = m_j + \delta_j$, where $\delta_j \in (-\frac{1}{2}, \frac{1}{2}]$, $m_j \in \mathbb{Z}$. Then $\|\delta_j\| = \|\gamma q^j\|$. Furthermore $\gamma q^{j+1} = qm_j + q\delta_j$, and so $\delta_{j+1} = q\delta_j$, $m_{j+1} = qm_j$ for every large j , and this contradicts to the fact that $\delta_j \neq 0$. The lemma is proved.

Lemma 3. Assume that (12) holds with $(h_1, \dots, h_k)$. Then

$$\frac{M(x)}{x} = c_{h_1}^{(1)} \dots c_{h_k}^{(k)} \frac{S_{h_1, \dots, h_k}(x)}{x} + o_x(1).$$

PROOF of Lemma 3. Repeating the argumentation of Lemma 1, we deduce that

$$\left| \frac{M(x)}{x} - d(h_1, \dots, h_k) \frac{S_{h_1, \dots, h_k}(x)}{x} \right| \leq c_1 k \Delta + \frac{ck}{\Delta K},$$

whence

$$\begin{aligned} & \left| \frac{M(x)}{x} - c_{h_1}^{(1)} \dots c_{h_k}^{(k)} \frac{S_{h_1, \dots, h_k}(x)}{x} \right| \\ & \leq c_1 k \Delta + \frac{ck}{\Delta K} + |d(h_1, \dots, h_k) - c_{h_1}^{(1)} \dots c_{h_k}^{(k)}|. \end{aligned}$$

Then, by $K \rightarrow \infty$, and $\Delta \rightarrow 0$ we obtain that

$$\lim_{x \rightarrow \infty} \left| \frac{M(x)}{x} - c_{h_1}^{(1)} \dots c_{h_k}^{(k)} \frac{S_{h_1, \dots, h_k}(x)}{x} \right| \rightarrow 0,$$

due to the fact that $d(h_1, \dots, h_k) \rightarrow c_{h_1}^{(1)} \dots c_{h_k}^{(k)}$ as $\Delta \rightarrow 0$.

Observe that $c_{h_1}^{(1)} \dots c_{h_k}^{(k)} \neq 0$.

From Lemma 3 we obtain that $\lim_{x \rightarrow \infty} \frac{M(x)}{x}$ exists if and only if $\lim_{x \rightarrow \infty} \frac{S_{h_1, \dots, h_k}(x)}{x}$ exists. Due to DELANGE's theorem [1] it exists and nonzero if and only if

$$\sum_{b \in \mathbb{A}} g(bq^j) e((h_1 \alpha_1 + \dots + h_k \alpha_k) bq^j) \neq 0$$

for $j = 0, 1, \dots$, and

$$\sum_{j=0}^{\infty} \sum_{b \in \mathbb{A}} (1 - g(bq^j) e((h_1 \alpha_1 + \dots + h_k \alpha_k) bq^j))$$

is convergent.

Hence Theorem 1 immediately follows.

3. On the distribution of q -additive functions

Theorem 2. Let $f \in \mathcal{A}_q$, $E(x) := \#\{n < x, n \in E\}$

$$F_x(y) := \frac{1}{E(x)} \#\{n < x, n \in E, f(n) < y\}.$$

The limit $\lim_{x \rightarrow \infty} F_x(y) = F(y)$ exists for almost all $y \in \mathbb{R}$, where F is a distribution function, if and only if the series

$$(14) \quad \sum_{j=0}^{\infty} \sum_{b \in \mathbb{A}} f(bq^j)$$

and

$$(15) \quad \sum_{j=0}^{\infty} \sum_{b \in \mathbb{A}} f^2(bq^j)$$

are convergent.

PROOF. Let $g_\tau(n) = e(\tau f(n))$, where $\tau \in \mathbb{R}$. Then $g_\tau(n) \in \overline{\mathcal{M}}_q$. Let $m_\tau(x) = \frac{1}{E(x)} \sum_{n < x} g_\tau(n) l(n)$. Assume first that (14), (15) are satisfied. Then $\sum_{j=0}^{\infty} \sum_{b \in \mathbb{A}} (1 - g_\tau(bq^j))$ is convergent, and by Theorem 1 we obtain that $m_\tau(x) \rightarrow m(\tau)$ ($x \rightarrow \infty$), where $m(\tau) \neq 0$ in a neighborhood of 0, i.e. if $|\tau| < c$. Thus, by a wellknown theorem in probability theory we obtain that $F_x(y) \rightarrow F(y)$, the characteristic function of F is $m(\tau)$.

Assume now that $\lim_{x \rightarrow \infty} F_x(y)$ exists. Then there exists $\lim_{x \rightarrow \infty} m_\tau(x) = m(\tau)$ in a suitable interval $|\tau| \leq c$. Applying Theorem 1, we obtain that

$$(16) \quad \sum_{l=0}^{\infty} \sum_{b \in \mathbb{A}} (1 - e(\tau f(bq^l) + (h_1(\tau)\alpha_1 + \cdots + h_k(\tau)\alpha_k)bq^l))$$

is convergent, where $h_1(\tau), \dots, h_k(\tau)$ are suitable integers. Consequently

$$\|\tau f(bq^l) + (h_1(\tau)\alpha_1 + \cdots + h_k(\tau)\alpha_k)bq^l\| \rightarrow 0,$$

and so

$$\|m\tau f(bq^l) + m(h_1(\tau)\alpha_1 + \cdots + h_k(\tau)\alpha_k)bq^l\| \rightarrow 0,$$

for every $m \in \mathbb{N}$. Furthermore,

$$\|m\tau f(bq^l) + (h_1(m\tau)\alpha_1 + \cdots + h_k(m\tau)\alpha_k)bq^l\| \rightarrow 0,$$

as $l \rightarrow \infty$.

Then, for every fixed $m \in \mathbb{N}$,

$$(17) \quad \sum_{l=0}^{\infty} \sum_{b \in \mathbb{A}} (1 - e(m\tau f(bq^l) + m(h_1(\tau)\alpha_1 + \cdots + h_k(\tau)\alpha_k)bq^l))$$

is convergent as well. Applying (16) for $m\tau$ instead of τ , we obtain that

$$(18) \quad \sum_{l=0}^{\infty} \sum_{b \in \mathbb{A}} (1 - e(m\tau f(bq^l) + (h_1(m\tau)\alpha_1 + \cdots + h_k(m\tau)\alpha_k)bq^l)),$$

(17) and (18) easily imply that

$$\sum_{l=0}^{\infty} \sum_{b \in \mathbb{A}} \|[(h_1(m\tau) - mh_1(\tau))\alpha_1 + \cdots + (h_k(m\tau) - mh_k(\tau))\alpha_k]bq^l\|^2 < \infty.$$

Applying the argument which was used in the proof of Lemma 2, and that $\alpha_1, \dots, \alpha_k$ are linearly independent, we obtain that $h_j(m\tau) = mh_j(\tau)$ ($j = 1, \dots, k$). Let now K be fixed, $|K| \leq c$. Then $h_j(K) = mh_j\left(\frac{K}{m}\right)$ holds for every $m = 1, 2, \dots$, and since $h_j\left(\frac{K}{m}\right) \in \mathbb{Z}$, therefore m divides $h_j(K)$ for every m . Thus $h_j(K) = 0$ ($j = 1, \dots, k$), $|K| \leq c$. Consequently,

$$(19) \quad \sum_{l=0}^{\infty} \sum_{b \in \mathbb{A}} (1 - e(\tau f(bq^l)))$$

is convergent for $|\tau| \leq c$.

Hence one can deduce that (14), (15) are convergent.

References

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