

Some remarks on the geometry of quasi-Banach spaces

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Abstract. We study the definitions of strict p -convexity and uniform p -convexity and the relations between them for quasi-Banach spaces when $0 < p < 1$.

1. Introduction

The study of quasi-normed spaces arises naturally as a generalization of normed spaces by substituting the triangular inequality of the norm by a weaker condition. The geometrical meaning of that generalization is that whereas the unit ball of a normed space is a convex set, the unit ball of a quasi-normed space needs not be convex. From a topological point of view, the topological vector spaces whose topology can be induced by a norm are those which have a convex bounded neighbourhood of zero, while the topological vector spaces whose topology can be induced by a quasi-norm are those which have a bounded neighbourhood of zero (see [3] and [7]). If that topology is complete, the normed spaces are called Banach spaces, and the quasi-normed spaces are called quasi-Banach spaces.

The geometry of the unit ball in Banach spaces has been widely studied and a great deal of information can be found in the mathematical literature. In this paper, we present some aspects of the geometry of the unit ball of those quasi-Banach spaces which are not Banach spaces. We will translate some concepts used to describe the characteristics of the unit ball of normed spaces into the setting of quasi-normed spaces and we will see to what extent we can generalize the properties relating those concepts.

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In Section 2, we define strict p -convexity of quasi-Banach spaces, for $0 < p < 1$, generalizing the classical concept of strict convexity of Banach spaces, and see some characterizations. We present the Lorentz sequence spaces $d(w, p)$ as an example of quasi-Banach spaces which are strictly p -convex but do not have an equivalent q -norm for any $0 < p < q \leq 1$.

In Section 3, we discuss the definition of uniform p -convexity ($0 < p < 1$) that best generalizes the classical concept of uniform convexity of Banach spaces.

We now introduce some definitions and notation we will be using in the sequel (see [3]).

Let X be a (real) vector space. A *quasi-norm* on X is a map $\|\cdot\| : X \rightarrow [0, +\infty)$ satisfying

- (i) $\|x\| > 0$ if $x \neq 0$,
- (ii) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$, $\alpha \in \mathbb{R}$,
- (iii) $\|x + y\| \leq C(\|x\| + \|y\|)$ for any $x, y \in X$, where $C \geq 1$ is a constant independent of x and y . If $C = 1$, $\|\cdot\|$ is a norm.

A quasi-norm is *p -subadditive* ($0 < p \leq 1$), and it is called *p -norm*, if

- (iv) $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for any $x, y \in X$.

A quasi-norm clearly defines a metrizable locally bounded vector topology on X . If such topology is complete then we say that $(X, \|\cdot\|)$ is a *quasi-Banach space*. If the quasi-norm is also p -subadditive then X is a *p -Banach space*.

Given a vector space X and $0 < p \leq 1$, we will denote by $[x, y]_p$ the *p -segment with ending points x, y* :

$$[x, y]_p = \{\lambda x + \mu y : \lambda, \mu \geq 0, \lambda^p + \mu^p = 1\}.$$

A subset C of X is *p -convex* (where $0 < p \leq 1$) if given $x, y \in C$ and $0 \leq t, s \leq 1$ with $s^p + t^p = 1$, then $tx + sy \in C$, i.e. C is p -convex if it contains every p -segment with ending points in C .

We recall that if $S \subset X$ and $0 < p \leq 1$, then the *p -convex hull* of S is the smallest p -convex set that contains S and can be described as

$$p\text{-co}(S) = \left\{ \sum_{k=1}^N \lambda_k x_k : \sum_{k=1}^N \lambda_k^p = 1, \lambda_k \geq 0, x_k \in S, k = 1, \dots, N, N \in \mathbb{N} \right\}.$$

When $0 < p < 1$, the p -convex hull of a set S can be written as well as

$$p\text{-co}(S) = \left\{ \sum_{k=1}^N \lambda_k x_k : 0 < \sum_{k=1}^N \lambda_k^p \leq 1, \lambda_k \geq 0, x_k \in S, \right. \\ \left. k = 1, \dots, N, N \in \mathbb{N} \right\}.$$

Therefore, if $0 < p < 1$ any non empty, p -convex, closed set contains 0. That leads to interesting situations. For instance, a set formed by one single element $x \neq 0$ is not p -convex if $0 < p < 1$. In this case,

$$p\text{-co}(\{x\}) = \{\lambda x : 0 < \lambda \leq 1\}.$$

Another particularity of the case $0 < p < 1$ is that $[x, y]_p$ needs not be a p -convex set: if x and y are not colinear then

$$p\text{-co}[x, y]_p = \{\lambda x + \mu y : \lambda, \mu \geq 0, 0 < \lambda^p + \mu^p \leq 1\} = \bigcup_{z \in [x, y]_p} \{\lambda z : 0 < \lambda \leq 1\}.$$

2. p -convexity and strict p -convexity

Whereas the unit ball of a normed space is always a convex set, the unit ball B_X of a quasi-normed space X needs not be convex. There are even such examples as $X = L_p([0, 1], dt)$, $0 < p < 1$, where the convex hull of B_X is the whole space. In fact, the unit ball B_X of a quasi-normed space X needs not be p -convex for any $0 < p \leq 1$ as the following example shows (see another example in [7], page 95). The set

$$B = \{(x, y) \in \mathbb{R}^2 : (|y| - 1) \leq (\log |x| - 1)^{-1} \leq 0\} \cup \{(0, y) : -1 \leq y \leq 1\}$$

is not p -convex for any $0 < p \leq 1$ but it is the unit ball of the quasi-Banach space $(\mathbb{R}^2, \mu_B)$, being μ_B the Minkowski functional of B :

$$\mu_B(x) = \inf\{t > 0 : x \in tB\}, \quad x \in \mathbb{R}^2.$$

Nevertheless, a theorem by AOKI ([1]) and ROLEWICZ ([6]) asserts that every quasi-norm is equivalent to a quasi-norm which is p -subadditive for some $p > 0$ (see Theorem 1.3 of [3] and Theorem 3.2.1 of [7]).

Proposition 2.2. *Let $0 < p < 1$.*

- (i) *A quasi-normed space $(X, \|\cdot\|)$ is strictly p -convex if and only if $\partial_p B_X = S_X$.*
- (ii) *If a quasi-normed space $(X, \|\cdot\|)$ is strictly p -convex then*

$$\left\| \frac{x+y}{2^{1/p}} \right\| < 1,$$

for any $x, y \in X$ such that $\|x\| = \|y\| = 1$. The converse is not true.

PROOF. (i) It is obvious that if $(X, \|\cdot\|)$ is strictly p -convex then $\partial_p B_X = S_X$. Reciprocally, if $(X, \|\cdot\|)$ is not strictly p -convex then there exist $x, y \in X$ different from 0_X such that

$$\|x+y\|^p = \|x\|^p + \|y\|^p.$$

We may assume that $\|x\| = 1$ and $\|y\| = \varepsilon \leq 1$; therefore, $\|x+y\| = \alpha = (1 + \varepsilon^p)^{1/p}$. Now, we observe that

$$\frac{x+y}{\alpha} = \frac{1}{\alpha}x + \frac{\varepsilon}{\alpha} \frac{y}{\varepsilon}, \quad \left(\frac{1}{\alpha}\right)^p + \left(\frac{\varepsilon}{\alpha}\right)^p = 1,$$

so $\frac{x+y}{\alpha} \in S_X \setminus \partial_p B_X$.

(ii) It is obvious. A counterexample of the converse for $0 < p < 1$ is the quasi-Banach space $(\mathbb{R}^2, \mu_B)$ whose unit ball is a set $B = B_p \cup 1/2B_\infty \subset \mathbb{R}^2$ formed by the union of the unit ball B_p of $\ell_p^{(2)}$ with the $\ell_\infty^{(2)}$ -ball of radius $1/2$. Given any $x = (x_1, x_2), y = (y_1, y_2) \in B$, we check that $\mu_B(x+y) < 2^{1/p}$: if $x \in B_p$ and $y \in 1/2B_\infty$ then $\mu_B(x+y) \leq (|x_1+y_1|^p + |x_2+y_2|^p)^{1/p} \leq ((1/2)^p + (3/2)^p)^{1/p} < 2^{1/p}$ for all $0 < p < 1$; if $x, y \in 1/2B_\infty$ then $\mu_B(x+y) \leq 2 \max(|x_1+y_1|, |x_2+y_2|) \leq 2 < 2^{1/p}$ for all $0 < p < 1$; if $x, y \in B_p$ and $x_1y_1 \neq 0$ or $x_2y_2 \neq 0$ then $\mu_B(x+y) \leq (|x_1+y_1|^p + |x_2+y_2|^p)^{1/p} < 2^{1/p}$ (see Lemma 2.4); and if $x, y \in B_p$ and $x_1y_1 = x_2y_2 = 0$ then $\mu_B(x+y) \leq \max(\mu_B(1,1), \mu_B(2,0)) = 2 < 2^{1/p}$ for all $0 < p < 1$. In this space,

$$\mu_B \left(\frac{x+y}{2^{1/p}} \right) < 1$$

for any $x, y \in \mathbb{R}^2$ such that $\mu_B(x) = \mu_B(y) = 1$, but it cannot be strictly p -convex because its unit sphere has points which are not p -extreme. $\square$

Strict p -convexity is an intermediate stage between p -convexity and q -convexity for $0 < p < q \leq 1$.

Proposition 2.3. *If a quasi-normed space $(X, \|\cdot\|)$ is q -convex for some $0 < q \leq 1$, then $(X, \|\cdot\|)$ is strictly p -convex for any $0 < p < q$.*

PROOF. It is a straightforward consequence of the following lemma. $\square$

Lemma 2.4. *If $0 < p < 1$, then*

$$|a + b|^p \leq |a|^p + |b|^p$$

for any real numbers a and b . Furthermore, $|a + b|^p = |a|^p + |b|^p$ if and only if $ab = 0$.

Next, we see that the converse of Proposition 2.3 is not true: there are strictly p -convex quasi-Banach spaces whose quasi-norm is not q -subadditive for any $p < q \leq 1$. The example is provided by the Lorentz sequence spaces.

For every $0 < p \leq 1$ and every non-increasing sequence of positive numbers $\omega = (\omega_n)_{n=1}^\infty$ so that $\omega_1 = 1$ we consider the Lorentz space $d(\omega, p)$ of all sequences of scalars $a = (a_n)_{n=1}^\infty$ for which

$$\|a\| = \sup_{\pi \in \Pi} \left(\sum_{n=1}^{\infty} |a_{\pi(n)}|^p \omega_n \right)^{1/p} < \infty,$$

the supremum being taken over the set Π of all permutations of the integers. It is known that $(d(\omega, p), \|\cdot\|)$ is a p -Banach space.

Theorem 2.5. *If $0 < p < 1$ then $(d(\omega, p), \|\cdot\|)$ is not q -convex for any $q > p$.*

If ω decreases then $(d(\omega, p), \|\cdot\|)$ is strictly p -convex.

PROOF. First, we check that $d(\omega, p)$ is not q -convex for any given $p < q \leq 1$. Let us take $x = (1, 0, 0, \dots)$ and $y = (b, b, 0, 0, \dots)$ so that

$\|y\| = 1$ (i.e., $b = \frac{1}{(1+w_2)^p}$). The points of the q -segment $[x, y]_q$ with ending points x and y are given by

$$\begin{aligned} & \lambda(b, b, 0, 0, \dots) + (1 - \lambda^q)^{1/q}(1, 0, 0, \dots) \\ &= (\lambda b + (1 - \lambda^q)^{1/q}, b\lambda, 0, 0, \dots), \quad \lambda \in [0, 1]. \end{aligned}$$

For the closed unit ball to be q -convex it must contain all the q -segments with ending points in it, in particular it must be verified that

$$\begin{aligned} & \|(\lambda b + (1 - \lambda^q)^{1/q}, b\lambda, 0, 0, \dots)\| \\ &= ((b\lambda + (1 - \lambda^q)^{1/q})^p + w_2(b\lambda)^p)^{1/p} \leq 1 \end{aligned}$$

for all $\lambda \in [0, 1]$. Let us see that for any $p < q \leq 1$, there exists $\lambda = \lambda(q) \in (0, 1)$ (closed enough to 0) such that $f(\lambda) > 1$. Call $\lambda^q = t$, and

$$f(t) = (bt^{1/q} + (1 - t)^{1/q})^p + w_2 b^p t^{p/q}, \quad t \in [0, 1].$$

Then

$$f'(t) = \frac{p}{q} (bt^{1/q} + (1 - t)^{1/q})^{p-1} (bt^{\frac{1}{q}-1} - (1 - t)^{\frac{1}{q}-1}) + \frac{p}{q} w_2 b^p t^{\frac{p}{q}-1}, \quad t \in (0, 1).$$

As $\lim_{t \rightarrow 0^+} f'(t) = +\infty$, it follows that there exists $\delta = \delta(q) > 0$ so that $t \in (0, \delta)$ implies $f'(t) > 1 > 0$. Then f is increasing in $[0, \delta]$. Now, $f(0) = 1$, so $f(t) > 1$ for every $t \in [0, \delta]$, or equivalently, $\|\lambda(b, b, 0, 0, \dots) + (1 - \lambda^q)^{1/q}(1, 0, 0, \dots)\| > 1$ if $\lambda \in [0, \delta^{1/p}]$.

Now, we see that if ω decreases then $d(\omega, p)$ is strictly p -convex. Let $x = (x_n)_{n=1}^\infty, y = (y_n)_{n=1}^\infty$ be any two elements in $d(\omega, p)$, and let $(|x_{\sigma(k)} + y_{\sigma(k)}|)_{k=1}^\infty$ be a decreasing rearrangement of the sequence $(|x_k + y_k|)_{k=1}^\infty$. Then,

$$\begin{aligned} \|x + y\|^p &= \sum_{k=1}^\infty |x_{\sigma(k)} + y_{\sigma(k)}|^p \omega_k \stackrel{(1)}{\leq} \sum_{k=1}^\infty |x_{\sigma(k)}|^p \omega_k \\ &\quad + \sum_{k=1}^\infty |y_{\sigma(k)}|^p \omega_k \stackrel{(2)}{\leq} \|x\|^p + \|y\|^p. \end{aligned}$$

The equality $\|x + y\|^p = \|x\|^p + \|y\|^p$ holds if and only if the inequalities (1), (2) are actually equalities. By Lemma 2.4, the inequality (1) is an

Case 2: If $(w_n)_{n=1}^\infty \in c_0 \setminus \ell_1$ then, by Lemma 2.6, there is a sequence $(u_n)_{n=1}^\infty \subset d(w, p)$ of elements of quasi-norm 1, and a constant C such that

$$CN^{1/p} \leq \left\| \sum_{n=1}^N u_n \right\|$$

for all $N \in \mathbb{N}$. If the space $d(w, p)$ had an equivalent q -subadditive quasi-norm then there would exist a constant D so that

$$C^q N^{q/p} \leq \|u_1 + u_2 + \cdots + u_N\|^q \leq D \sum_{n=1}^N \|u_n\|^q = DN$$

for all $N \in \mathbb{N}$, but the previous inequality cannot be true for all $N \in \mathbb{N}$ if $q > p$. $\square$

3. Uniform p -convexity

The aim of this section is to find the most appropriate definition of uniform p -convexity when $0 < p < 1$ in the sense that it should coincide with the concept of uniform convexity for $p = 1$ and it should be an intermediate property between strict p -convexity and q -convexity for $q > p$.

A Banach space $(X, \|\cdot\|)$ is called *uniformly convex* if for every $\varepsilon > 0$ there is a number $\delta = \delta(\varepsilon)$ such that

$$\inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon \right\} = \delta > 0.$$

If $(X, \|\cdot\|)$ is a p -Banach space and $0 < p < 1$, then for any $x, y \in X$ such that $\|x\| = \|y\| = 1$ and $\|x-y\| < \varepsilon$ it holds that

$$1 - \left\| \frac{x+y}{2^{1/p}} \right\| \geq 1 - \frac{(2^p + \varepsilon^p)^{1/p}}{2^{1/p}} \xrightarrow{\varepsilon \rightarrow 0} 1 - 2^{(p-1)/p} > 0.$$

Therefore, the obvious generalization of uniform convexity is the following:

Definition. Let $0 < p < 1$. A quasi-Banach space $(X, \|\cdot\|)$ has the *p -midpoint property* if

$$\inf \left\{ 1 - \left\| \frac{x+y}{2^{1/p}} \right\| : \|x\| = \|y\| = 1 \right\} = \delta > 0.$$

Definition. Let $0 < p \leq 1$. A quasi-Banach space $(X, \|\cdot\|)$ is *uniformly p -convex* if $\text{diam } R_p(a) \xrightarrow{\|a\| \rightarrow 1} 0$ uniformly on $\|a\|$.

Next, we check that uniform p -convexity is an intermediate property between strict p -convexity and q -convexity for $q > p$ as well as its relation with the p -midpoint property.

Theorem 3.3. *If $(X, \|\cdot\|)$ is q -convex quasi-Banach space for some $p < q \leq 1$ then $(X, \|\cdot\|)$ is uniformly p -convex. The converse is not true.*

PROOF. For each $a \in X$ with $\|a\| > 1$, if $x_1, x_2 \in R_p(a)$ then

$$x_1 = (1 - s^p)^{1/p}a + sy_1, \quad x_2 = (1 - t^p)^{1/p}a + sy_2,$$

for some $y_1, y_2 \in B_X$, $s, t \in [0, 1]$. Since

$$1 \leq \|(1 - s^p)^{1/p}a + sy_1\|^q \leq (1 - s^p)^{q/p}\|a\|^q + s^q$$

hence

$$1 \leq \frac{1 - s^q}{(1 - s^p)^{q/p}} \leq \|a\|^q.$$

It follows that $s \rightarrow 0$ when $\|a\| \rightarrow 1$. Analogously, $t \rightarrow 0$ when $\|a\| \rightarrow 1$. Therefore,

$$\begin{aligned} \|x_1 - x_2\|^q &= \|(1 - s^p)^{1/p}a + sy_1 - (1 - t^p)^{1/p}a - sy_2\|^q \\ &\leq |(1 - s^p)^{1/p} - (1 - t^p)^{1/p}|^q \|a\|^q + s^q + t^q \xrightarrow{\|a\| \rightarrow 1} 0. \end{aligned}$$

The 2-dimensional Lorentz spaces $d(\omega, p)$ with $\omega = (1, \omega_2)$, $0 < \omega_2 < 1$, are a counterexample of the converse. $\square$

Theorem 3.4. *If $0 < p < 1$ and $(X, \|\cdot\|)$ is uniformly p -convex then it is strictly p -convex. The converse is not true.*

PROOF. First, we prove that $(X, \|\cdot\|)$ is p -convex. Suppose it is not and take $a \in X$ with $\|a\| > 1$ such that $a \in p - \text{co}(B_X)$. Then $-\lambda a, a \in R_p(-\lambda a)$ for all $\lambda > \|a\|^{-1}$ and $\text{diam } R_p(-\lambda a) \geq \|\lambda a + a\| = (\lambda + 1)\|a\| > 1$ for all $\lambda > \|a\|^{-1}$.

Now, suppose that $(X, \|\cdot\|)$ is not strictly p -convex. Then, there is a non p -extreme point in S_X , i.e. there exist $0 < \beta < \alpha < 1$, and $x, y \in S_X$ such that $\alpha^p + \beta^p = 1$ and $\|\alpha x + \beta y\| = 1$. Given $\varepsilon > 0$, denote

$$C = \left(\frac{\alpha^p}{(1 + \varepsilon)^p} + \beta^p \right)^{-1/p}, \quad \lambda = \frac{\alpha C}{1 + \varepsilon}, \quad \mu = (1 - \lambda^p)^{1/p} = \beta C.$$

These numbers satisfy that $\lambda^p + \mu^p = 1$ and, since $1 < C < 1 + \varepsilon$,

$$\frac{\alpha}{1 + \varepsilon} < \lambda < \alpha, \quad \beta < \mu < (1 + \varepsilon)\beta.$$

Since $\|\lambda(1 + \varepsilon)x + \mu y\| = C\|\alpha x + \beta y\| = C > 1$, we have that $\lambda(1 + \varepsilon)x + \mu y \in R_p((1 + \varepsilon)x)$, and

$$\begin{aligned} (\text{diam } R_p((1 + \varepsilon)x))^p &\geq \|\lambda(1 + \varepsilon)x + \mu y - (1 + \varepsilon)x\|^p \\ &\geq (1 - \lambda)^p(1 + \varepsilon)^p - (1 - \lambda^p) \xrightarrow{\varepsilon \rightarrow 0} (1 - \alpha)^p - 1 + \alpha^p > 0. \end{aligned}$$

A strictly p -convex quasi-Banach space which is not uniformly p -convex is the space $(\ell_1(\ell_{p_n}^n), \|\cdot\|)$ where $(p_n)_{n=1}^\infty$ is a decreasing sequence of numbers converging to p . This space does not have the p -midpoint property which is a necessary condition for uniform p -convexity, as we see in the next theorem. $\square$

Theorem 3.5. *If $0 < p < 1$ and $(X, \|\cdot\|)$ is uniformly p -convex then it has the p -midpoint property. The converse is not true.*

PROOF. For any $x, y \in S_X$, and any $\varepsilon > 0$,

$$\frac{x + y}{2^{1/p}} = \alpha_\varepsilon(\lambda_\varepsilon(1 + \varepsilon)x + \mu_\varepsilon y)$$

with

$$\lambda_\varepsilon = \frac{1}{\alpha_\varepsilon 2^{1/p}(1 + \varepsilon)}, \quad \mu_\varepsilon = \frac{1}{\alpha_\varepsilon 2^{1/p}}, \quad \alpha_\varepsilon = \left(\frac{(1 + \varepsilon)^p + 1}{2(1 + \varepsilon)^p} \right)^{1/p}.$$

These numbers satisfy the following

$$\frac{1}{1 + \varepsilon} < \alpha_\varepsilon < 1, \quad \frac{1}{2^{1/p}(1 + \varepsilon)} < \lambda_\varepsilon < \frac{1}{2^{1/p}} < \mu_\varepsilon < \frac{1 + \varepsilon}{2^{1/p}}, \quad \lambda_\varepsilon^p + \mu_\varepsilon^p = 1.$$

Therefore, $\lambda_\varepsilon(1 + \varepsilon)x + \mu_\varepsilon y \in D_p((1 + \varepsilon)x, B_X)$; and since

$$\begin{aligned} \|\lambda_\varepsilon(1 + \varepsilon)x + \mu_\varepsilon y - (1 + \varepsilon)x\|^p &= \|(1 - \lambda_\varepsilon)(1 + \varepsilon)x - \mu_\varepsilon y\|^p \\ &\geq (1 - \lambda_\varepsilon)^p(1 + \varepsilon)^p - \mu_\varepsilon^p \\ &> \left(1 - \frac{1}{2^{1/p}}\right)^p (1 + \varepsilon)^p - \frac{(1 + \varepsilon)^p}{2} > \left(1 - \frac{1}{2^{1/p}}\right)^p - \frac{1}{2} \end{aligned}$$

for all $\varepsilon > 0$, there must exist $\varepsilon_0 > 0$ such that $\|\lambda_\varepsilon(1 + \varepsilon)x + \mu_\varepsilon y\| \leq 1$ for all $\varepsilon \leq \varepsilon_0$. Otherwise, $\text{diam } R_p((1 + \varepsilon)x) \xrightarrow{\varepsilon \rightarrow 0} 0$. Then,

$$\left\| \frac{x + y}{2^{1/p}} \right\| = \alpha_{\varepsilon_0} \|\lambda_{\varepsilon_0}(1 + \varepsilon_0)x + \mu_{\varepsilon_0}y\| \leq \alpha_{\varepsilon_0} < 1.$$

The counterexample in Proposition 2.2 is an example of quasi-Banach space with the p -midpoint property that is not strictly p -convex and, therefore, it is not uniformly p -convex. $\square$

Finally, we see if spaces with the p -properties that we have studied can be renormed with an equivalent q -subadditive quasi-norm for some $p < q \leq 1$.

Theorem 3.6. *If $0 < p < 1$ and $(X, \|\cdot\|)$ has the p -midpoint property then there exist $p < q \leq 1$ and a q -subadditive quasi-norm $\|\cdot\|_q$ on X which is equivalent to $\|\cdot\|$. In particular, uniformly p -convex quasi-Banach spaces can be renormed with an equivalent q -subadditive quasi-norm for some $p < q \leq 1$. The converse is not true.*

PROOF. Let

$$\delta = \inf \left\{ 1 - \left\| \frac{x + y}{2^{1/p}} \right\| : \|x\| = \|y\| = 1 \right\} > 0,$$

and fix $0 < \varepsilon < p\delta$. If $\|x\| = 1$ and $\|y\| \leq 1 - 2\varepsilon$, then

$$\left\| \frac{x + y}{2^{1/p}} \right\|^p \leq \frac{1 + (1 - 2\varepsilon)^p}{2} < \frac{2 - 2p\varepsilon}{2} = 1 - p\varepsilon.$$

If $\|x\| = 1$ and $1 - 2\varepsilon < \|y\| \leq 1$, then

$$\left\| \frac{x + y}{2^{1/p}} \right\|^p = \left\| \frac{x + \frac{y}{\|y\|}}{2^{1/p}} - \frac{y}{\|y\|} \frac{1 - \|y\|}{2^{1/p}} \right\|^p \leq (1 - \delta)^p + \varepsilon < 1 - p\delta + \varepsilon.$$

We have proved that there exists $0 < \gamma < 1$ such that

$$\|x + y\| \leq 2^{1/p} \gamma \max\{\|x\|, \|y\|\}$$

for all $x, y \in X$.

Now, take $p < q \leq 1$ such that $2^{1/p}\gamma = 2^{1/q}$, by the theorem of AOKI and ROLEWICZ's (see Theorem 1.3 of [3] and Theorem 3.2.1 of [7]),

$$\|x\|_q = \inf \left\{ \left(\sum_{i=1}^n \|x_i\|^q \right)^{1/q} : \sum_{i=1}^n x_i = x, n \in \mathbb{N} \right\}$$

defines a q -subadditive quasi-norm on X which is equivalent to $\|\cdot\|$.

Any finite dimensional ℓ_p space with $0 < p < 1$ is a counterexample of the converse. $\square$

Theorem 3.7. *For every $0 < p < 1$, there are quasi-normed spaces that can be renormed with an equivalent norm (or 1-subadditive quasi-norm) but fail to be strictly p -convex, and there are strictly p -convex quasi-normed spaces that cannot be renormed with an equivalent q -subadditive quasi-norm for any $p < q \leq 1$.*

PROOF. Any finite dimensional quasi-normed space can be renormed with an equivalent norm.

The space $(\ell_1(\ell_{p_n}^n), \|\cdot\|)$, where $(p_n)_{n=1}^\infty$ is a decreasing sequence of numbers converging to p , is strictly p -convex but cannot be renormed with an equivalent q -subadditive quasi-norm for any $p < q \leq 1$. $\square$

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