Publ. Math. Debrecen
61 / 1-2 (2002), 145–155

On τ-injective hulls of modules

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Abstract. Let τ be a hereditary torsion theory on the category R-mod of left modules over an associative unitary ring R. We will study the τ-injective hull of certain modules, giving an explicit description as well. The main goal will be the connection between the τ-injective hull of R/p, for some prime ideal p of a commutative ring R, and the annihilator of p in the injective hull of R/p. Some torsion theories generalizing the Dickson torsion theory in the case of a commutative ring R will be considered and further properties will be established.

1. Preliminaries

Since injective hulls for modules were defined [7], numerous attempts to give their structure or to characterize them have been made. But even in particular cases, such as for a simple module, still there are things to say on the injective hull [10]. An important problem, that will be discussed here as well, has been finding explicit descriptions of injective hulls of modules (see for instance [12, [16]).

As far as the torsion-theoretic version is concerned, we mention [2], [8], [14], but the injective hull has been studied mainly in connection with the τ-localization functor (see [9] for a more complete survey on the subject). In recent years, there have been established results on the relationship between injective hulls or indecomposable injectives and (τ-closed) prime ideals of some rings in the context of a hereditary torsion theory [1], [11], [18].

The goal of the present paper is to establish some further results on the injective hull of a module relative to a hereditary torsion theory. Thus,

Mathematics Subject Classification: Primary 16S90; Secondary 16D70.
Key words and phrases: hereditary torsion theory, τ-injective hull.

it will be proved that if p is a prime ideal of a commutative ring R and R/p is τ-cocritical, then the τ-injective hull of R/p is isomorphic to the field of fractions of R/p. As a consequence, every τ-torsionfree minimal τ-injective module is isomorphic to the field of fractions of a domain R/p for some prime ideal p of R. Moreover, if R/p is τ-cocritical noetherian, then the τ-injective hull of R/p is locally noetherian.

Most of the results established in the literature on torsion theories have been detailed for some specific torsion theories, such as the Goldie torsion theory or the Dickson torsion theory [5]. But since injectivity with respect to the Goldie torsion theory and usual injectivity are the same, the Dickson torsion theory seems to be the most convenient for going further. Having it as starting point [4] and main motivation, we will discuss in the last section some other "appropriate" torsion theories, generated by the modules of Krull dimension at most n, that generalize the Dickson torsion theory in the case of a commutative ring. Modules having (relative) Krull dimension have often been present in the literature related to (relative) injective hulls of modules [1], [3].

Throughout the paper we denote by R an associative ring with non-zero identity and all modules are left unital R-modules.

We denote by $\mathrm{Spec}(R)$ the set of all prime ideals of a commutative ring R. For a prime ideal p of a commutative ring R, we denote by $\dim p$ the (Krull) dimension of the ring R/p. A prime ideal p of a commutative ring R is called N-prime if R/p is noetherian [17, p. 52].

A module A is called locally noetherian if every finitely generated submodule of A is noetherian [6, p. 10]. As usual, a module A is said to be $\sum$-quasi-injective if any direct sum of copies of A is quasi-injective.

Throughout the paper we denote by τ a hereditary torsion theory on the category R-mod of left R-modules.

A submodule B of a module A is said to be τ-dense (τ-closed) in A if A/B is τ-torsion (τ-torsionfree). A non-zero module A is said to be τ-cocritical if A is τ-torsionfree and each of its non-zero submodules is τ-dense in A. A module is called τ-noetherian if it satisfies the ascending chain condition for τ-closed submodules.

A module A is said to be τ-injective if it is injective with respect to every monomorphism having a τ-torsion cokernel or equivalently if it satisfies a Baer-type criterion with respect to the left ideals of the associated Gabriel filter. In this paper, a non-zero module that is the τ-injective hull of each of its non-zero submodules will be called minimal τ-injective. It is clear that every minimal τ-injective module is either τ-torsion or τ-cocritical.

For a module A we will denote by $E(A)$ and $E_\tau(A)$ the injective hull and the τ-injective hull of A respectively.

A torsion theory τ is called stable if the class of τ-torsion modules is closed under taking injective hulls.

For additional information on torsion theories we refer to [9].

2. τ-injective hulls

In the study of the τ-injective hull of a module an important part is played by minimal τ-injective modules. Notice that every minimal τ-injective module is either τ-torsion or τ-torsionfree and τ-torsionfree minimal τ-injective modules are exactly τ-cocritical τ-injective modules.

We mention here a preliminary result that will be frequently used.

Lemma 2.1. *Let A be a τ-cocritical module over a commutative ring R. Then:*

(i) $\operatorname{Ann}_R a = \operatorname{Ann}_R A$ *for every non-zero element* $a \in A$*;*

(ii) $\operatorname{Ann}_R A \in \operatorname{Spec}(R)$*;*

(iii) $R/\operatorname{Ann}_R A$ *is τ-cocritical.*

Proof. Straightforward. □

Let us now refer to minimal τ-injective modules, considering first the τ-torsion case.

Proposition 2.2. *Let $\mathcal{A}$ be a class of modules closed under homomorphic images and let τ be the hereditary torsion theory generated by $\mathcal{A}$. Then the following statements are equivalent for a non-zero τ-torsion module A:*

(i) *A is minimal τ-injective;*

(ii) *$A = E_\tau(B)$, where $B \in \mathcal{A}$ and B is uniform.*

Proof. Suppose first that A is minimal τ-injective. Since A is τ-torsion, there exists a non-zero submodule B of A such that $B \in \mathcal{A}$. Then $A = E_\tau(B)$. Since A is uniform, it follows that B is uniform.

Suppose now that the statement (ii) holds. Let C be a non-zero submodule of A. Since A is τ-torsion, $E_\tau(C)$ is τ-dense in A. It follows that $E_\tau(C)$ is a direct summand of A. But A is uniform, hence $E_\tau(C) = A$. Therefore A is minimal τ-injective. $\square$

In the sequel we will assume the ring R to be commutative, unless stated otherwise.

Let us now consider τ-torsionfree minimal τ-injective modules.

Proposition 2.3. *Let A be a τ-torsionfree minimal τ-injective module. Then $A \cong E_\tau(R/p)$, where $p = \operatorname{Ann}_R A \in \operatorname{Spec}(R)$.*

Proof. Notice that A is τ-cocritical. By Lemma 2.1, $p \in \operatorname{Spec}(R)$ and R/p is τ-cocritical. Let a be a non-zero element of A. Then again by Lemma 2.1 we have $Ra \cong R/\operatorname{Ann}_R a = R/p$. Since A is minimal τ-injective, $A = E_\tau(Ra) \cong E_\tau(R/p)$. $\square$

Now we are able to give the structure of the τ-injective hull of R/p, where p is a prime ideal of R such that R/p is τ-cocritical.

For reader's convenience we mention first the following known result.

Proposition 2.4 [17, Proposition 2.27]**.** *Let I be a two-sided ideal of a not necessarily commutative ring R and let E be an injective R-module. Then $\operatorname{Ann}_E I$ is injective as an R/I-module. Moreover, if E is the injective hull of an R-module A, then $\operatorname{Ann}_E I$ is an injective hull of $A \cap \operatorname{Ann}_E I$ considered as an R/I-module.*

Theorem 2.5. *Let $p \in \operatorname{Spec}(R)$ and assume that R/p is τ-cocritical. Then:*

(i) $E_\tau(R/p) = \operatorname{Ann}_{E(R/p)} p$;

(ii) *There exists an R/p-isomorphism (and hence an R-isomorphism) between $E_\tau(R/p)$ and the field of fractions of R/p;*

(iii) *If p is not maximal, then $R/p \neq E_\tau(R/p)$.*

Proof. (i) Denote $A = E_\tau(R/p)$. Notice that A is τ-torsionfree minimal τ-injective. We have seen in the proof of Proposition 2.3 that $\operatorname{Ann}_R A = p$. It follows that $A \subseteq \operatorname{Ann}_{E(R/p)} p$.

On the other hand, it is known that the collection of all annihilators of non-zero elements of the R-module $E(R/p)$ has a unique maximal member, namely p itself [17, Lemma 2.31]. Therefore we have $\operatorname{Ann}_R d \subseteq p$ for every non-zero element $d \in E(R/p)$. Let $b \in \operatorname{Ann}_{E(R/p)} p$ be a non-zero element. Then $\operatorname{Ann}_R b = p$ and $Rb \cong R/p$ is τ-cocritical. But since $E(R/p)$ is uniform, A is the maximal τ-cocritical submodule of $E(R/p)$ [9, Proposition 14.12]. It follows that $Rb \subseteq A$, hence $\operatorname{Ann}_{E(R/p)} p \subseteq A$. Therefore, $E_\tau(R/p) = \operatorname{Ann}_{E(R/p)} p$.

(ii) By (i) and Proposition 2.4, it follows that $E_\tau(R/p) = \operatorname{Ann}_{E(R/p)} p$ is an injective hull of R/p considered as an R/p-module. Since R/p is a domain, $E_\tau(R/p)$, considered as an R/p-module, is isomorphic to the field of fractions of R/p.

(iii) It follows by (ii), because R/p is not a field. □

Remarks. (1) Notice that the hypothesis needed in the proof is for $E_\tau(R/p)$ to be τ-torsionfree minimal τ-injective. In the light of Proposition 2.3 and Theorem 2.5, every τ-torsionfree minimal τ-injective module is isomorphic to the field of fractions of a domain R/p for some prime ideal p of R.

(2) Consider the context of Theorem 2.5 and suppose that R is a domain and $p \neq 0$. Then $\operatorname{Ann}_R E(R/p) = 0$ [17, Proposition 2.26, Corollary 1]. On the other hand, by Lemma 2.1, $\operatorname{Ann}_R E_\tau(R/p) = p$. Hence the τ-injective hull does not coincide here with the injective hull.

(3) An example of determining the τ-injective hull of a τ-cocritical module R/p by using Theorem 2.5 will be given in the next section.

The following theorem is a partial converse of Theorem 2.5.

Theorem 2.6. *Let $p \in \operatorname{Spec}(R)$. If $E_\tau(R/p) = \operatorname{Ann}_{E(R/p)} p$, then $E_\tau(R/p)$ is minimal τ-injective.*

Proof. Suppose that $E_\tau(R/p)$ is not minimal τ-injective. Then there exists a non-zero proper τ-injective submodule A of $E_\tau(R/p) = \operatorname{Ann}_{E(R/p)} p$. Let a be a non-zero element of A. Then it follows that $\operatorname{Ann}_R a = p$ and

$$E_\tau(Ra) \cong E_\tau(R/p) = \operatorname{Ann}_{E(R/p)} p.$$

Thus $E_\tau(Ra)$ is a proper submodule of $\operatorname{Ann}_{E(R/p)} p$, hence $E_\tau(Ra)$ is a proper R/p-submodule of $\operatorname{Ann}_{E(R/p)} p$. By Proposition 2.4, $\operatorname{Ann}_{E(R/p)} p$ is the injective hull of R/p considered as an R/p-module. Moreover, both

$\operatorname{Ann}_{E(R/p)} p$ and $E_\tau(Ra)$ are indecomposable injective R/p-modules. Then $E_\tau(Ra)$ is a direct summand of $\operatorname{Ann}_{E(R/p)} p$, which is a contradiction. $\square$

It is known that if $p \in \operatorname{Spec}(R)$, then R/p is either τ-torsionfree or τ-torsion. As far as the former case is concerned, we will see the relationship between $E_\tau(R/p)$ and $\operatorname{Ann}_{E(R/p)} p$ in some particular situations in the next section. Considering the latter case, it is easy to see that $\operatorname{Ann}_{E(R/p)} p \subseteq E_\tau(R/p)$, since everything is τ-torsion.

We will end this section with two properties of the τ-injective hull of R/p for some specific prime ideal p of R.

Theorem 2.7. *Let p be an N-prime ideal of R such that R/p is τ-cocritical. Then $E_\tau(R/p)$ is locally noetherian and $\sum$-quasi-injective.*

Proof. Let A be a non-zero finitely generated submodule of $E_\tau(R/p)$. Then A is τ-cocritical. Since $E_\tau(R/p) = \operatorname{Ann}_{E(R/p)} p$ by Theorem 2.5, we have $\operatorname{Ann}_R A = \operatorname{Ann}_R a = p$, for every non-zero element $a \in E_\tau(R/p)$ [17, Lemma 2.31]. If $\{a_1, \dots, a_n\}$ is a set of generators of A, each $Ra_k \simeq R/p$ is a noetherian R-module, so that $A = \sum_{k=1}^n Ra_k$ is noetherian. Therefore, $E_\tau(R/p)$ is locally noetherian.

Since $E_\tau(R/p)$ is τ-torsionfree minimal τ-injective, it follows that it is $\sum$-quasi-injective [13, Theorem 3]. $\square$

3. The torsion theories τ_n

Throughout this section we will assume the ring R to be commutative, unless stated otherwise. We will consider some torsion theories generalizing the Dickson torsion theory.

For each positive integer n, let $\mathcal{A}_n$ be the class consisting of all modules isomorphic to factor modules U/V, where U and V are ideals of R containing a prime ideal p of R with $\dim p \leq n$. Denote by τ_n the hereditary torsion theory generated by $\mathcal{A}_n$. Equivalently, τ_n may be seen as the torsion theory generated by the modules of Krull dimension at most n.

Notice that τ_0 is the hereditary torsion theory generated by the class $\mathcal{A}_0$ consisting of all simple modules, i.e. the Dickson torsion theory [5]. In the sequel set a positive integer n and, in order to ensure that the study is not vacuous, assume that $\dim R \geq n$.

Lemma 3.1. *Let $p \in \operatorname{Spec}(R)$.*

(i) *If $\dim p \leq n$, then R/p is τ_n-torsion.*

(ii) *If $\dim p \geq n+1$, then R/p is τ_n-torsionfree.*

Proof. (i) Obvious.

(ii) Assume $\dim p \geq n+1$. It is a routine verification to show that $\operatorname{Hom}_R(U/V, R/p) = 0$ for every ideals U, V of R such that there exists a prime ideal q of R with $\dim q \leq n$ and $q \subseteq V \subseteq U$. Thus, R/p is τ_n-torsionfree. □

Proposition 3.2. *Let A be a τ_n-cocritical module. Then $\operatorname{Ann}_R A \in \operatorname{Spec}(R)$ and $\dim \operatorname{Ann}_R A = n+1$.*

Proof. Denote $p = \operatorname{Ann}_R A$. By Lemma 2.1, $p \in \operatorname{Spec}(R)$ and R/p is τ_n-cocritical, hence R/p is τ_n-torsionfree. By Lemma 3.1, $\dim p \geq n+1$. Suppose that $\dim p > n+1$. Then there exists $q \in \operatorname{Spec}(R)$ with $\dim q = n+1$ and $p \subset q$. Moreover, again by Lemma 3.1, R/q is τ_n-torsionfree. On the other hand, $R/q \cong (R/p)/(q/p)$ is τ_n-torsion, which is a contradiction. Hence $\dim p = n+1$. □

Corollary 3.3. *Let $p \in \operatorname{Spec}(R)$ such that R/p is τ-noetherian. Then R/p is τ_n-cocritical if and only if $\dim p = n+1$.*

Proof. The “if” part follows by Proposition 3.2.

Suppose now that $\dim p = n+1$. Then R/p is τ_n-torsionfree. Since the R-module R/p is τ-noetherian, there exists an ideal q of R such that $p \subseteq q$ and R/q is τ_n-cocritical [9, Proposition 20.3]. By Proposition 3.2, $q = \operatorname{Ann}_R(R/q) \in \operatorname{Spec}(R)$ and $\dim q = n+1$. Then $p = q$, hence R/p is τ_n-cocritical. □

As far as the torsion theories τ_n are concerned, we obtain some stronger results for minimal τ_n-injective modules.

Theorem 3.4. *Let $p \in \operatorname{Spec}(R)$. Then:*

(i) *$E_{\tau_n}(R/p)$ is τ_n-torsion minimal τ_n-injective if and only if $\dim p \leq n$.*

(ii) *If $E_{\tau_n}(R/p)$ is τ_n-torsionfree minimal τ_n-injective, then $\dim p = n+1$.*

(iii) *If $E_{\tau_n}(R/p)$ is τ_n-torsionfree, but not minimal τ_n-injective, then $\dim p \geq n+1$.*

(iv) *If* $\dim p \geq n+2$, *then* $E_{\tau_n}(R/p)$ *is* τ_n*-torsionfree, but not minimal* τ_n*-injective.*

Proof. (i) Apply Lemma 3.1 and Proposition 2.2.

(ii) Since $E_{\tau_n}(R/p)$ is τ_n-cocritical, then by Proposition 3.2, $q = \operatorname{Ann}_R E_{\tau_n}(R/p)$ is a prime ideal and $\dim q = n+1$. On the other hand, by Lemma 2.1, we have $q = \operatorname{Ann}_R a = p$ for every non-zero element $a \in R/p$. Hence $\dim p = n+1$.

(iii) Apply Lemma 3.1.

(iv) Apply Lemma 3.1 and (ii). □

Over a noetherian ring R we are able to establish the form of minimal τ_n-injective modules.

Theorem 3.5. (i) *If* R *is noetherian, then a module* A *is* τ_n*-torsion minimal* τ_n*-injective if and only if* $A \cong E(R/p)$, *where* $p \in \operatorname{Spec}(R)$ *and* $\dim p \leq n$.

(ii) *If* R *is* τ_n*-noetherian, then a module* A *is* τ_n*-torsionfree minimal* τ_n*-injective if and only if* $A \cong E_{\tau_n}(R/p)$, *where* $p \in \operatorname{Spec}(R)$ *and* $\dim p = n+1$.

Proof. (i) In this case τ_n is stable [15, Chapter III, Corollary 3.4.6]. The result now follows by Lemma 3.1 and by the fact that $E(R/p)$ is indecomposable.

(ii) The "if" part follows by Proposition 2.3 and Theorem 3.4. Suppose now that $A \cong E_{\tau_n}(R/p)$, where $p \in \operatorname{Spec}(R)$ and $\dim p = n+1$. Then by Corollary 3.3, R/p is τ_n-cocritical. Hence $A \cong E_{\tau_n}(R/p)$ is τ_n-torsionfree minimal τ_n-injective. □

Remark. As a consequence of Theorem 3.5, we have the form of minimal τ_0-injective modules, i.e. minimal injective modules relative to the Dickson torsion theory, over a commutative noetherian ring. But notice that by using Proposition 2.2, we may immediately obtain a more general result for minimal injective modules relative to the Dickson torsion theory τ_D.

Corollary 3.6. *The following statements are equivalent for a module* D *over a not necessarily commutative ring:*

(i) D *is minimal* τ_D*-injective;*

(ii) $D = E_{\tau_D}(A)$, *where* A *is either* τ_D*-cocritical or simple.*

Let us now give an example of determining the τ_n-injective hull of a τ_n-cocritical module R/p for some $p \in \operatorname{Spec}(R)$ by using Theorem 2.5.

Example 3.7. Consider the polynomial ring $R = K[X_1, \ldots, X_m]$ ($m \geq 2$), where K is a field, and the prime ideal $p = (X_1, \ldots, X_{m-n-1})$ of R with $n < m-1$. Then we have the ring isomorphism $K[X_1, \ldots, X_m]/(X_1, \ldots, X_{m-n-1}) \cong K[X_{m-n}, \ldots, X_m]$. But $K[X_{m-n}, \ldots, X_m]$ has both a structure of R/p-module and R-module by restriction of scalars. Since R is noetherian and $\dim p = n+1$, R/p is a τ_n-cocritical R-module by Corollary 3.3. Then $E_{\tau_n}(R/p)$ is τ_n-torsionfree minimal τ_n-injective. Now by Theorem 2.5 we have the R-isomorphism

$$E_{\tau_n}(K[X_1, \ldots, X_m]/(X_1, \ldots, X_{m-n-1})) \cong K(X_{m-n}, \ldots, X_m),$$

where $K(X_{m-n}, \ldots, X_m)$ is the field of fractions of $K[X_{m-n}, \ldots, X_m]$.

Let us now analyze the connection between $E_{\tau_n}(R/p)$ and $\operatorname{Ann}_{E(R/p)} p$ for some $p \in \operatorname{Spec}(R)$.

Theorem 3.8. *Let $p \in \operatorname{Spec}(R)$.*

(i) *If $\dim p \leq n$, then $\operatorname{Ann}_{E(R/p)} p \subseteq E_{\tau_n}(R/p)$.*

(ii) *Let R be τ_n-noetherian. If $\dim p = n+1$, then $\operatorname{Ann}_{E(R/p)} p = E_{\tau_n}(R/p)$.*

(iii) *Let R be noetherian. If $\dim p \geq n+2$, then $\operatorname{Ann}_{E(R/p)} p \supset E_{\tau_n}(R/p)$.*

Proof. (i) It is clear even for an arbitrary τ.

(ii) It follows by Theorems 3.5 and 2.5.

(iii) Let us denote $A = \operatorname{Ann}_{E(R/p)} p$. We will prove first that A is τ_n-injective by showing that $E(R/p)/A$ is τ_n-torsionfree [9, Proposition 8.2]. We may assume that $p \neq 0$. Suppose that $E(R/p)/A$ is not τ_n-torsionfree. Then there exists a non-zero submodule B of $E(R/p)/A$ such that $B \in \mathcal{A}_n$. It follows that $B \cong U/V$, where U and V are ideals of R containing a prime ideal q of R with $\dim q \leq n$. Then $qU \subseteq V$, hence $qB = 0$. On the other hand, there exists an element $b \in E(R/p) \setminus A$ such that $b + A \in B$. Now let $r \in q \setminus p$ and let $s \in p^m$ such that $sb \neq 0$. Then $rb \in A$, so that $srb = 0$. Since multiplication by r on $E(R/p)$ is an automorphism [19, p. 83], we have $rsb \neq 0$, which is a contradiction. Hence $\operatorname{Ann}_{E(R/p)} p$ is τ_n-injective.

Now since $R/p \subseteq A$, it follows that $E_{\tau_n}(R/p)$ is an essential submodule of A. But $\dim p \geq n+2$, hence $E_{\tau_n}(R/p)$ is not minimal τ_n-injective by Theorem 3.4. Now by Theorem 2.6, it follows that $E_{\tau_n}(R/p) \neq A$. Therefore $E_{\tau_n}(R/p)$ is a proper submodule of $\mathrm{Ann}_{E(R/p)}\, p$. □

Remark. For a noetherian ring R, Theorem 3.8 offers a complete picture of the relationship between $E_{\tau_n}(R/p)$ and $\mathrm{Ann}_{E(R/p)}\, p$, depending on the dimension of the prime ideal p of R.

Acknowledgements. The author is thankful to the referee, whose valuable comments and suggestions have been helpful to improve the presentation of results.

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(Received July 3, 2001; revised December 19, 2001)