

On Fermat's problem in matrix rings and groups

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Abstract. We describe the periodic elements in $GL_2(\mathbb{Z})$ and give the answer to some problems concerning Fermat's equation $X^m + Y^m = Z^m$ (F) in matrix groups and in irreducible elements of matrix rings, proposed by L. N. Vaserstein and A. Khazanov. Namely: (1) equation (F) has solutions in $GL_2(\mathbb{C})$ for every m ; (2) if $m = 3$ or $m \equiv \pm 1 \pmod{3}$, then (F) has solutions in $SL_3(\mathbb{Z})$; (3) if m is odd, then equation (F) has solutions in $M_2(\mathbb{Z})$ in irreducible elements; (4) if $m \equiv \pm 1 \pmod{3}$ or $m = n \geq 2$, then (F) has solutions in irreducible elements of matrix rings $M_3(\mathbb{Z})$ and $M_n(\mathbb{Z})$ respectively.

1. Introduction

We consider the solution of Fermat's equation

$$(F) \quad X^m + Y^m = Z^m \quad (m \in \mathbb{N})$$

in the ring $M_n(\mathbb{Z})$ of $n \times n$ -matrices over the ring $\mathbb{Z}$ of integers.

It is proved in [4] that equation $X^4 + Y^4 = Z^4$ is solvable in $M_2(\mathbb{Z})$. It is also easy to see that if $n \geq 2$, then there are such idempotent elements A and B that $A + B = E$, where E is the identity matrix, and so $A^m + B^m = E^m$ for every $m \geq 1$.

Equation (F) was studied with distinct restrictions in papers [1]–[14] in the ring $M_n(\mathbb{Z})$ and in $M_n(R)$ over a commutative ring R with unit element.

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The question about the solvability of (F) in the group $GL_2(\mathbb{Z})$ was studied at first by L. N. VASERSTEIN in [14].

The solvability of (F) in the set of positive integer powers of a matrix A with elements $a_{11} = 0, a_{12} = a_{21} = a_{22} = 1$ was studied in [5] and [6].

In [8] and [11] a general result has been proved: if $A \in M_2(\mathbb{Z})$ and $n > 2$, then the equation (F) has solution (X, Y, Z) with $X, Y, Z \in \mathcal{A} = \{A^k \mid k \in \mathbb{N}\}$ if and only if A is nilpotent or $\text{tr } A = \det A = 1$. Evidently $X, Y, Z \in SL_2(\mathbb{Z})$ and we can (by [8], [10]) determine effectively all such solutions.

Paper [7] contains the following result: let $A \in M_n(\mathbb{C})$, $n \geq 2$ and $A^x + A^y = A^z$ for some positive integers x, y, z . If A has at least one real eigenvalue $\alpha > \sqrt{2}$, then $\max\{x - z, y - z\} = -1$. Many interesting consequences can be obtained from this assertion.

A. KHAZANOV in [9] proved that in $GL_3(\mathbb{Z})$ solutions do not exist if m is a multiple of either 21 or 96, and in $SL_3(\mathbb{Z})$ solutions do not exist if m is a multiple of 48. Paper [12] gives another proof of Khazanov's result (see [9], Corollary 4) on the solvability of (F) in $SL_2(\mathbb{Z})$.

L. N. Vaserstein proposed the following problem: *how about solutions of the equation (F) in $SL_2(\mathbb{Z})$ or in $GL_3(\mathbb{Z})$; or in irreducibles X, Y, Z of the ring $M_2(\mathbb{Z})$?* Later A. Khazanov called attention to the fact that *the equation $X^3 + Y^3 = Z^3$ is still unsolved in $SL_3(\mathbb{Z})$ and in $GL_3(\mathbb{Z})$ as well as in $SL_3(\mathbb{Q})$.*

We give the answers to some of these questions and their generalizations.

Here $GL_n(\mathbb{Z})$ is the group of units of the ring $M_n(\mathbb{Z})$ and

$$SL_n(\mathbb{Z}) = \{A \in M_n(\mathbb{Z}) : \det A = 1\}.$$

An element x of a ring R is called *irreducible*, if it is neither a unit nor the product yz of two elements y, z of R , both not units. A matrix X in $M_n(\mathbb{Z})$ is irreducible if and only if $\det X = \pm p$ for a prime p .

2. Description of the elements in $GL_2(\mathbb{Z})$

In this part we give a description of the elements in $GL_2(\mathbb{Z})$ by proving the following theorem:

Theorem 2.1. *Let $A \in GL_2(\mathbb{Z})$ and A is a periodic element. Then we have:*

- 1° if $\text{ord } A = 1$, then $A = E$.
- 2° if $\text{ord } A = 2$, then either $\text{tr } A = -2$ and $A = -E$, or $\text{tr } A = 0$ and $A = \pm \begin{pmatrix} 1 & 0 \\ z & -1 \end{pmatrix}$, or $A = \begin{pmatrix} z & v \\ (1 - z^2)v^{-1} & -z \end{pmatrix}$, where $z \in \mathbb{Z}$ and $v \mid 1 - z^2$.
- 3° if $\text{ord } A = 3$, then $A = \begin{pmatrix} z & -v \\ (1 + z + z^2)v^{-1} & -1 - z \end{pmatrix}$ for some $z \in \mathbb{Z}$ and $v \mid 1 + z + z^2$.
- 4° if $\text{ord } A = 4$, then $A = \begin{pmatrix} z & -v \\ (1 + z^2)v^{-1} & -z \end{pmatrix}$, where $z \in \mathbb{Z}$ and $v \mid 1 + z^2$.
- 5° if $\text{ord } A = 6$, then $A = \begin{pmatrix} -z & -v \\ (1 + z + z^2)v^{-1} & 1 + z \end{pmatrix}$ for some $z \in \mathbb{Z}$ and $v \mid 1 + z + z^2$.

PROOF. It is well-known (see for example [7] or [9]) that any periodic matrix in $GL_2(\mathbb{Z})$ has order 1, 2, 3, 4 or 6 and if A is an arbitrary matrix of $M_2(\mathbb{Z})$, $t = \text{tr } A$ and $d = \det A$, then for every natural n the n -th power of A can be written in the form $A^n = u_n A - du_{n-1} E$, where $u_0 = 0$, $u_1 = 1$, $u_2 = t$ and $u_n = tu_{n-1} - du_{n-2}$ for $n \geq 3$. Therefore, for a nondiagonal matrix $A \in M_2(\mathbb{Z})$ and for some nonzero $k \in \mathbb{Z}$ the equality $A^n = kE$ holds if and only if in the series $u_0, u_1, u_2, u_3, \dots$ the element u_n is zero. Indeed, suppose $A^n = kE$ and $A = (a_{ij})$. Then $u_n A = A^n + du_{n-1} E = (k + du_{n-1})E$ and $u_n a_{12} = u_n a_{21} = 0$. Since $a_{12} \neq 0$ or $a_{21} \neq 0$, we have $u_n = 0$. Conversely, in case $u_n = 0$, $A^n = -du_{n-1} E = kE$ with $k = -du_{n-1}$.

Let $\text{ord } A = 2$ and A diagonal. Then $A = -E$ or $A = \pm \begin{pmatrix} 1 & 0 \\ z & -1 \end{pmatrix}$. Let A be a nondiagonal. Then $u_2 = \text{tr } A = 0$, so $A = \begin{pmatrix} z & v \\ u & -z \end{pmatrix}$. If $v = 0$, then $z^2 = 1$ and $A = \pm \begin{pmatrix} 1 & 0 \\ u & -1 \end{pmatrix}$, $u \in \mathbb{Z}$. If $v \neq 0$, then $z^2 + uv = 1$, u and v are divisors of $1 - z^2$ and $A = \begin{pmatrix} z & v \\ (1 - z^2)v^{-1} & -z \end{pmatrix}$, where $z \in \mathbb{Z}$ and $v \mid 1 - z^2$.

Let $\text{ord } A = 3$. Then $d \cdot t = -1$, $0 = u_3 = t^2 - d$. This implies $d = 1$, $t = -1$ and $A = \begin{pmatrix} z & -v \\ u & -1 - z \end{pmatrix}$. So $1 = d = -z - z^2 + uv$ and $uv = 1 + z + z^2$.

Let $\text{ord } A = 4$. Then $u_4 = t^3 - 2dt = 0$, $du_3 = d(t^2 - d) = -1$. The case $t \neq 0$ is impossible, since from $t^2 - 2d = 0$ follows, so $t^2 - d = d$ and $d^2 = -1$. Therefore $t = 0$, $A = \begin{pmatrix} z & -v \\ u & -z \end{pmatrix}$, $A^2 = \begin{pmatrix} z^2 - uv & 0 \\ 0 & z^2 - uv \end{pmatrix} = -E$ and $uv = 1 + z^2$.

Let $\text{ord } A = 6$. Then $0 = u_6 = t^5 - 4dt^3 + 3d^2t$, $1 = du_5 = d(t^4 - 3dt^2 + d^2)$. Since $A^3 = -E$, $A^2 \neq E$, it follows $d = 1$, $t = 1$. It is easy to prove that in this case A has form $A = \begin{pmatrix} -z & -v \\ (1 + z + z^2)v^{-1} & 1 + z \end{pmatrix}$, where $z \in \mathbb{Z}$ and $v \mid 1 + z + z^2$. The theorem is proved.

Theorem 2.2. *Let $\mathbb{C}$ be the field of complex numbers. The equation (F) has infinitely many solutions in the group $GL_2(\mathbb{C})$ for every m .*

PROOF. Let u be an arbitrary integer. The matrices

$$A = \begin{pmatrix} u & 1 \\ -1 - u - u^2 & -1 - u \end{pmatrix}, \quad B = \begin{pmatrix} -1 - u & -1 \\ 1 + u + u^2 & u \end{pmatrix}$$

are elements of order 3 in $GL_2(\mathbb{C})$ and $A + B = -E$. By [3] for every m there exist some A_m , B_m and C_m in $M_2(\mathbb{C})$, for which $(A_m)^m = A$, $(B_m)^m = B$ and $(C_m)^m = -E$. The matrices A_m , B_m and C_m belong to $GL_2(\mathbb{C})$ and (A_m, B_m, C_m) is a solution of the equation (F).

Theorem 2.3. *If $UT_n(\mathbb{Z})$ is the subgroup of upper (or lower) triangular matrices of $GL_n(\mathbb{Z})$ and $m \geq 1$, then the equation (F) has no solutions in $UT_n(\mathbb{Z})$.*

PROOF. Suppose that $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ and (A, B, C) is a solution of (F). Since for every $A \in UT_n(\mathbb{Z})$ the equation $\det A = a_{11}a_{22} \cdots a_{nn} = \pm 1$ holds, it follows that $a_{ii} \in \{-1; 1\}$, and similarly, $b_{ii} \in \{-1; 1\}$, $c_{ii} \in \{-1; 1\}$. From the equation $A^m + B^m = C^m$ it follows that $a_{ii}^m + b_{ii}^m = c_{ii}^m$. Then $c_{ii} \in \{-2; 0; 2\}$, which contradicts $c_{ii} \in \{-1; 1\}$. The theorem is proved.

3. Fermat's equation in $SL_3(\mathbb{Z})$

Theorem 3.1. *If $m \equiv \pm 1 \pmod{3}$, then the equation (F) has solutions in $SL_3(\mathbb{Z})$.*

PROOF. Obviously

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

are the elements of order 3 in $SL_3(\mathbb{Z})$ and $A + B = C$.

If $m \equiv 1 \pmod{3}$, then $A^m + B^m = A + B = C = C^m$ and so (A, B, C) is a solution of (F).

Let $m \equiv -1 \pmod{3}$. Then (A^2, B^2, C^2) is a solution of (F). The theorem is proved.

Theorem 3.2. *The equation $X^3 + Y^3 = Z^3$ has solutions in $SL_3(\mathbb{Z})$.*

PROOF. It is easy to verify that (A, B, C) with the following elements A , B and C is a solution of the equation $X^3 + Y^3 = Z^3$ in $SL_3(\mathbb{Z})$. We also give the elements A^3 , B^3 , C^3 below.

$$1) \quad A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$A^3 = \begin{pmatrix} 0 & -1 & 2 \\ -1 & -3 & 3 \\ 2 & 3 & -1 \end{pmatrix}, \quad B^3 = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 4 & -2 \\ -1 & -3 & 2 \end{pmatrix}, \quad C^3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

and A , B , C are not periodic elements.

$$2) \quad A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix},$$

$$A^3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B^3 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad C^3 = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$

A , B are elements of order 4 and C is not a periodic element.

$$3) \quad A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

$$A^3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, B^3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, C^3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

A is an element of order 4 and B, C are not periodic elements.

$$4) \quad A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$

$$A^3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}, B^3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, C^3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

A is not a periodic element, B is an element of order 2 and C has order 4.

$$5) \quad A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix},$$

$$A^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, B^3 = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}, C^3 = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$

A is an element of order 6, B, C are not periodic elements. The theorem is proved.

4. Fermat's equation in irreducible elements of the rings $M_2(\mathbb{Z})$ and $M_3(\mathbb{Z})$

Let us now consider the solution of (F) in irreducibles X, Y, Z of the rings $M_2(\mathbb{Z})$ and $M_3(\mathbb{Z})$.

Theorem 4.1. *If m is odd, then the equation (F) has solutions in $M_2(\mathbb{Z})$ in irreducible elements.*

PROOF. Let

$$A = \begin{pmatrix} -i+1 & -i^2+2i+2 \\ 1 & i-1 \end{pmatrix}, \quad B = \begin{pmatrix} -i-2 & -i^2-4i-1 \\ 1 & i+2 \end{pmatrix},$$

$$C = \begin{pmatrix} -2i-1 & -2i^2-2i+1 \\ 2 & 2i+1 \end{pmatrix}.$$

Then $\det A = \det B = \det C = -3$, $A + B = C$ and

$$A^2 = B^2 = C^2 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = 3E$$

for every $i \in \mathbb{Z}$. If $m = 2k+1$ ($k \in \mathbb{N}$), then (for example) $A^m = A^{2k+1} = (A^2)^k A = (3E)^k A = 3^k A$. Therefore $A^{2k+1} + B^{2k+1} = 3^k A + 3^k B = 3^k C = C^{2k+1}$ and the proof is complete.

Theorem 4.2. *If $m \equiv \pm 1 \pmod{3}$, then the equation (F) has solutions in $M_3(\mathbb{Z})$ in irreducible elements.*

PROOF. Let

$$A = \begin{pmatrix} 0 & 1 & -i \\ 0 & 0 & 1 \\ 2 & 2i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} i-1 & i^2-i & i+1 \\ -1 & -i & -1 \\ -2 & -2i-1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} i-1 & i^2-i+1 & 1 \\ -1 & -i & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Then $\det A = \det B = \det C = 2$, $A + B = C$ and $A^3 = B^3 = C^3 = 2E$ for every $i \in \mathbb{Z}$.

If $m = 3k+1$, then (as in the proof of Theorem 4.1)

$$A^{3k+1} + B^{3k+1} = 2^k A + 2^k B = 2^k C = C^{3k+1}.$$

Let $m = 3k-1$ ($k = 1, 2, \dots$). Then A^2, B^2, C^2 is a solution of (F). Indeed, using equation $(A^2)^{3k-1} = A^{6k-2} = A^{3(2k-1)+1} = 2^{2k-1} A$ it is easy to see that $(A^2)^{3k-1} + (B^2)^{3k-1} = 2^{2k-1} A + 2^{2k-1} B = 2^{2k-1} C = (C^2)^{3k-1}$. The proof is complete.

Theorem 4.3. *If $m = n \geq 2$, then the equation (F) has solutions in $M_n(\mathbb{Z})$ in irreducible elements.*

PROOF. Let E_{n-1} denote the $(n-1) \times (n-1)$ identity matrix, $\mathbf{0}$ the $n-1$ -dimensional vector-column, $\mathbf{0}^*$ the $(n-1)$ -dimensional vector-line and let p be a prime. Then the element $A_p = \begin{pmatrix} \mathbf{0} & E_{n-1} \\ p & \mathbf{0}^* \end{pmatrix}$ is irreducible in $M_n(\mathbb{Z})$, $\det A_p = p$ and it is easy to prove that $(A_p)^n = pE_n$. Therefore, for example, (A_2, A_5, A_7) is a solution of (F) and the proof is complete.

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