

Fixed point of a Ljubomir Ćirić's quasi-contraction mapping in a generalized metric space

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Abstract. We prove a fixed point theorem for a quasi-contraction mapping in a generalized metric space, a concept recently introduced in [1].

1. Introduction

If (X, d) is a complete metric space and $T : X \rightarrow X$ is a contraction, i.e. $d(Tx, Ty) \leq \alpha \cdot d(x, y)$, $0 < \alpha < 1$ for all $x, y \in X$ then the widely known Banach's fixed point theorem tells that T has a unique fixed point in X . During the last four decades, this theorem has undergone various generalizations either by relaxing the condition on contractivity or withdrawing the requirement of completeness or sometimes even both.

Recently [1] a very interesting generalization was obtained by changing the structure of the space itself. BRANCIARI [1] introduced the concept of a generalized metric space by replacing the triangle inequality by a more general inequality. As such, any metric space is a generalized metric space but the converse is not true [1]. He obtained Banach's fixed point theorem in such a space.

Under the situation, it is now reasonable to consider if some of the remarkable generalizations of Banach's theorem may be obtained in a generalized metric space. In this paper we shall prove ĆIRIĆ's fixed point theorem [3] on quasi-contraction mapping, which generalizes Banach's theorem with respect to both the mapping and the space.

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2. Definitions and known theorem

Let $\mathbb{R}^+$ denote the set of all non-negative real numbers and $\mathbb{N}$ denote the set of all positive integers.

Definition 1 [1]. Let X be a set and $d : X^2 \rightarrow \mathbb{R}^+$ a mapping such that for all $x, y \in X$ and for all distinct points $z, w \in X$, each of them different from x and y , one has

- (i) $d(x, y) = 0$ if and only if $x = y$,
- (ii) $d(x, y) = d(y, x)$,
- (iii) $d(x, y) \leq d(x, z) + d(z, w) + d(w, y)$

then we shall say that (X, d) is a generalized metric space (or shortly g.m.s.).

Any metric space is a g.m.s. but the converse is not true [1].

As in the metric setting, any g.m.s. X is a topological space with neighbourhood basis given by

$$B = \{B(x, r) \mid x \in X, r \in \mathbb{R}^+ - \{0\}\}$$

where $B(x, r) = \{y \in X; d(x, y) < r\}$ is the open ball with centre x and radius r .

Definition 2 [1]. Let (X, d) be a g.m.s. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for all $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, $n \geq n_\epsilon$, one has $d(x_m, x_n) < \epsilon$. (X, d) is called complete if every Cauchy sequence in X is convergent in X .

Let $T : X \rightarrow X$ be a mapping where X is a g.m.s. For $A \subset X$ let $\delta(A) = \sup\{d(a, b) \mid a, b \in A\}$ and for each $x \in X$ let

$$O(x, n) = \{x, Tx, T^2x, \dots, T^n x\}, \quad n = 1, 2, \dots;$$

$$O(x, \infty) = \{x, Tx, T^2x, \dots\}.$$

We now consider the following definitions.

Definition 3 (cf. [2], [3]). A g.m.s. X is said to be T -orbitally complete if and only if every Cauchy sequence which is contained in $O(x, \infty)$ for some $x \in X$ converges in X .

Note 1. Considering $X = [0, 1)$ with the usual metric and $Tx = \frac{x}{10}$, $x \in X$ we see that a T -orbitally complete g.m.s. need not be complete.

Definition 4 (cf. [3]). A mapping $T : X \rightarrow X$ where X is a g.m.s. is said to be a quasi-contraction if and only if there exists a number q , $0 \leq q < 1$ such that

$$(1) \quad d(Tx, Ty) \leq q \max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\}$$

hold for all $x, y \in X$.

Ćirić's fixed point theorem. *Let T be a quasi-contraction on a metric space M and let M be T -orbitally complete. Then*

- (a) T has a unique fixed point u in M ,
- (b) $\lim_n T^n x = u$, and
- (c) $d(T^n x, u) \leq \frac{q^n}{1-q} d(x, Tx)$ for every $x \in M$.

Clearly a contraction mapping is a quasi-contraction but the converse is not true (see [3]).

Throughout this paper we shall assume that X is a generalized metric space (g.m.s.).

3. Lemmas

The following lemmas on quasi-contractions will be needed. The proof of Lemma 1 is omitted as it is similar to the proof of Lemma 1 in [3].

Lemma 1. *Let T be a quasi-contraction on X and let n be any positive integer. Then for each $x \in X$ and all positive integers i and j , $i, j \in \{1, 2, \dots, n\}$ we have $d(T^i x, T^j x) \leq q \cdot \delta[O(x, n)]$.*

Remark 1. From Lemma 1 it follows that if $x \in X$, then for every positive integer n , there is a positive integer $k \leq n$ such that $\delta[O(x, n)] = d(x, T^k x)$.

Lemma 2. *If T is a quasi-contraction on X then*

$$\delta[O(x, \infty)] \leq \frac{1}{1-q} \max\{d(x, Tx), d(x, T^2 x)\}$$

holds for all $x \in X$.

PROOF. Let $x \in X$. Since

$$\delta[O(x, 1)] \leq \delta[O(x, 2)] \leq \dots \leq \delta[O(x, n)] \leq \dots,$$

we can write

$$\delta[O(x, \infty)] = \sup\{\delta[O(x, n)]; n \in \mathbb{N}\}.$$

The proof will follow if we show that

$$(2) \quad \delta[O(x, n)] \leq \frac{1}{1-q} \max\{d(x, Tx), d(x, T^2x)\}$$

for all $n \in \mathbb{N}$.

Clearly we can assume that $n \geq 3$. By Remark 1 there is a k , $1 \leq k \leq n$ such that $\delta[O(x, n)] = d(x, T^kx)$. If $k = 1$ or 2 , then the proof follows. So let $k \geq 3$. We note that if $x = Tx$ or $x = T^2x$ or $Tx = T^2x$ then $\delta[O(x, n)] = d(x, Tx)$ and (2) is obtained. So we can assume that x, Tx, T^2x are all distinct while Tx, T^2x are also different from T^kx . Then from (1) and Lemma 1

$$\begin{aligned} d(x, T^kx) &\leq d(x, Tx) + d(Tx, T^2x) + d(T^2x, T^kx) \\ &\leq d(x, Tx) + q \max\{d(x, Tx), d(x, T^2x)\} + d(TTx, T^{k-1}Tx) \\ &\leq (1+q) \max\{d(x, Tx), d(x, T^2x)\} + q\delta[O(Tx, k-1)] \\ &= (1+q) \max\{d(x, Tx), d(x, T^2x)\} + qd(Tx, T^mTx), \quad (m \leq k-1) \\ &\leq (1+q) \max\{d(x, Tx), d(x, T^2x)\} + qq\delta[O(x, m+1)] \\ &\leq (1+q) \max\{d(x, Tx), d(x, T^2x)\} + q^2\delta[O(x, n)] \\ &= (1+q) \max\{d(x, Tx), d(x, T^2x)\} + q^2d(x, T^kx), \end{aligned}$$

i.e.

$$(1 - q^2)d(x, T^kx) \leq (1 + q) \max\{d(x, Tx), d(x, T^2x)\}.$$

So, $\delta[O(x, n)] = d(x, T^kx) \leq \frac{1}{1-q} \max\{d(x, Tx), d(x, T^2x)\}$ and this completes the proof. $\square$

4. Theorem

In this section we prove the main result of the paper.

Theorem 1. *Let T be a quasi-contraction on X and let X be T -orbitally complete. Then*

- a) T has a unique fixed point u in X ,

- b) $\lim_{n \rightarrow \infty} T^n x = u$ for every $x \in X$,
 c) $d(T^n x, u) \leq \frac{q^n}{1-q} \max\{d(x, Tx), d(x, T^2 x)\}$ for all $n \in \mathbb{N}$.

PROOF. We first note that if T has a fixed point, it must be unique because T is a quasi-contraction. Let $x \in X$ and m and n ($m > n$) be arbitrary positive integers. From Lemma 1 we obtain

$$(3) \quad \begin{aligned} d(T^n x, T^m x) &= d(TT^{n-1}x, T^{m-n+1}T^{n-1}x) \\ &\leq q \cdot \delta[O(T^{n-1}x, m-n+1)]. \end{aligned}$$

By Remark 1, there is a k_1 , $1 \leq k_1 \leq m-n+1$ such that

$$(4) \quad \delta[O(T^{n-1}x, m-n+1)] = d(T^{n-1}x, T^{k_1}T^{n-1}x).$$

Again by Lemma 1 we have

$$\begin{aligned} d(T^{n-1}x, T^{k_1}T^{n-1}x) &= d(TT^{n-2}x, T^{k_1+1}T^{n-2}x) \\ &\leq q \cdot \delta[O(T^{n-2}x, k_1+1)] \leq q \cdot \delta[O(T^{n-2}x, m-n+2)]. \end{aligned}$$

Hence from (3) and (4) we obtain

$$\begin{aligned} d(T^n x, T^m x) &\leq q \cdot \delta[O(T^{n-1}x, m-n+1)] \\ &\leq q^2 \cdot \delta[O(T^{n-2}x, m-n+2)]. \end{aligned}$$

Proceeding in this way we get

$$d(T^n x, T^m x) \leq q \cdot \delta[O(T^{n-1}x, m-n+1)] \leq \dots \leq q^n \cdot \delta[O(x, m)].$$

From Lemma 2 it now follows that

$$(5) \quad d(T^n x, T^m x) \leq \frac{q^n}{1-q} \max\{d(x, Tx), d(x, T^2 x)\}.$$

Since $\lim_{n \rightarrow \infty} q^n = 0$, $\{T^n x\}$ is a Cauchy sequence. Since X is T -orbitally complete,

$$(6) \quad \lim_{n \rightarrow \infty} T^n x = u \text{ (say).}$$

Making $m \rightarrow \infty$ in (5) and noting that $\{T^n x\}$ is a Cauchy sequence, we can easily show that

$$(7) \quad d(T^n x, u) \leq \frac{q^n}{1-q} \max\{d(x, Tx), d(x, T^2 x)\}.$$

We shall now show that $Tu = u$. From (1) we have

$$d(Tu, TT^n x) \leq q \max\{d(u, Tu), d(T^n x, T^{n+1} x), d(u, T^{n+1} x), \\ d(T^n x, Tu), d(u, T^n x)\}.$$

Taking the limit as n tends to infinity, by (6) and the fact that $\{T^n x\}$ is a Cauchy sequence, we get $d(Tu, u) \leq q \cdot d(Tu, u)$.

Hence, as $q < 1$, $d(Tu, u) = 0$; hence $Tu = u$. This in conjunction with (6) and (7) completes the proof. $\square$

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References

- [1] A. BRANCIARI, A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces, *Publ. Math. Debrecen* **57** 1–2 (2000), 31–37.
- [2] LJ. B. ĆIRIĆ, On contraction type mappings, *Math. Balkanica* **1** (1971), 52–57.
- [3] LJ. B. ĆIRIĆ, A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.* (45) **2** (1974), 267–273.

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