

On the second variation formula for biharmonic maps to a sphere

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Abstract. We compute the nullity for the following weakly stable biharmonic maps: the identity map $\mathbf{1} : \mathbb{S}^n \rightarrow \mathbb{S}^n$ and the canonical inclusion $\mathbf{i} : \mathbb{S}^m \rightarrow \mathbb{S}^n$.

1. Introduction

A map $\phi : (M, g) \rightarrow (N, h)$ between two Riemannian manifolds is *harmonic* if it is a critical point of the *energy* $E(\phi) = \frac{1}{2} \int_M |d\phi|^2 v_g$. The map ϕ is harmonic if and only if its tension field $\tau(\phi) = \text{trace } \nabla d\phi$ vanishes. In the same way, as suggested by J. EELLS and J. H. SAMPSON in [6], a map ϕ is *biharmonic* if it is a critical point of the *bienergy* $E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 v_g$. G. Y. JIANG has obtained in [7], [8] the first and second variation formula. He has proved that the map ϕ is biharmonic if and only if

$$\tau_2(\phi) = J(\tau(\phi)) = 0,$$

where J is the Jacobi operator of ϕ . Of course, any harmonic map is biharmonic.

B. Y. CHEN and S. ISHIKAWA have shown in [3] that there are no nonharmonic biharmonic submanifolds of $\mathbb{R}^3$. In the same way, in [2], the authors have proved that there are no such submanifolds in $N^3(-1)$, where $N^3(-1)$ is a 3-dimensional manifold with negative constant sectional curvature -1 .

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In [1] the authors have given the classification of nonharmonic biharmonic submanifolds of $\mathbb{S}^3$. They are: circles, spherical helices and parallel spheres. Then, in [2], the authors have given some methods to construct examples of nonharmonic biharmonic submanifolds of the unit n -dimensional sphere $\mathbb{S}^n$, for $n > 3$. In this case the family of such submanifolds is much larger.

A harmonic map is an absolute minimum of the bienergy and hence stable. The goal of this paper is to find the second variation formula for biharmonic maps $\phi : (M, g) \rightarrow \mathbb{S}^n$ and then to compute the nullity for the simplest two biharmonic maps: the identity map $\mathbf{1} : \mathbb{S}^n \rightarrow \mathbb{S}^n$ and the canonical inclusion $i : \mathbb{S}^m \rightarrow \mathbb{S}^n$ (Theorem 2.4 and Theorem 2.5).

Notation. We shall work in the C^∞ category, i.e. manifolds, metrics, connections, maps will be assumed to be smooth. By (M^m, g) we shall indicate a connected manifold of dimension m , without boundary, endowed with a Riemannian metric g . We shall denote by ∇ the Levi-Civita connection of (M, g) . For vector fields X, Y, Z on M we define the Riemann curvature operator by $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z$. The indices i, j, k, l take the values $1, 2, \dots, m$.

2. The second variation formula of the bienergy

Let $\phi : (M, g) \rightarrow (N, h)$ be a smooth map between two Riemannian manifolds. Assume that M is compact and orientable. The tension field of ϕ is given by $\tau(\phi) = \text{trace } \nabla d\phi$ and the *bienergy* is defined by

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 v_g.$$

The map ϕ is called *biharmonic* if it is a critical point of the bienergy. As we said in the introduction, the first variation formula is given by

$$\left. \frac{d}{dt} \right|_{t=0} E_2(\phi_t) = \int_M \langle \tau_2(\phi), V \rangle v_g,$$

where v_g is the volume element, V is the variational vector field corresponding to the variation $\{\phi_t\}_{t \in \mathbb{R}}$ of ϕ , and

$$(2.1) \quad \tau_2(\phi) = -\Delta \tau(\phi) - \text{trace } R^N(d\phi \cdot, \tau(\phi))d\phi \cdot.$$

Now, let $\phi : (M, g) \rightarrow \mathbb{S}^n$ be a biharmonic map. We consider a smooth variation $\{\phi_{s,t}\}_{s,t \in \mathbb{R}}$ of ϕ with two parameters s and t , i.e. we consider the smooth map Φ given by

$$\Phi : \mathbb{R} \times \mathbb{R} \times M \rightarrow \mathbb{S}^n, \quad \Phi(s, t, p) = \phi_{s,t}(p),$$

where $\Phi(0, 0, p) = \phi_{0,0}(p) = \phi(p)$, $\forall p \in M$.

The corresponding variational vector fields V and W are given by

$$V(p) = \left. \frac{d}{ds} \right|_{s=0} \phi_{s,0}(p) = d\Phi_{(0,0,p)} \left(\frac{\partial}{\partial s} \right) \in T_{\phi(p)} \mathbb{S}^n,$$

and

$$W(p) = \left. \frac{d}{dt} \right|_{t=0} \phi_{0,t}(p) = d\Phi_{(0,0,p)} \left(\frac{\partial}{\partial t} \right) \in T_{\phi(p)} \mathbb{S}^n.$$

V and W are sections of $\phi^{-1}T\mathbb{S}^n$, i.e. $V, W \in C(\phi^{-1}T\mathbb{S}^n)$.

The Hessian of E_2 at its critical point ϕ is defined by

$$H(E_2)_\phi(V, W) = \left. \frac{\partial^2}{\partial s \partial t} \right|_{(s,t)=(0,0)} E_2(\phi_{s,t}).$$

Theorem 2.1. *Let $\phi : (M, g) \rightarrow \mathbb{S}^n$ be a biharmonic map. Then the Hessian of the bienergy E_2 at ϕ is given by*

$$H(E_2)_\phi(V, W) = \int_M \langle I(V), W \rangle v_g,$$

where

$$\begin{aligned} (2.2) \quad I(V) = & \Delta(\Delta V) + \Delta\{\text{trace}\langle V, d\phi \cdot \rangle d\phi \cdot - |d\phi|^2 V\} \\ & + 2\langle d\tau(\phi), d\phi \rangle V + |\tau(\phi)|^2 V \\ & - 2\text{trace}\langle V, d\tau(\phi) \cdot \rangle d\phi \cdot - 2\text{trace}\langle \tau(\phi), dV \cdot \rangle d\phi \cdot \\ & - \langle \tau(\phi), V \rangle \tau(\phi) + \text{trace}\langle d\phi \cdot, \Delta V \rangle d\phi \cdot \\ & + \text{trace}\langle d\phi \cdot, \text{trace}\langle V, d\phi \cdot \rangle d\phi \cdot \rangle d\phi \cdot \\ & - 2|d\phi|^2 \text{trace}\langle d\phi \cdot, V \rangle d\phi \cdot \\ & + 2\langle dV, d\phi \rangle \tau(\phi) - |d\phi|^2 \Delta V + |d\phi|^4 V. \end{aligned}$$

PROOF. We start by computing $\frac{\partial}{\partial t}\big|_{t=0} E_2(\phi_{s,t})$. We have

$$\begin{aligned} \frac{\partial}{\partial t}\big|_{t=0} E_2(\phi_{s,t}) &= \frac{\partial}{\partial t}\big|_{t=0} \frac{1}{2} \int_M |\tau(\phi_{s,t})|^2 v_g \\ &= \int_M \langle \nabla_{\frac{\partial}{\partial t}} \tau(\phi_{s,t}), \tau(\phi_{s,t}) \rangle \big|_{t=0} v_g. \end{aligned}$$

In order to obtain $\nabla_{\frac{\partial}{\partial t}} \tau(\phi_{s,t})$, let $\{X_i\}_{i=1}^m$ be a geodesic frame field around an arbitrary point $p \in M$. We obtain

$$\begin{aligned} \nabla_{\frac{\partial}{\partial t}} \tau(\phi_{s,t}) &= \nabla_{\frac{\partial}{\partial t}} \left\{ \sum_{i=1}^m (\nabla_{X_i} d\phi_{s,t}(X_i) - d\phi_{s,t}(\nabla_{X_i} X_i)) \right\} \\ &= \nabla_{\frac{\partial}{\partial t}} \left\{ \sum_{i=1}^m (\nabla_{X_i} d\Phi_s(X_i) - d\Phi_s(\nabla_{X_i} X_i)) \right\}, \end{aligned}$$

where $\Phi_s(t, p) = \Phi(s, t, p)$. Using the formula

$$\nabla_{\tilde{X}} d\Phi_s(\tilde{Y}) - \nabla_{\tilde{Y}} d\Phi_s(\tilde{X}) = d\Phi_s([\tilde{X}, \tilde{Y}]), \quad \forall \tilde{X}, \tilde{Y} \in C(\Phi_s^{-1} T\mathbb{S}^n),$$

we obtain, at p and for $t = 0$, the following

$$\begin{aligned} \nabla_{\frac{\partial}{\partial t}} \tau(\phi_{s,t}) &= \sum_{i=1}^m \left\{ \nabla_{\frac{\partial}{\partial t}} \nabla_{X_i} d\Phi_s(X_i) - \nabla_{\frac{\partial}{\partial t}} d\Phi_s(\nabla_{X_i} X_i) \right\} \\ &= \sum_{i=1}^m \left\{ \nabla_{\frac{\partial}{\partial t}} \nabla_{X_i} d\Phi_s(X_i) - \nabla_{\nabla_{X_i} X_i} d\Phi_s\left(\frac{\partial}{\partial t}\right) - d\Phi_s\left(\left[\frac{\partial}{\partial t}, \nabla_{X_i} X_i\right]\right) \right\} \\ &= \sum_{i=1}^m \nabla_{\frac{\partial}{\partial t}} \nabla_{X_i} d\Phi_s(X_i) = \sum_{i=1}^m \left\{ R^{\mathbb{S}^n}(d\Phi_s\left(\frac{\partial}{\partial t}\right), d\Phi_s(X_i)) d\Phi_s(X_i) \right. \\ &\quad \left. + \nabla_{X_i} \nabla_{\frac{\partial}{\partial t}} d\Phi_s(X_i) + \nabla_{[\frac{\partial}{\partial t}, X_i]} d\Phi_s(X_i) \right\} \\ &= \sum_{i=1}^m \left\{ R^{\mathbb{S}^n}(W_s, d\Phi_s(X_i)) d\Phi_s(X_i) \right. \\ &\quad \left. + \nabla_{X_i} \left(\nabla_{X_i} d\Phi_s\left(\frac{\partial}{\partial t}\right) + d\Phi_s\left(\left[\frac{\partial}{\partial t}, X_i\right]\right) \right) \right\} \\ &= \sum_{i=1}^m R^{\mathbb{S}^n}(W_s, d\Phi_s(X_i)) d\Phi_s(X_i) + \sum_{i=1}^m \nabla_{X_i} \nabla_{X_i} W_s \end{aligned}$$

$$= -\Delta W_s - \sum_{i=1}^m R^{\mathbb{S}^n}(d\Phi_s(X_i), W_s) d\Phi_s(X_i),$$

where

$$W_s(p) = \frac{d}{dt} \Big|_{t=0} \phi_{s,t}(p) = d\Phi_{s,(0,p)} \left(\frac{\partial}{\partial t} \right), \quad W_s \in C(\phi_{s,0}^{-1} T\mathbb{S}^n), \quad W_0 = W.$$

Thus $\frac{\partial}{\partial t} \Big|_{t=0} E_2(\phi_{s,t})$ is given by

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=0} E_2(\phi_{s,t}) &= \int_M \langle -\Delta W_s - \text{trace } R^{\mathbb{S}^n}(d\phi_{s,0}\cdot, W_s) d\phi_{s,0}\cdot, \tau(\phi_{s,0}) \rangle v_g \\ &= \int_M \langle -\Delta \tau(\phi_{s,0}) - \text{trace } R^{\mathbb{S}^n}(d\phi_{s,0}\cdot, \tau(\phi_{s,0})) d\phi_{s,0}\cdot, W_s \rangle v_g. \end{aligned}$$

Since ϕ is biharmonic, from (2.1) we obtain

$$\begin{aligned} H(E_2)_\phi(V, W) &= \frac{\partial}{\partial s} \Big|_{s=0} \int_M \langle -\Delta \tau(\phi_{s,0}) - \text{trace } R^{\mathbb{S}^n}(d\phi_{s,0}\cdot, \tau(\phi_{s,0})) d\phi_{s,0}\cdot, W_s \rangle v_g \\ &= \int_M \langle \nabla_{\frac{\partial}{\partial s}} \{ -\Delta \tau(\phi_{s,0}) - \text{trace } R^{\mathbb{S}^n}(d\phi_{s,0}\cdot, \tau(\phi_{s,0})) d\phi_{s,0}\cdot \} \Big|_{s=0}, W \rangle v_g \\ &= \int_M \langle I(V), W \rangle v_g, \end{aligned}$$

where

$$(2.3) \quad I(V) = \nabla_{\frac{\partial}{\partial s}} \{ -\Delta \tau(\phi_{s,0}) - \text{trace } R^{\mathbb{S}^n}(d\phi_{s,0}\cdot, \tau(\phi_{s,0})) d\phi_{s,0}\cdot \} \Big|_{s=0}.$$

Next, since

$$\nabla_{\frac{\partial}{\partial s}} \tau(\phi_{s,0}) \Big|_{s=0} = -\Delta V - \text{trace } R^{\mathbb{S}^n}(d\phi\cdot, V) d\phi\cdot$$

and

$$\text{trace } R^{\mathbb{S}^n}(d\phi\cdot, V) d\phi\cdot = \text{trace} \langle V, d\phi\cdot \rangle d\phi\cdot - |d\phi|^2 V,$$

we get

$$\begin{aligned} \nabla_{\frac{\partial}{\partial s}} \{ -\Delta \tau(\phi_{s,0}) \} \Big|_{s=0} &= \Delta(\Delta V) + \Delta \{ \text{trace} \langle V, d\phi\cdot \rangle d\phi\cdot - |d\phi|^2 V \} \\ (2.4) \quad &+ 2 \langle d\tau(\phi), d\phi \rangle V + |\tau(\phi)|^2 V + \text{trace} \langle \tau(\phi), d\phi\cdot \rangle dV\cdot \\ &- 2 \text{trace} \langle V, d\tau(\phi)\cdot \rangle d\phi\cdot - \text{trace} \langle \tau(\phi), dV\cdot \rangle d\phi\cdot - \langle \tau(\phi), V \rangle \tau(\phi), \end{aligned}$$

and

$$\begin{aligned}
 & \nabla_{\frac{\partial}{\partial s}} \left\{ -\text{trace } R^{\mathbb{S}^n}(d\phi_{s,0}, \tau(\phi_{s,0}))d\phi_{s,0} \right\} \Big|_{s=0} \\
 &= -\text{trace} \langle \tau(\phi), dV \cdot \rangle d\phi + \text{trace} \langle d\phi, \Delta V \rangle d\phi \\
 (2.5) \quad &+ \text{trace} \langle d\phi, \text{trace} \langle V, d\phi \rangle d\phi \rangle d\phi \\
 &- |d\phi|^2 \text{trace} \langle d\phi, V \rangle d\phi - \text{trace} \langle \tau(\phi), d\phi \rangle dV \\
 &+ 2 \langle dV, d\phi \rangle \tau(\phi) - |d\phi|^2 \Delta V - |d\phi|^2 \text{trace} \langle V, d\phi \rangle d\phi + |d\phi|^4 V.
 \end{aligned}$$

Now, replacing (2.4) and (2.5) in (2.3), we obtain (2.2). $\square$

Remark 2.2. We note that formula (2.2) can be also deduced from formula (5.8) in [8].

Corollary 2.3. *Let $\phi : (M, g) \rightarrow \mathbb{S}^n$ be a harmonic Riemannian immersion. Then the operator I of ϕ is symmetric, positive semi-definite and*

$$(2.6) \quad \ker I = \{V \in C(\phi^{-1}T\mathbb{S}^n) \mid \Delta V = mV - V^T\},$$

where $V = V^T + V^N$, $V^T \in C(TM)$ and $V^N \in C(NM)$.

PROOF. From (2.2) it follows

$$I(V) = \Delta(\Delta V) - 2m\Delta V + m^2V + \Delta V^T + (\Delta V)^T + (1 - 2m)V^T.$$

First we shall prove that I is symmetric, i.e. $(I(V), W) = (V, I(W))$, $\forall V, W \in C(\phi^{-1}T\mathbb{S}^n)$, where $(V, W) = \int_M \langle V, W \rangle v_g$ is the usual inner product on the real vector space $C(\phi^{-1}T\mathbb{S}^n)$. Since Δ is a symmetric operator and $\langle V^T, W \rangle = \langle W^T, V \rangle$, in order to prove that I is symmetric we must show that

$$\int_M \langle \Delta V^T + (\Delta V)^T, W \rangle v_g = \int_M \langle \Delta W^T + (\Delta W)^T, V \rangle v_g.$$

But

$$\begin{aligned}
 \int_M \langle \Delta V^T, W \rangle v_g &= \int_M \langle V^T, \Delta W \rangle v_g = \int_M \langle V^T, (\Delta W)^T \rangle v_g \\
 &= \int_M \langle V, (\Delta W)^T \rangle v_g,
 \end{aligned}$$

and

$$\begin{aligned}\int_M \langle (\Delta V)^T, W \rangle v_g &= \int_M \langle (\Delta V)^T, W^T \rangle v_g = \int_M \langle \Delta V, W^T \rangle v_g \\ &= \int_M \langle V, \Delta W^T \rangle v_g.\end{aligned}$$

So I is a symmetric operator.

In order to prove that J is positive semi-definite, i.e. $(I(V), V) \geq 0$, we start with the following remarks

$$\int_M \langle \Delta V^T, V \rangle v_g = \int_M \langle (\Delta V)^T, V \rangle v_g,$$

and

$$\begin{aligned}I(V) &= \Delta \Delta V^T + \Delta \Delta V^N - 2m \Delta V^T - 2m \Delta V^N + m^2 V^T + m^2 V^N \\ &\quad + \Delta V^T + (\Delta V)^T + (1 - 2m) V^T.\end{aligned}$$

Thus we have

$$\begin{aligned}(I(V), V) &= \int_M \{ \langle \Delta(\Delta V^T), V \rangle + 2(1 - m) \langle \Delta V^T, V \rangle + (m - 1)^2 \langle V^T, V \rangle \\ &\quad + \langle \Delta(\Delta V^N), V \rangle - 2m \langle \Delta V^N, V \rangle + m^2 \langle V^N, V \rangle \} v_g \\ &= \int_M \{ \langle \Delta(\Delta V^T), V^T \rangle + 2(1 - m) \langle \Delta V^T, V^T \rangle + (m - 1)^2 |V^T|^2 \\ &\quad + \langle \Delta(\Delta V^N), V^N \rangle - 2m \langle \Delta V^N, V^N \rangle + m^2 |V^N|^2 \\ &\quad + \langle \Delta(\Delta V^T), V^N \rangle + 2(1 - m) \langle \Delta V^T, V^N \rangle \\ &\quad + \langle \Delta(\Delta V^N), V^T \rangle - 2m \langle \Delta V^N, V^T \rangle \} v_g \\ &= \int_M \{ |\Delta V^T + (1 - m) V^T|^2 + |\Delta V^N - m V^N|^2 \\ &\quad + 2(\langle \Delta V^T, \Delta V^N \rangle + (1 - 2m) \langle \Delta V^T, V^N \rangle) \} v_g \\ &= \int_M |\Delta V^T + (1 - m) V^T + \Delta V^N - m V^N|^2 v_g \\ &= \int_M |\Delta V - m V + V^T|^2 v_g.\end{aligned}$$

From the above relation it follows that I is positive semi-definite and $\ker I$ is given by (2.6). $\square$

In the following we shall consider the simplest two cases of biharmonic maps $\phi : (M, g) \rightarrow \mathbb{S}^n$. These maps are harmonic Riemannian immersions, so they are weakly-stable, i.e. the operator I is positive semi-definite.

Theorem 2.4. *The identity map $\mathbf{1} : \mathbb{S}^n \rightarrow \mathbb{S}^n$ is weakly-stable and*

- a) *if $n = 2$ then $\text{nullity}(\mathbf{1}) = 6$,*
- b) *if $n > 2$ then $\text{nullity}(\mathbf{1}) = \frac{n(n+1)}{2}$.*

PROOF. In this case $C(\mathbf{1}^{-1}T\mathbb{S}^n) = C(T\mathbb{S}^n)$ and $\Delta V = -\text{trace } \nabla^2 V$. We shall use X to denote a tangent vector field on $\mathbb{S}^n$. By Corollary 2.3, the operator I is given by

$$I(X) = \Delta(\Delta X) - 2(n-1)\Delta X + (n-1)^2 X,$$

and

$$I(X) = 0 \iff \Delta X = (n-1)X.$$

The Hodge decomposition theorem for $C(T\mathbb{S}^n)$ states that

$$C(T\mathbb{S}^n) = \{X \in C(T\mathbb{S}^n) \mid \text{div } X = 0\} \oplus \{\text{grad } f \mid f \in C^\infty(\mathbb{S}^n)\}.$$

This decomposition of $C(T\mathbb{S}^n)$ is orthogonal with respect to the scalar product on the real vector space $C(T\mathbb{S}^n)$, and Δ_H preserves invariantly these subspaces, where, using the musical isomorphisms,

$$\Delta_H(X) = (\overline{\Delta} X^\flat)^\sharp,$$

$\overline{\Delta}$ being the Laplacian which acts on $\Lambda^1(\mathbb{S}^n)$.

It is known that

$$\Delta X = \Delta_H(X) - (n-1)X$$

(see [5], [11]), so

$$I(X) = 0 \iff \Delta_H(X) = 2(n-1)X.$$

From the Hodge decomposition theorem we write $X = Y + \text{grad } f$, $\text{div } Y = 0$ and we obtain

$$\Delta_H(X) = 2(n-1)X \iff \begin{cases} \Delta_H(Y) = 2(n-1)Y \\ \Delta_H \text{grad } f = 2(n-1) \text{grad } f. \end{cases}$$

Consequently

$$I(X) = 0 \iff \begin{cases} Y \text{ is a Killing vector field} \\ \Delta f = 2(n-1)f. \end{cases}$$

It is known that the first eigenvalues of Δ which acts on $C^\infty(\mathbb{S}^n)$ are 0, n , $2(n+1)$, and the eigenvalue n has the multiplicity $n+1$. So $2(n-1)$ is an eigenvalue if and only if $n=2$, and in this case its multiplicity is 3.

It is well known too that

$$\dim\{Y \in C(T\mathbb{S}^n) \mid Y \text{ is a Killing vector field}\} = \frac{n(n+1)}{2}.$$

Now, the theorem follows. $\square$

Theorem 2.5. *The canonical inclusion $\mathbf{i} : \mathbb{S}^m \rightarrow \mathbb{S}^n$ is weakly-stable and*

- a) *if $m=2$ then $\text{nullity}(\mathbf{i}) = 3n$,*
- b) *if $m > 2$ then $\text{nullity}(\mathbf{i}) = (n-m)(m+1) + \frac{m(m+1)}{2}$.*

PROOF. Let $V \in C(N\mathbb{S}^m)$ and $X, Y \in C(T\mathbb{S}^m)$. As $\mathbf{i}$ is a totally geodesic map, it results that

$$\nabla_X V = \nabla_X^\perp V, \quad \Delta V = \Delta^\perp V, \quad \nabla_X Y = \nabla_X^{\mathbb{S}^m} Y, \quad \Delta X = -\text{trace}^{\mathbb{S}^m} \nabla^2 X.$$

Again, by Corollary 2.3, the operator I is given by

$$\begin{cases} I(V) = \Delta^\perp(\Delta^\perp V) - 2m\Delta^\perp V + m^2 V \in C(N\mathbb{S}^m) \\ I(X) = \Delta(\Delta X) - 2(m-1)\Delta X + (m-1)^2 X \in C(T\mathbb{S}^m), \end{cases}$$

and

$$\begin{cases} I(V) = 0 \iff \Delta^\perp V = mV \\ I(X) = 0 \iff \Delta X = (m-1)X. \end{cases}$$

Now, let $\{E_{m+1}, \dots, E_n\}$ be the vector fields which give the trivialisation of $N\mathbb{S}^m$. We have

$$(2.7) \quad \nabla_X E_{m+1} = \dots = \nabla_X E_n = 0, \quad \forall X \in C(T\mathbb{S}^m)$$

(see [10]). Since any $V \in C(N\mathbb{S}^m)$ can be written as

$$V = f^1 E_{m+1} + \dots + f^{n-m} E_n,$$

where $f^1, \dots, f^{n-m} \in C^\infty(\mathbb{S}^m)$, from (2.7) we obtain

$$\Delta^\perp V = mV \iff \Delta f^1 = m f^1, \dots, \Delta f^{n-m} = m f^{n-m}.$$

So we have

$$\dim\{V \in C(N\mathbb{S}^m) \mid I(V) = 0\} = (n-m)(m+1).$$

Now, using Theorem 2.4 and the fact that the kernel of I splits in the direct sum of the kernel of I restricted to $C(N\mathbb{S}^m)$ and the kernel of I restricted to $C(T\mathbb{S}^m)$, we conclude. $\square$

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